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Dynamic behaviour for a class of FitzHugh-Nagumo network model

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Abstract
The present paper studies the dynamic behaviour for a class of coupled FHN network models with discrete delays. Two criteria are provided to reveal the existence of periodic oscillations for this model, which is simpler than the bifurcation method. Numerical simulation is presented to demonstrate the effectiveness of this method.

Keywords: A coupled FHN network model, delay, instability, oscillation

Introduction
In 1961, FitzHugh provided a nerve membrane model as follows.

$$\begin{cases} \varepsilon x'(t) = x(t) - \frac{1}{3}x^3(t) - y(t) \\ y'(t) = x(t) - by(t) + a \end{cases} \tag{1}$$

Where x means the activator, and y the inhibitor state of a neuron; ε, a, b are some constants. Since then various FHN models have been studied. For example, Rybalova *et al*. considered the following *n-coupled* model [2].

$$\begin{cases} \varepsilon {u_i}' = u_i - \frac{1}{3}u_i^3 - v_i + \frac{\sigma}{2R}\sum_{j=i-R}^{i+R}\left(b_{uu}(u_j - u_i) + b_{vv}(v_j - v_i)\right) \\ {v_i}' = u_i + a_i + \frac{\sigma}{2R}\sum_{j=i-R}^{i+R}\left(b_{uu}(u_j - u_i) + b_{vv}(v_j - v_i)\right) \end{cases} \tag{2}$$

The authors explored the phase space structure, calculated basins of attraction, and analyzed the system's parameter space. Plotnikov and Fradkov also considered a n coupled network model [3].

$$\begin{cases} \varepsilon {x_i}' = x_i - \frac{1}{3}x_i^3 - y_i + \sum_{j=1}^{n} c_{ij}x_j + u, \\ {y_i}' = x_i - by_i + a, i = 1,2,\dots,n. \end{cases} \tag{3}$$

The FHN model (3) with random topology and different coupling values has been discussed. It is known that many network models which include the FHN network should incorporate time delays due to the finite propagation speeds of signals. Therefore, some delayed FHN network models have been investigated [4-15]. For instance, Lu *et al*. considered the following FHN model [4].

$$\begin{cases} x_1'(t) = x_1(t)(x_1(t) - a)\left(1 - x_1(t)\right) - y_1(t) + M\left(x_2(t-\tau) - x_1(t)\right) + \xi(t) \\ y_1'(t) = bx_1(t) - cy_1(t) + M_1\left(y_2(t-\tau) - y_1(t)\right) \\ x_2'(t) = x_2(t)(x_2(t) - a)\left(1 - x_2(t)\right) - y_2(t) + M\left(x_1(t-\tau) - x_2(t)\right) + \xi(t) \\ y_2'(t) = bx_2(t) - cy_2(t) + M_1\left(y_1(t-\tau) - y_2(t)\right) \end{cases} \tag{4}$$

It was found that time delays can induce the phase synchronization and mode transition of oscillation. Zhen and Xu considered a coupled FHN neural system with delay as follows [5].

$$\begin{cases} u_1'(t) = -u_1(t)(u_1(t)-1)(u_1(t)-a) - u_2(t) + tanh\big(u_3(t-\tau)\big) \\ u_2'(t) = b\big(u_1(t) - ru_2(t)\big) \\ u_3'(t) = -u_3(t)(u_3(t)-1)(u_3(t)-a) - u_4(t) + tanh\big(u_1(t-\tau)\big) \\ u_4'(t) = b\big(u_3(t) - ru_4(t)\big) \end{cases} \tag{5}$$

By means of the normal form method, steady-state Tate bifurcation of the coupled FHN model (5) was investigated. In order to describe the model as more reasonable, Fan and Hong [6] introduced second time delay in model (5).

$$\begin{cases} u_1'(t) = -u_1(t)(u_1(t)-1)(u_1(t)-a) - u_2(t) + tanh\big(u_3(t-\tau_1)\big) \\ u_2'(t) = b\big(u_1(t) - ru_2(t)\big) \\ u_3'(t) = -u_3(t)(u_3(t)-1)(u_3(t)-a) - u_4(t) + tanh\big(u_1(t-\tau_2)\big) \\ u_4'(t) = b\big(u_3(t) - ru_4(t)\big) \end{cases} \tag{6}$$

An investigation of stability and Hop bifurcation of the coupled system (6) was presented. Similarly, the bifurcating periodic solution and stability for two coupled FHN models, one can see [7-13]. In [14], Zhen and Xu considered a three-coupled FHN model.

$$\begin{cases} u_1{}'(t) = -\frac{1}{3}u_1^3(t) + cu_1^2(t) + du_1(t) - u_2(t) + \alpha u_1^2(t) + \beta[f(u_3(t-\tau) + f(u_5(t-\tau))] \\ u_2'(t) = \varepsilon\big(u_1(t) - bu_2(t)\big) \\ u_3{}'(t) = -\frac{1}{3}u_3^3(t) + cu_3^2(t) + du_3(t) - u_4(t) + \alpha u_3^2(t) + \beta[f(u_1(t-\tau) + f(u_5(t-\tau))] \\ u_4'(t) = \varepsilon\big(u_3(t) - bu_4(t)\big) \\ u_5{}'(t) = -\frac{1}{3}u_5^3(t) + cu_5^2(t) + du_5(t) - u_6(t) + \alpha u_5^2(t) + \beta[f(u_1(t-\tau) + f(u_3(t-\tau))] \\ u_6'(t) = \varepsilon\big(u_5(t) - bu_6(t)\big) \end{cases} \tag{7}$$

A group of sufficient conditions were given to present Bautin bifurcation of the synchronous system by applying the Bautin bifurcation theorem. In [15], Iqbal *et al*. discussed the following four coupled FHN models:

$$\begin{cases} x_1'(t) = x_1(x_1-1)(1-r_1x_1) - y_1 - g_1[(x_1-x_2) + (x_1-x_4)] + I_{ext,1} + d_{ext,1} \\ y_1'(t) = b_1x_1 \\ x_2'(t) = x_2(x_2-1)(1-r_2x_2) - y_2 - g_2[(x_2-x_1) + (x_2-x_3)] + I_{ext,2} + d_{ext,2} + u_{x1} \\ y_2'(t) = b_2x_2 + u_{y1} \\ x_3'(t) = x_3(x_3-1)(1-r_3x_3) - y_3 - g_3[(x_3-x_2) + (x_3-x_4)] + I_{ext,3} + d_{ext,3} + u_{x2} \\ y_3'(t) = b_3x_3 + u_{y2} \\ x_4'(t) = x_4(x_4-1)(1-r_4x_4) - y_4 - g_4[(x_4-x_3) + (x_4-x_1)] + I_{ext,1} + d_{ext,1} + u_{x3} \\ y_4'(t) = b_4x_4 + u_{y3} \end{cases} \tag{8}$$

The dynamical behaviour analysis, and synchronization for model (8) with unknown parameters linked were exploited. For a single neuron of memristive FHN network, Njitacke *et al*. presented the dynamical analysis [16]. Semenov *et al*. discussed a stochastic FHN model with time-delayed feedback by means of simulations [17]. For a modified FHN neuron model with external electric fields, Zhang *et al*. studied the complex motions of the neuron by means of the discrete mapping method [18]. Motivated by the above models, in this paper, we will consider the following five coupled FHN models with discrete delays.

$$\begin{cases} u_1{}'(t) = -\frac{1}{3}u_1^3(t) + c_1u_1^2(t) + d_1u_1(t) - r_1u_2(t) + \alpha_1u_1^2(t) \\ \quad + \sum_{j=2}^{5}\beta_{1,2j-1}\left[f(u_1(t-\tau_1) - f(u_{2j-1}(t-\tau_{2j-1}))\right] \\ u_2'(t) = \varepsilon_1u_1(t) - b_1u_2(t) \\ u_3{}'(t) = -\frac{1}{3}u_3^3(t) + c_2u_3^2(t) + d_2u_3(t) - r_2u_4(t) + \alpha_2u_3^2(t) \\ \quad + \sum_{j=1,j\neq 2}^{5}\beta_{3,2j-1}\left[f(u_3(t-\tau_3) - f(u_{2j-1}(t-\tau_{2j-1}))\right] \\ u_4'(t) = \varepsilon_2u_3(t) - b_2u_4(t) \\ u_5{}'(t) = -\frac{1}{3}u_5^3(t) + c_3u_5^2(t) + d_3u_5(t) - r_3u_6(t) + \alpha_3u_5^2(t) \\ \quad + \sum_{j=1,j\neq 3}^{5}\beta_{5,2j-1}\left[f(u_5(t-\tau_5) - f(u_{2j-1}(t-\tau_{2j-1}))\right] \\ u_6'(t) = \varepsilon_3u_5(t) - b_3u_6(t) \\ u_7{}'(t) = -\frac{1}{3}u_7^3(t) + c_4u_7^2(t) + d_4u_7(t) - r_4u_8(t) + \alpha_4u_7^2(t) \\ \quad + \sum_{j=1,j\neq 4}^{5}\beta_{7,2j-1}\left[f(u_7(t-\tau_7) - f(u_{2j-1}(t-\tau_{2j-1}))\right] \\ u_8'(t) = \varepsilon_4u_7(t) - b_4u_8(t) \\ u_9{}'(t) = -\frac{1}{3}u_9^3(t) + c_5u_9^2(t) + d_5u_9(t) - r_5u_{10}(t) + \alpha_5u_9^2(t) \\ \quad + \sum_{j=1}^{4}\beta_{9,2j-1}\left[f(u_9(t-\tau_9) - f(u_{2j-1}(t-\tau_{2j-1}))\right] \\ u_{10}'(t) = \varepsilon_5u_9(t) - b_5u_{10}(t) \end{cases} \tag{9}$$

Our goal is to consider the dynamic behaviour of the model (9). Since system (9) has five delays, and if those delays are different positive numbers, then the bifurcating method is not easy to deal with model (9) due to the complexity of bifurcating equation. In order to discuss the dynamic behaviour of the solutions for system (9), we adopt the generalized Chafee's criterion [19, 20].

Preliminaries

For the activation functions $f(u)$, we assume that $f(u)$ is continuous bounded differentiable function, satisfying:

$f(0) = 0, uf(u) > 0 \ (u \neq 0)$ (10)

The general activation functions such as tanh (x), arctan (x) satisfy condition (10). The linearized system of (9) around the zero point is the following:

$$\begin{cases} u_1{}'(t) = d_1u_1(t) - r_1u_2(t) + b_{11}u_1(t-\tau_1) - b_{13}u_3(t-\tau_3) \\ \quad -b_{15}u_5(t-\tau_5) - b_{17}u_7(t-\tau_7) - b_{19}u_9(t-\tau_9) \\ \quad u_2'(t) = \varepsilon_1u_1(t) - b_1u_2(t) \\ u_3{}'(t) = d_2u_3(t) - r_2u_4(t) - b_{31}u_1(t-\tau_1) + b_{33}u_3(t-\tau_3) \\ \quad -b_{35}u_5(t-\tau_5) - b_{37}u_7(t-\tau_7) - b_{39}u_9(t-\tau_9) \\ \quad u_4'(t) = \varepsilon_2u_3(t) - b_2u_4(t) \\ u_5{}'(t) = d_3u_5(t) - r_3u_6(t) - b_{51}u_1(t-\tau_1) - b_{53}u_3(t-\tau_3) \\ \quad +b_{55}u_5(t-\tau_5) - b_{57}u_7(t-\tau_7) - b_{59}u_9(t-\tau_9) \\ \quad u_6'(t) = \varepsilon_3u_5(t) - b_3u_6(t) \\ u_7{}'(t) = d_4u_7(t) - r_4u_8(t) - b_{71}u_1(t-\tau_1) - b_{73}u_3(t-\tau_3) \\ \quad -b_{75}u_5(t-\tau_5) + b_{77}u_7(t-\tau_7) - b_{79}u_9(t-\tau_9) \\ \quad u_8'(t) = \varepsilon_4u_7(t) - b_4u_8(t) \\ u_9{}'(t) = d_5u_9(t) - r_5u_{10}(t) - b_{91}u_1(t-\tau_1) - b_{93}u_3(t-\tau_3) \\ \quad -b_{95}u_5(t-\tau_5) - b_{97}u_7(t-\tau_7) + b_{99}u_9(t-\tau_9) \\ \quad u_{10}'(t) = \varepsilon_5u_9(t) - b_5u_{10}(t) \end{cases} \tag{11}$$

Where $b_{ii} = \sum_{j=1,j\neq i}^{5} \beta_{ij} f'(0), b_{ij} = \beta_{ij} f'(0) \ (i,j = 1,3,5,7,9)$. System (11) can be written in a matrix form:

$U'(t) = AU(t) + BU(t-\tau)$ (12)

Where $U(t) = [u_1(t), u_2(t), \dots, u_{10}(t)]^T$, $U(t-\tau) = [u_1(t-\tau_1), 0, u_3(t-\tau_3), \dots, u_9(t-\tau_9), 0]^T$. Both A and B are 10×10 matrices as follows:

$$A = (a_{ij})_{10\times10} = \begin{pmatrix} d_1 & -r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_1 & -b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_2 & -r_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & -b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_3 & -r_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_3 & -b_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_4 & -r_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_4 & -b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_5 & -r_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_5 & -r_5 \end{pmatrix},$$

$$B = (b_{ij})_{10\times10} = \begin{pmatrix} b_{11} & 0 & -b_{13} & 0 & -b_{15} & 0 & -b_{17} & 0 & -b_{19} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{31} & 0 & b_{33} & 0 & -b_{35} & 0 & -b_{37} & 0 & -b_{39} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{51} & 0 & -b_{53} & 0 & b_{55} & 0 & -b_{67} & 0 & -b_{59} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{71} & 0 & -b_{73} & 0 & -b_{75} & 0 & b_{77} & 0 & -b_{79} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{91} & 0 & -b_{93} & 0 & -b_{95} & 0 & -b_{97}\,0 & 0 & b_{99} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus system (9) can be written as a matrix form

$U'(t) = AU(t) + BU(t-\tau) + \Phi(U)$ (13)

Where $\Phi(U)$ is a ten-by-one vector?

$$\Phi(U) = [\Phi_1(U), \Phi_2(U), \dots, \Phi_{10}(U)]^T$$

$$= \Bigg[-\frac{1}{3}u_1^3(t) + (c_1 + \alpha_1)u_1^2(t) + \sum_{j=2}^{5} \beta_{1,2j-1}\left[f(u_1(t-\tau_1) - f(u_{2j-1}(t-\tau_{2j-1}))\right] - \sum_{j=2}^{5} \beta_{1,2j-1}[f_1'(0) - f_{2j-1}'(0)], 0, \dots, -\frac{1}{3}u_9^3(t) + (c_5 + \alpha_5)u_9^2(t) + \sum_{j=1}^{4} \beta_{9,2j-1}\left[f(u_9(t-\tau_9) - f(u_{2j-1}(t-\tau_{2j-1}))\right] - \sum_{j=1}^{4} \beta_{9,2j-1}[f_9'(0) - f_{2j-1}'(0)], 0\Bigg]^T.$$

Lemma 1 All solutions of system (9) are bounded.

Proof From condition (10), the activation functions are bounded, and assuming that $|f(u_i)| \le N_i$ (i = 1, 3, 5, 7, 9). Construct a Lyapunov function $V(t) = \sum_{i=1}^{10} \frac{1}{2}u_i^2(t)$, then calculating the derivative of $V(t)$ through system (9) we have

$$V'(t)|_{(9)} = \sum_{i=1}^{10} u_i'(t)\, u_i(t)$$

$$= -\frac{1}{3}u_1^4(t) + c_1u_1^3(t) + d_1u_1^2(t) - r_1u_1(t)u_2(t) + \alpha_1u_1^3(t)$$
$$+u_1(t)\sum_{j=2}^{5} \beta_{1,2j-1}\left[f(u_1(t-\tau_1) - f(u_{2j-1}(t-\tau_{2j-1}))\right] + \cdots$$
$$-\frac{1}{3}u_9^4(t) + c_5u_9^3(t) + d_5u_9^2(t) - r_5u_9(t)u_{10}(t) + \alpha_5u_9^3(t)$$
$$+u_9(t)\sum_{j=1}^{4} \beta_{9,2j-1}\left[f(u_9(t-\tau_9) - f(u_{2j-1}(t-\tau_{2j-1}))\right] +$$
$$\varepsilon_5u_9(t)u_{10}(t) - b_5u_{10}^2(t)$$
$$\le -\frac{1}{3}\left(u_1^4(t) + \cdots + u_9^4(t)\right) + |c_1u_1^3(t)| + \cdots + |c_5u_9^3(t)|$$
$$+\cdots + \textstyle\sum_{j=1}^{4} |\beta_{9,2j-1}|\, N_{2j-1}|u_9(t)| + |\varepsilon_5||u_9(t)u_{10}(t)| + |b_5|u_{10}^2(t) \tag{14}$$

Noting that $u_i^4(t)$ are higher order infinity than $u_i^3(t)$, $u_i^2(t)$, $|u_i(t)u_j(t)|$ as $u_i \to \infty$, and $-\frac{1}{3} < 0$. Therefore, there exists a sufficiently large positive number L such that $V'(t)|_{(9)} < 0$ as $|u_i(t)| > L$, implying that all solutions of system (9) are bounded.

Lemma 2 If the matrix $A + B = C$ is a nonsingular matrix, then system (9) has a unique equilibrium point.

Proof If $U^* = [u_1^*, u_2^*, \dots, u_{10}^*]^T$ is an equilibrium point of system (9), then we have the following algebraic equation:

$$\begin{cases} -\frac{1}{3}u_1^{*3} + c_1u_1^{*2} + d_1u_1^* - r_1u_2^* + \alpha_1u_1^{*2} + \sum_{j=2}^{5}\beta_{1,2j-1}\left[f(u_1^*) - f(u_{2j-1}^*)\right] = 0 \\ \varepsilon_1u_1^* - b_1u_2^* = 0 \\ -\frac{1}{3}u_3^{*3} + c_2u_3^{*2} + d_2u_3^* - r_2u_4^* + \alpha_2u_3^{*2} + \sum_{j=1,j\ne2}^{5}\beta_{3,2j-1}\left[f(u_3^*) - f(u_{2j-1}^*)\right] = 0 \\ \varepsilon_2u_3^* - b_2u_4^* = 0 \\ -\frac{1}{3}u_5^{*3} + c_3u_5^{*2} + d_3u_5^* - r_3u_6^* + \alpha_3u_5^{*2} + \sum_{j=1,j\ne3}^{5}\beta_{5,2j-1}\left[f(u_5^*) - f(u_{2j-1}^*)\right] = 0 \\ \varepsilon_3u_5^* - b_3u_6^* = 0 \\ -\frac{1}{3}u_7^{*3} + c_4u_7^{*2} + d_4u_7^* - r_4u_8^* + \alpha_4u_7^{*2} + \sum_{j=1,j\ne4}^{5}\beta_{7,2j-1}\left[f(u_7^*) - f(u_{2j-1}^*)\right] = 0 \\ \varepsilon_4u_7^* - b_4u_8^* = 0 \\ -\frac{1}{3}u_9^{*3} + c_5u_9^{*2} + d_5u_9^* - r_5u_{10}^* + \alpha_5u_9^{*2} + \sum_{j=1}^{4}\beta_{9,2j-1}\left[f(u_9^*) - f(u_{2j-1}^*)\right] = 0 \\ \varepsilon_5u_9^* - b_5u_{10}^* = 0 \end{cases} \tag{15}$$

From condition (10), the zero is a solution of system (15). System (15) can be written as

$$CU^* = -\Phi(U^*) \tag{16}$$

Corresponding (16), consider system

$$CU^* = 0 \quad (17)$$

Obviously, system (17) has a unique trivial solution since C is a nonsingular matrix. Based on the Cramer's Rule, system (16) also has a unique solution. Now the even number equations in system (16) and (17) are the same. This means that the unique solution in system (16) is only the trivial solution. In other words, system (9) has a unique equilibrium point, it is exactly the zero point.

The existence of periodic oscillatory solution

Theorem 1 Assume that the system (9) has a unique equilibrium point for a selected set of parameters. Let the eigenvalues of matrices A and B are $\alpha_1, \alpha_2, \dots, \alpha_{10}$ and $\beta_1, \beta_2, \dots, \beta_{10}$, respectively. If there exists at least one eigenvalue of matrix A, say α_1, with the real part Re(α_1) > 0, or Re($\alpha_1 + \beta_1$) > 0, then the trivial solution of system (12) is unstable, implying that the trivial solution of system (9) is unstable, and system (9) generates a periodic oscillatory solution.

Proof In order to prove the existence of the periodic oscillatory solution for system (9), it is necessary to first consider an auxiliary system of (12) as follows:

$$U'(t) = AU(t) + BU(t - \tau_*) \quad (18)$$

Where $\tau_* = \min\{\tau_1, \tau_3, \dots, \tau_9\}$, and $U(t - \tau_*) = [u_1(t - \tau_*), 0, u_3(t - \tau_*), \dots, u_9(t - \tau_*), 0]^T$. Since $\alpha_1, \alpha_2, \dots, \alpha_{10}$ and $\beta_1, \beta_2, \dots, \beta_{10}$ are eigenvalues of matrices A and B respectively, system (18) has the following characteristic equation:

$$\prod_{i=1}^{10}\left(\lambda - \alpha_i - \beta_i\, e^{-\lambda\tau_*}\right) = 0 \quad (19)$$

Noting that matrix B has five rows which are zeros. Therefore, there are five eigenvalues of matrix B which are zeros, assuming that $\beta_1 = 0$. Thus we have

$$\lambda - \alpha_1 - \beta_1\, e^{-\lambda\tau_*} = \lambda - \alpha_1 = 0 \quad (20)$$

Equation (20) means that there is an eigenvalue which is a positive real part of a complex number of systems (18), implying that the trivial solution of system (18) is unstable. If Re ($\alpha_1 + \beta_1$) > 0, we will prove that there exists a positive real part of the characteristic root for equation:

$$\lambda - \alpha_1 - \beta_1\, e^{-\lambda\tau_*} = 0 \quad (21)$$

Indeed, set $f(\lambda) = \lambda - \alpha_1 - \beta_1\, e^{-\lambda\tau_*}$, then $f(\lambda)$ is a continuous function of λ. Since Re ($\alpha_1 + \beta_1$) > 0, so $f(0) = Re(-\alpha_1 - \beta_1\,) < 0$. Noting that $e^{-\lambda} \ll 0$ as λ (> 0) sufficiently large. Therefore, there exists a λ_1 such that $f\big(Re(\lambda_{1)}\big) = Re(\lambda_1) - \alpha_1 - \beta_1\, e^{-Re(\lambda_1)\tau_*} > 0$. Based on the Intermediate value theorem of continuous function, there is a $\lambda_0 \in (0, \lambda_1)$ such that $f\big(Re(\lambda_{0)}\big) = Re(\lambda_0) - \alpha_1 - \beta_1\, e^{-Re(\lambda_0)\tau_*} = 0$. In other words, equation (21) has a positive real part characteristic root. Therefore, the trivial solution of system (18) is unstable.

Noting that $\tau_* \leq \tau_i\ (i = 1,3,5,7,9)$. now from the basic theory of functional differential equation [21], for a small delay, the trivial solution of system (18) is unstable, then the trivial solution is still unstable when delays are increased in system (12). In other words, the instability of the trivial solution of system (18) implies the instability of the trivial solution of system (12). One can see that system (12) is a linearized system of (9). The instability of the trivial solution of system (12) indicates that the trivial solution of system (9) is unstable. Since system (9) has a unique unstable equilibrium point, all solutions are bounded, it will force system (9) to generate a limit cycle, namely, a periodic oscillatory solution [19, 20].

Theorem 2 Assume that the system (9) has a unique equilibrium point for a selected set of parameters. The following condition

$$\det(-A - B) < 0 \quad (22)$$

Holds, then the trivial solution of system (12) is unstable, implying that the trivial solution of system (9) is unstable, and system (9) generates a periodic oscillatory solution.

Proof Similar to Theorem 1, we only need to prove that the trivial solution of auxiliary system (18) is unstable. The characteristic equation associated with system (18) is the following:

$$\det\left(\lambda I_{10} - A - Be^{-\lambda\tau_*}\right) = 0 \quad (23)$$

Where I_{10} is the ten-by-ten identity matrix? Setting

$$g(\lambda) = \det\left(\lambda I_{10} - A - Be^{-\lambda\tau_*}\right) \quad (24)$$

Then $g(\lambda)$ is the characteristic polynomial of system (18). Obviously, $g(\lambda)$ is a continuous function of λ. We will show that $g(\lambda)$ has a positive characteristic root indeed, according to condition (22).

$$g(0) = \det(-A - B) < 0 \quad (25)$$

On the other hand, there is a sufficiently large λ, say λ^* such that

$$g(\lambda^*) = \det\left(\lambda^* I_{10} - A - Be^{-\lambda^*\tau_*}\right) > 0 \qquad (26)$$

Again based on the Intermediate value theorem of continuous function, there exists a $\lambda_* \in (0, \lambda^*)$ such that $g(\lambda_*) = \det\left(\lambda_* I_{10} - A - Be^{-\lambda_*\tau_*}\right) = 0$. In other words, $g(\lambda)$ has a positive characteristic root λ_*. So the trivial solution of system (18) is unstable, implying that the trivial solution of system (9) is unstable. This means that system (9) generates a periodic oscillatory solution based on the generated Chafee's criterion. The proof is completed.

Simulation result

Firstly we select the activation function as $f(u) = \arctan(u)$. Then $f'(u) = \frac{1}{1+u^2}$, so $f'(0) = 1$ and $b_{ij} = \beta_{ij}$. The parameters are $c_1 = 0.56, c_2 = 0.54, c_3 = 0.58, c_4 = 0.52, c_5 = 0.55$, $d_1 = -0.15, d_2 = -0.18, d_3 = -0.12, d_4 = -0.16, d_5 = -0.14, r_1 = 1.25, r_2 = 1.14, r_3 = 1.12, r_4 = 1.28, r_5 = 1.15$, $= 0.15, b_2 = 0.28, b_3 = 0.25, b_4 = 0.24, b_5 = 0.18$, $\varepsilon_1 = 1.35, \varepsilon_2 = 1.45, \varepsilon_3 = 1.55, \varepsilon_4 = 1.48, \varepsilon_5 = 1.52, \beta_{13} = 0.95, \beta_{15} = -1.16, \beta_{17} = 0.78, \beta_{19} = -0.56, \beta_{31} = 0.95, \beta_{35} = -0.24, \beta_{37} = 0.52, \beta_{39} = -0.64, \beta_{51} = 0.15, \beta_{53} = 0.35, \beta_{57} = 0.12, \beta_{59} = 1.75, \beta_{71} = 0.28, \beta_{73} = -0.75, \beta_{75} = 0.26, \beta_{79} = -1.25, \beta_{91} = 0.85, \beta_{93} = -1.98, \beta_{95} = 0.45, \beta_{97} = 1.86, \alpha_1 = -0.25, \alpha_2 = -0.15, \alpha_3 = -0.24, \alpha_4 = -0.42, \alpha_5 = -0.32$. We see that the characteristic values of matrix A are $-0.1500 \pm 1.2990\, i$, $-0.1500 \pm 1.4464\, i, -0.1850 \pm 1.3579\, i$, $-0.2000 \pm 1.3541\, i, -0.2300 \pm 1.3870\, i$, the characteristic values of matrix B are $3.0835, 0.0368, -0.1668, 0.0413 \pm 0.7153\, i, 0, 0, 0, 0, 0$. obviously, $A + B = C$ is a nonsingular matrix. Noting that matrix B has a characteristic value 3.0835, and $3.0835 + (-0.2300) = 2.8535 > 0$. Based on Theorem 1, there is a periodic solution (see Fig. 1 and Fig. 2). Then we change the activation function as $f(u) = \tanh(u)$. we see that $f'(u) = 1 - \tanh^2 u$, so $f'(0) = 1$ still holds and $b_{ij} = \beta_{ij}$. We select $c_1 = 0.36, c_2 = 0.25, c_3 = 0.38, c_4 = 0.42, c_5 = 0.35, d_1 = -0.25, d_2 = -0.28, d_3 = -0.32, d_4 = -0.26$, $d_5 = -0.24, r_1 = 0.85, r_2 = 0.76, r_3 = 0.82, r_4 = 0.78, r_5 = 0.75, b_1 = 0.45, b_2 = 0.38, b_3 = 0.35, b_4 = 0.45, b_5 = 0.48, \varepsilon_1 = 1.85, \varepsilon_2 = 1.75, \varepsilon_3 = 1.95, \varepsilon_4 = 1.78, \varepsilon_5 = 1.92$, $\beta_{13} = 0.95, \beta_{15} = -1.16, \beta_{17} = 0.78, \beta_{19} = -0.56, \beta_{31} = 1.15, \beta_{35} = -0.52, \beta_{37} = 0.48, \beta_{39} = -0.74, \beta_{51} = 0.35, \beta_{53} = 0.45, \beta_{57} = 0.25, \beta_{59} = 1.85, \beta_{71} = 0.48, \beta_{73} = -0.95, \beta_{75} = 0.56, \beta_{79} = -1.55, \beta_{91} = 0.95, \beta_{93} = -1.28, \beta_{95} = 0.75, \beta_{97} = 1.25, \alpha_1 = -0.15, \alpha_2 = -0.18, \alpha_3 = -0.12, \alpha_4 = -0.16, \alpha_5 = -0.20$. The characteristic values of matrix A are $-0.3500 \pm 1.2500\, i$, $-0.3300 \pm 1.1522\, i$, $-0.3350 \pm 1.2644\, i$, $-0.3550 \pm 1.1745\, i, -0.3600 \pm 1.1940\, i$, the characteristic values of matrix B are $3.9947, 1.4542, -0.2964, 0.6987 \pm 1.3279\, i, 0, 0, 0, 0, 0$. Therefore, $A + B = C$ is a nonsingular matrix. We see that $\det\,(-A - B) = -0.3092 < 0$. The condition of Theorem 2 is satisfied, when time delays are selected as 0.18, 0.16, 0.15, 0.14, 0.12, system (9) generates periodic oscillations (see Fig. 3).

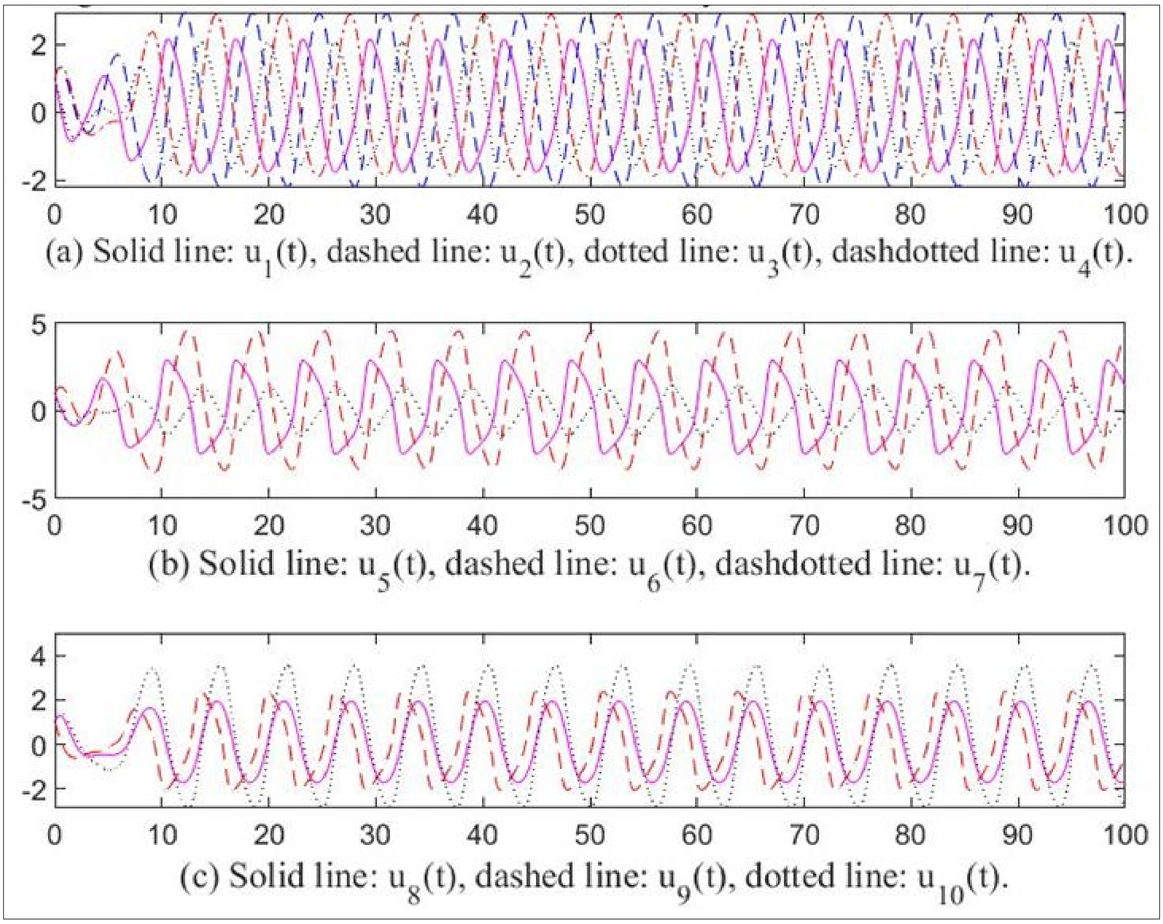
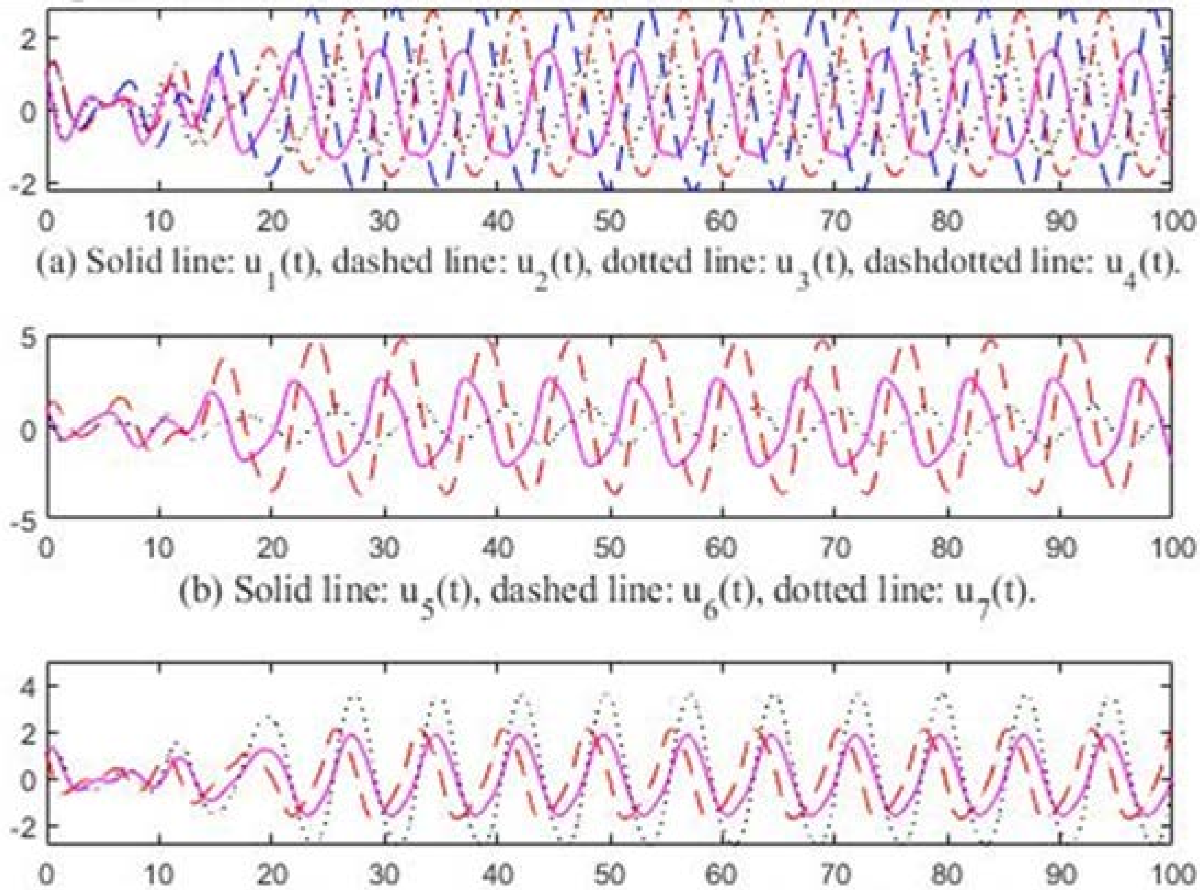


Fig 1: Periodic oscillation of the solutions, delays: 0.28, 0.26, 0.24, 0.25, 0.22

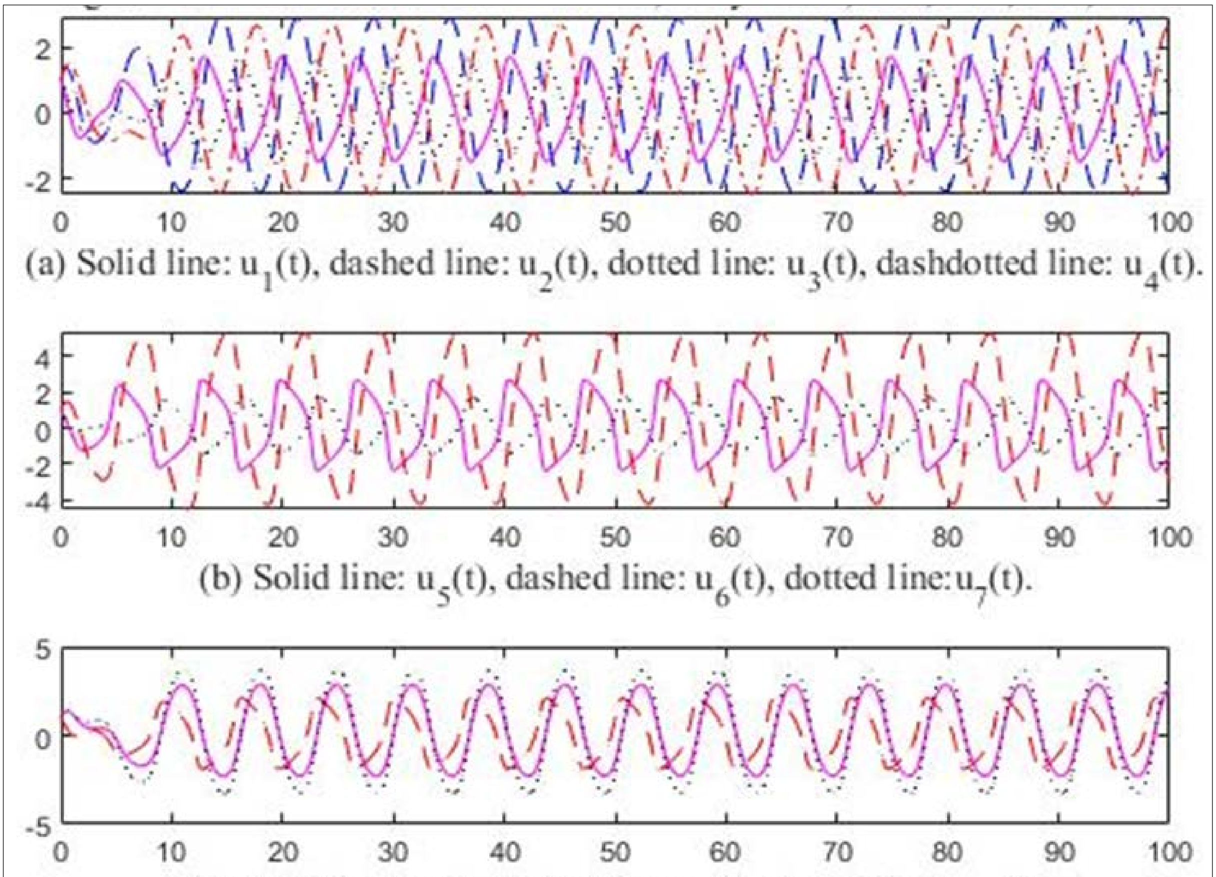


(a) Solid line: $u_1(t)$, dashed line: $u_2(t)$, dotted line: $u_3(t)$, dashdotted line: $u_4(t)$.

(b) Solid line: $u_5(t)$, dashed line: $u_6(t)$, dotted line: $u_7(t)$.

(c) Solid line: $u_8(t)$, dashed line: $u_9(t)$, dotted line: $u_{10}(t)$.

Fig 2: Periodic oscillation of the solutions, delays: 0.78, 0.76, 0.75, 0.74, 0.72



(a) Solid line: $u_1(t)$, dashed line: $u_2(t)$, dotted line: $u_3(t)$, dashdotted line: $u_4(t)$.

(b) Solid line: $u_5(t)$, dashed line: $u_6(t)$, dotted line: $u_7(t)$.

(c) Solid line: $u_8(t)$, dashed line: $u_9(t)$, dotted line: $u_{10}(t)$.

Fig 3: Periodic oscillation of the solutions, delays: 0.18, 0.16, 0.15, 0.14, 0.12

Conclusion

This paper considers a class of FitzHugh-Nagumo network model which includes five different discrete delays. Two sufficient conditions to guarantee the existence of periodic oscillatory solutions are obtained. A specific selection of parameters is used to demonstrate the present results. Our criterion to guarantee the existence of permanent oscillations is different bifurcation method.

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