Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2020; 1(1): 49-51

www.mathematicaljournal.com
Received: 16-11-2019
Accepted: 20-12-2019

Umida Ashurova
Department of Higher Mathematics, Bukhara Technological Institute of Engineering, Bukhara, Uzbekistan

Correspondence
Umida Ashurova
Department of Higher Mathematics, Bukhara Technological Institute of Engineering, Bukhara, Uzbekistan

Extensions of quasi-filiform leibniz algebras of maximum length in small dimensions

Umida Ashurova

Abstract
In this paper we consider Leybniz algebra with a quasi-filiform Leibniz algebras maximum length small dimension nilradical. It proved that such an algebra is decomposed as a direct sum of its nilradical and two ore one dimensional complementary subspace.

Keywords: Nilradical, Leybniz algebra, quasi-filiform Leibniz, algebra, Lie algebra, non-isomorphic

Introduction

Definition 1. An algebra L over field F is called Lie algebra, if for any $x, y, z \in L$ the following identities hold:

$$[x,\ x] = \text{ - anticommutativity},$$

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0,$$

where $[,]$ is multiplication in L.

Definition 2. An algebra L over field F is called a Leibniz algebra, if for any $x, y, z \in L$, the Leibniz identity holds:

$$[x,\ [y,z]]=[[x,\ y],\ z]-[[x,z],\ y],$$

where $[,]$ is multiplication in L.
Note that if the identity $[x,x]=0$ holds in L, then the Leibniz identity is transformed into the Jacobi identity. Thus Leibniz algebras are 'non-commutative' analogs of Lie algebras.
The study of Leibniz algebras from the structural point of view is one of the topical problems in the theory of Lie algebras and Leibniz algebras.
A subalgebra H of a Leibniz algebra L is called a two-sided ideal if $[L,H]\subseteq L$ and $[H,L]\subseteq L$.
For an arbitrary Leibniz algebra L, using the corresponding two-sided ideals recursively define the lower central and derived series, respectively, by the sequences

$$L^1=L,\quad L^{n+1}=[L^n,\ L^1],\ n\geq 1 \text{ and } L^{[1]}=L,\ L^{[n+1]}=[L^{[n]},\ L^{[n]}],\ n\geq 1.$$

Definition 3. Leibniz algebra L is called decidable if there exists $m\in N$ such that $L^{[m]}=0$. A natural number m is called the index of solvability of the algebra L if $L^{[m-1]}\neq 0$ and $L^{[m]}=0$.
Leibniz algebra L is called nilpotent if there exists $s\in N$ such that $L^s=0$. The minimal number s with this property is called the nilpotency index (nilindex) of the algebra L, that is $L^{s-1}\neq 0$ and $L^s=0$.

Remark. It is easy to see that the nilpotency index of an arbitrary n -dimensional nilpotent algebra does not exceed $n+1$.

Definition 4. The maximal nilpotent ideal of Leibniz algebra is called the nilradical of this algebra.

Definition 5. Linear transformation d of Leibniz algebra L is called derivation if

$$d([x,y])=[d(x),\ y]+[x,\ d(y)]$$

for any $x,y\in L$.

Definition 6. [5]. Let $d_1,d_2,\ldots,d_n$ be derivations of the Leibniz algebra *L*. Derivations $d_1,d_2,\ldots,d_n$ are called nil-independent if $\alpha_1 d_1+\alpha_2 d_2+\ldots+\alpha_n d_n$ is not nilpotent for any scalars $\alpha_1,\alpha_2,\ldots,\alpha_n\in F$.
In other words, if for all scalars $\alpha_1,\alpha_2,\ldots,\alpha_n\in F$, there is a natural number k such that if $(\alpha_1 d_1+\alpha_2 d_2+\ldots+\alpha_n d_n)^k=0$ then $\alpha_1=0,\alpha_2=0,\ldots,\alpha_n=0$.
Let *R* be a solvable Leibniz algebra. Then it can be represented in the form of the decomposition $R=N\oplus Q$, where *N* is a nilradical, and *Q* is a vector space-complement to *N* [5].

Theorem 1. Let *R* be solvable Leibniz algebra and *N* its nilradical. Then the dimension of the vector space-complement to *N* is not more than the maximum number of nil-independent derivations *N* [5].
The notion of the length of a Lie algebra was introduced by Gomes, Jimenez-Mershan, and Reyes in [3]. In this paper they introduced and studied algebras admitting a grading with the largest number of nonzero subspaces, which they called algebras of maximum length.
The article [4] describes and studies the properties of Leibniz algebras of maximum length.
In this paper, we study solvable Leibniz algebras that have, as a nilradical, a Leibniz algebra of maximum length in dimensions four and five.

Theorem 2. [4]: Let *L* be a 5-dimensional Leibniz algebra of maximum length. Then *L* is isomorphic to some algebra from the following pairwise non-isomorphic families:

$$M^{1,0}:\begin{cases}[e_1,e_1]=e_5,\\ [e_4,e_1]=e_2,\\ [e_2,e_1]=e_5;\end{cases}\quad M^{2,\lambda}:\begin{cases}[e_i,e_1]=e_{i+1}, 1\le i\le 2,\\ [e_4,e_1]=e_5,\\ [e_1,e_4]=\lambda e_5,\ \lambda\in\mathrm{C};\end{cases}\quad M^{3,0}:\begin{cases}[e_1,e_1]=e_2,\\ [e_i,e_1]=e_{i+1},\ 3\le i\le 4\\ [e_1,e_i]=-e_{i+1},\ 3\le i\le 4.\end{cases}$$

Consider an algebra of type $M^{1,0}$.
Find all derivations of this algebra. We seek differentiations in the form

$$d(e_1)=\sum_{i=1}^{5}\alpha_i e_i,\quad d(e_4)=\sum_{i=1}^{5}\beta_i e_i.$$

By definition of differentiation: $d([e_1,e_2])=[d(e_1),e_1]+[e_1,d(e_1)]$.
With this in mind

$$d(e_5)=d(e_1,e_1)=\left(\sum_{i=1}^{5}\alpha_i e_i\right)e_1+e_1\left(\sum_{i=1}^{5}\beta_i e_i\right)=\alpha_4 e_2+\alpha_2 e_3+2\alpha_1 e_5.$$

Similarly, by direct calculations we find

$$0=\left(\sum_{i=1}^{5}\beta_i e_i\right)e_4+e_4\left(\sum_{i=1}^{5}\beta_i e_i\right)e_4,\quad 0=0+\beta_1 e_2\Rightarrow\beta_1=0\Rightarrow d(e_4)=\sum_{i=2}^{5}\beta_i e_i,$$

$$d(e_1,e_4)=d(e_1)e_4+e_1 d(e_4),\ 0=\left(\sum_{i=1}^{5}\alpha_i e_i\right)e_4+e_1\left(\sum_{i=1}^{5}\beta_i e_i\right),\ 0=0,$$

$$d(e_1,e_4)=d(e_4)e_1+e_4d(e_1),$$

$$d(e_2)=d(e_4,e_1)=\left(\sum_{i=1}^{5}\beta_i e_i\right)e_1+e_1\left(\sum_{i=1}^{5}\alpha_i e_i\right)=\beta_2e_3+\beta_4e_2+\alpha_1e_2\ ,$$

$$d(e_2)=(\alpha_1+\beta_4)e_2+\beta_2e_3\ ,\quad d(e_1,e_4)=d(e_4)e_1+e_4d(e_1),$$

$$d(e_3)=(\alpha_1+\beta_4)e_3+e_3\sum_{i=1}^{5}\alpha_i e_i\ ,\ d(e_3)=(\alpha_1+\beta_4)e_3+\alpha_1e_3.$$

Thus, all derivations of the considered algebra of type $M^{1,0}$ in matrix form have the form:

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5\\ 0 & \alpha_1+\beta_4 & \beta_2 & 0 & 0\\ 0 & 0 & 2\alpha_1+\beta_4 & 0 & 0\\ 0 & \beta_4 & \beta_3 & \beta_4 & \beta_5\\ 0 & \alpha_2 & \alpha_3 & 0 & 2\alpha_1\end{pmatrix}.$$

Based on this table and from the definition of non-independent derivations, it is easy to prove that the number of non-independent derivations is equal to two.

Similar calculations show that for the algebras $M^{2,\lambda}$ and $M^{3,0}$ the number of non-independent derivations is also two.

The same result holds for the four-dimensional Leibniz algebras of maximum length given in [4]. These algebras are characterized by the following structural relations:

$$N^{1,\alpha}:\begin{cases}[e_1,e_1]=e_2,\\ [e_3,e_1]=e_4,\\ [e_1,e_3]=\alpha e_4,\ \alpha\in\mathrm{C};\end{cases}\qquad N^2:\begin{cases}[e_1,e_1]=e_2,\\ [e_1,e_3]=e_4.\end{cases}$$

From this it is easy to see that the number of non-independent derivations is equal to two and, taking into account Theorem 1, we come to the conclusion that

Theorem 3. Any solvable Leibniz algebra L having as the nilradical $N(R)$ of the Leibniz algebra of maximum length of dimensions four or five can be represented in decomposition $L=N(R)\oplus Q$, where Q is a vector space-complement to $N(R)$ with dimension equal to two or one.

References

1. Sh. Ayupov A, Omirov BA. On some classes of nil-component Leibniz algebras, Siberian Math. Mat. J 2001;42(1):18-29.
2. Barnes DW. On Levi‘s theorem for Leibniz algebras, ar Xiv:1109.1060 v 1.
3. G´omez JR, Jim´enez-Merch´an A, Reyes J. Quasi-filiform Lie algebras of maximum length//Linear Algebra Appl 2001;335:119-135.
4. Camacho LM, Canete EM, Gomez JR, Omirov BA. Quasi-filiform Leibniz Algebras of maximum length arXiv:1009.2148v1.
5. Casas JM, Ladra M, Omirov BA, Karimjanov IA. Classification of solvable Leibniz Algebras with null-filiform Nilradical. Linear Algebra Appl 2013;438(7):2973-3000.
6. Mamatov T. Weighted Zygmund estimates for mixed fractional integration. Case Studies Journal 2018;7(5):82-88.
7. Mamatov T. Mixed Fractional Integration In Mixed Weighted Generalized Hölder Spaces. Case Studies Journal 2018;7(6):61-68.
8. Loday JL. Une version non commutative des algebras de Lie: les algebras de Leibniz, Enseign. Math 1993;(2)39(3-4):269-293.
9. Malcev AI. Solvable Lie algebras, Amer. Math. Soc. Translation 1950, (27).
10. Mubarakzjanov GM. On solvable Lie algebras, Izv. Vysš. Učehn. Zaved. Matematika 1963;1(32):114-123.