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On the number of zeroes of a polynomial with restricted real Co-efficient

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Abstract
The well-known theorem in theory of distribution of zeros of a polynomial due to Enestrom – Kakeya states that if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ be a polynomial of degree n, satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$ Then all the zeroes of $P(z)$ lie in $|z| \leq 1$. In this paper we will extend Enestrom – Kakeya theorem by relaxing the restrictions on the coefficients of a polynomial in several ways and thereby present a result on zero free region of a polynomial to certain condition.

Keywords: Polynomial, zeroes, enestrom – kakeya theorem

1. Introduction

If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree n, then concerning the distribution of zero's of $P(z)$, Enestrom – Kakeya [2, 3], Proved the following elegant result.

Theorem A: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree n, satisfying

$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$

Then $P(z)$ has its zero's in $|z| \leq 1$.

Many attempts have been made by various researchers and scholars in the field of the distribution of the zero's of a polynomial and have extended and generalized the Enestrom – Kakeya theorem by relaxed the hypothesis in several ways Joyal *et al.* [1], extended theorem A to the polynomial whose coefficients are monotonic not necessarily non – negative in fact they proved the following result.

Theorem B: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree n, satisfying $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0$

Then $P(z)$ has its zero's lie in

$$|z| \leq \frac{1}{|a_n|} (|a_n| - a_0 + |a_0|)$$

Recently Dr. Mushtaq Ahmad Shah and H. G. Hyun [4], relaxed the hypothesis in various ways, in fact they proved.

Theorem C: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0$ is a polynomial of degree n, satisfying

$$a_n \geq a_{n-1} \geq \cdots \geq a_p, 0 \leq \rho \leq n \text{ and } M_p = \sum_{j=0}^{p} |a_j - a_{j-1}|$$

Then $P(z)$ has its zero's lie in

$$|z| \leq \frac{a_n - a_p + M_p}{|a_n|}$$

In this paper, we prove some extensions and generalizations of theorem A, B, C, in fact we prove the following results.

2: Main results

Theorem 1: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + a_q z^q \ldots + a_1 z + a_0$ is a polynomial of degree n, satisfying

$$a_n \geq a_{n-1} \geq \cdots \geq a_p, a_q \geq a_{q-1} \geq \cdots \geq a_1 \geq a_0, p \geq q$$

and $M_{(p,q)} = \sum_{j=q+1}^{p} |a_j - a_{j-1}|$

Then $P(z)$ has all its zeroes in

$$|z| \leq \frac{[a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}]}{|a_n|}$$

Remark: If $P = q = n$ then theorem it reduced to theorem B.

Theorem 2: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \cdots + a_q z^q \ldots + a_1 z + a_0$ is a polynomial of degree n, satisfying

$$ka_n \geq a_{n-1} \geq \cdots \geq a_p, a_q \leq a_{q-1} \leq \cdots \leq a_1 \leq \rho a_0 \; and \; k \geq 1, 0 < \rho < 1$$

and $M_{(p,q)} = \sum_{j=q+1}^{p} |a_j - a_{j-1}|$

Then all the zeroes of $P(z)$ lie in

$$|z + k - 1| \leq \frac{ka_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)}}{|a_n|}$$

Theorem 3: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \cdots + a_q z^q + \cdots + a_1 z + a_0$ is a polynomial of degree n, satisfying

$$a_n \geq a_{n-1} \geq \cdots \geq a_p, a_q \leq a_{q-1} \leq \cdots \leq a_1 \leq a_0, p \geq q$$

and $M_{(p,q)} = \sum_{j=q+1}^{p} |a_j - a_{j-1}|$

Then $P(z)$ does not vanish in

$$|z| < min \left(1, \frac{|a_0|}{|a_n| + a_n - a_0 + (a_q - a_p) + M_{(p,q)}}\right)$$

3: Proof of theorems

Proof of theorem 1: Consider the polynomial

$F(z) = (1 - z)P(z)$

$$= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_{p+1} z^{p+1} + a_p z^p + \cdots + a_{q+1} z^{q+1} + a_q z^q + \ldots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1} + \cdots + (a_{q+1} - a_q) z^{q+1} + (a_q - a_{q-1}) z^q + \ldots + (a_1 - a_0) z + a_0$$

$$|F(z)| = \left| -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1} + (a_p - a_{p-1}) z^p + \cdots + (a_{q+1} - a_q) z^{q+1} + (a_q - a_{q-1}) z^q + \cdots + (a_1 - a_0) z + a_0 \right|$$

$$|F(z)| \geq |a_n||z|^{n+1} - \{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{p+1} - a_p||z|^{p+1} + |a_p - a_{p-1}||z|^p + \cdots + |a_{q+1} - a_q||z|^{q+1} + |a_q - a_{q-1}||z|^q + \cdots + |a_1 - a_0||z| + |a_0|\}$$

$$\geq |z|^n\left[|a_n||z| - \left\{\begin{matrix} |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}|\frac{1}{|z|} + \cdots + |a_{p+1} - a_p|\frac{1}{|z|^{n-p-1}} + |a_p - a_{p-1}|\frac{1}{|z|^{n-p}} \\ + \cdots + |a_{q+1} - a_q|\frac{1}{|z|^{n-q-1}} + |a_q - a_{q-1}|\frac{1}{|z|^{n-q}} + \cdots + |a_1 - a_o|\frac{1}{|z|^{n-1}} + |a_o|\frac{1}{|z|^n}\end{matrix}\right\}\right]$$

For $|z| \geq 1$ so that $\frac{1}{|z|}$, we have,

$$|F(z)| \geq |z|^n\left[|a_n||z| - \left\{\begin{matrix} |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| + |a_p - a_{p-1}| \\ + \cdots + |a_{q+1} - a_q| + |a_q - a_{q-1}| + \cdots + |a_1 - a_o| + |a_o| \end{matrix}\right\}\right]$$

$$\geq |z|^n[|a_n||z| - \{a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_{p+2} - a_{p+1} + a_{p+1} - a_p + |a_p - a_{p-1}| + \ldots + |a_{q+1} - a_q| + a_q - a_{q-1} + a_{q-1} - a_{q-2} + \cdots + a_2 - a_1 + a_1 - a_0 + |a_0|\}]$$

$$\geq |z|^n\left[|a_n||z| - \left\{a_n - a_p + \sum_{j=q+1}^{p} |a_j - a_{j-1}| + a_q - a_0 + |a_0|\right\}\right]$$

$$\geq |z|^n[|a_n||z| - \{a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}\}]$$

Where $M_{(p,q)} = \sum_{j=q+1}^{p} |a_j - a_{j-1}|$

$$\geq |z|^n\left[|a_n||z| - \frac{1}{|a_n|}\{a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}\}\right] > 0$$

If $|z| - \frac{1}{|a_n|}\{a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}\} > 0$

i.e $|z| > \frac{1}{|a_n|}(a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)})$

Thus all the zeros of $F(z)$, whose modulus is greater than 1 lie in

$|z| \leq \frac{1}{|a_n|}(a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)})$.

But those zeros of $F(z)$, whose modulus is less than or equal to 1 already satisfy above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$, hence it follows that all the zero's of $P(z)$ lie

$$|z| \leq \frac{1}{|a_n|}(a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)})$$

Hence it completes the proof of theorem 1.

Proof of theorem 2: Consider the polynomial
$F(z) = (1 - z)P(z)$

$$= (1 - z)(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_{p+1}z^{p+1} + a_p z^p + \cdots + a_{q+1}z^{q+1} + a_q z^q + \ldots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} - ka_n z^n + a_n z^n + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \cdots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + \rho a_0 z - a_0 z + a_0$$

$$= -z^n a_n(z + k - 1) + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \cdots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0$$

$$|F(z)| = \left|z^n a_n(z+k-1) + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \cdots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0\right|$$

$$|F(z)| \geq \Big[|z|^n|a_n(z+k-1)| - \big\{|ka_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{p+1} - a_p||z|^{p+1} + |a_p - a_{p-1}||z|^p + \cdots + |a_{q+1} - a_q||z|^{q+1} + |a_q - a_{q-1}||z|^q + \cdots + |a_1 - \rho a_0||z| + (\rho - 1)|a_0||z| + |a_0|\big\}\Big]$$

$$\geq |z|^n|a_n|\left[|z+k-1| - \frac{1}{|a_n|}\left\{\begin{matrix} |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}|\frac{1}{|z|} + \cdots + |a_{p+1} - a_p|\frac{1}{|z|^{n-p-1}} + |a_p - a_{p-1}|\frac{1}{|z|^{n-p}} \\ + \cdots + \\ |a_{q+1} - a_q|\frac{1}{|z|^{n-q-1}} + |a_q - a_{q-1}|\frac{1}{|z|^{n-q}} + \cdots + |a_1 - \rho a_o|\frac{1}{|z|^{n-1}} + (1-\rho)|a_o|\frac{1}{|z|^{n-1}} + |a_o|\frac{1}{|z|^n} \end{matrix}\right\}\right]$$

For $|z| \geq 1$, so that $\frac{1}{|z|} \leq 1$, we have

$$|F(z)| \geq |z|^n|a_n|\left[|z+k-1| - \frac{1}{|a_n|}\left\{\begin{matrix} |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| + |a_p - a_{p-1}| \\ + \cdots + |a_{q+1} - a_q| + |a_q - a_{q-1}| + \cdots + |a_1 - \rho a_o| + (1-\rho)|a_o| + |a_o| \end{matrix}\right\}\right]$$

$$\geq |z|^n|a_n|\left[|z+k-1| - \frac{1}{|a_n|}\left\{\begin{matrix} ka_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_{p+1} - a_p + |a_p - a_{p-1}| \\ + \cdots + |a_{q+1} - a_q| + a_{q-1} - a_q + \cdots + \rho a_o - a_1 + (1-\rho)|a_o| + |a_o| \end{matrix}\right\}\right]$$

$$\geq |z|^n|a_n|\left[|z+k-1| - \frac{1}{|a_n|}\{ka_n - a_p + M_{(p,q)} - a_q + \rho a_o + (1-\rho)|a_o| + |a_o|\}\right]$$

Where $M_{(p,q)} = \sum_{j=q+1}^{p} |a_j - a_{j-1}|$

$$\geq |z|^n|a_n|\left[|z+k-1| - \frac{1}{|a_n|}\{ka_n - (a_p + a_q) - \rho(|a_o| - a_o) + 2|a_o| + M_{(p,q)}\}\right] > 0$$

If $\quad |z+k-1| - \frac{1}{|a_n|}\{ka_n - (a_p + a_q) - \rho(|a_o| - a_o) + 2|a_o| + M_{(p,q)}\} > 0$

i.e $\quad |z+k-1| > \frac{1}{|a_n|}\{ka_n - (a_p + a_q) - \rho|a_o| - a_o + 2|a_o| + M_{(p,q)}\}$

Thus all the zeros of $F(z)$, whose modulus is greater than 1 lie in

$$|z+k-1| \leq \frac{1}{|a_n|}\{ka_n - (a_p + a_q) - \rho(|a_o| - a_o) + 2|a_o| + M_{(p,q)}\}$$

But those zeros of (z), whose modulus is less than or equal to 1 already satisfy above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$, hence it follows that all the zero's of $P(z)$ lie

$$|z+k-1| \leq \frac{ka_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)}}{|a_n|}$$

Hence it completes the proof of theorem 2.

Theorem 3: Consider the reciprocal polynomial

$$J(z) = z^n P\left(\frac{1}{z}\right)$$
$$= a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + a_{q+1} z^{n-q-1} + \ \ldots + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \cdots + a_{n-1} z + a_n$$

Let $R(z) = (1-z)J(z)$
$= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{q-1} - a_q)z^{n-q+1} + (a_q - a_{q+1})z^{n-q} + \cdots + (a_{p-1} - a_p)z^{n-p+1} + (a_p - a_{p+1})z^{n-p} + \cdots + (a_{n-1} - a_n)z + a_n$

This gives

$$|R(z)| \geq |a_0||z|^{n+1} - \{|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} + |a_q - a_{q+1}||z|^{n-q} + \cdots + |a_{p-1} - a_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n|\}$$

$$|R(z)| \geq |z|^n \Big\{|a_0||z| - \Big(|a_0 - a_1| + |a_1 - a_2|\frac{1}{|z|} + \cdots + |a_{q-1} - a_q|\frac{1}{|z|^{q-1}} + |a_q - a_{q+1}|\frac{1}{|z|^q} + \cdots + |a_{p-1} - a_p|\frac{1}{|z|^{p-1}} + |a_p - a_{p+1}|\frac{1}{|z|^p} + \cdots + |a_{n-1} - a_n|\frac{1}{|z|^{n-1}} + |a_n|\frac{1}{|z|^n}\Big)\Big\}$$

For $|z| \geq 1$, so that $\frac{1}{|z|} \leq 1$, we have

$$|R(z)| \geq |z|^n\{|a_0||z| - (|a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{q-1} - a_q| + |a_q - a_{q+1}| + \cdots + |a_{p-1} - a_p| + |a_p - a_{p+1}| + \cdots + |a_{n-1} - a_n| + |a_n|)\}$$

$$\geq |z|^n\{|a_0||z| - (|a_1 - a_0| + |a_2 - a_1| + \cdots + |a_q - a_{q-1}| + |a_{q+1} - a_q| + \cdots + |a_p - a_{p-1}| + |a_{p+1} - a_p| + \cdots + |a_n - a_{n-1}| + |a_n|)\}$$

$$\geq |z|^n\{|a_0||z| - (a_1 - a_0 + a_2 - a_1 + \cdots + a_q - a_{q-1} + |a_{q+1} - a_q| + \cdots + |a_p - a_{p-1}| + a_{p+1} - a_p + \ldots + a_n - a_{n-1} + |a_n|)\}$$

$$\geq |z|^n\{|a_0||z| - (M_{(p,q)} - a_0 + a_q - a_p + a_n + |a_n|)\} > 0$$

If $|z| > \frac{1}{|a_0|}\{|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)}\}$

Thus all the zeros of $R(z)$, whose modulus is greater than 1 lie in

$$|z| \leq \frac{1}{|a_0|}\{|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)}\}$$

Hence all the zeroes of $R(z)$ and hence $J(z)$ lie in

$$|z| \leq \text{Max}\left\{1, \frac{1}{|a_0|}(|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)})\right\}$$

Therefore zero's of $P(z)$ lie in

$$|z| \geq \text{Min}\left\{1, \frac{|a_0|}{(|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)})}\right\}$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < Min\left\{1, \frac{|a_0|}{(|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)})}\right\}$$

Hence it completes the proof of theorem 3.

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