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On the number of zeroes of a polynomial with restricted real Co-efficient

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Abstract

The well-known theorem in theory of distribution of zeros of a polynomial due to Enestrom – Kakeya states that if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree n , satisfying $a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0$ Then all the zeroes of $P(z)$ lie in $|z| \leq 1$. In this paper we will extend Enestrom – Kakeya theorem by relaxing the restrictions on the coefficients of a polynomial in several ways and thereby present a result on zero free region of a polynomial to certain condition.

Keywords: Polynomial, zeroes, enestrom – kakeya theorem

1. Introduction

If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is a polynomial of degree n , then concerning the distribution of zero's of $P(z)$, Enestrom – Kakeya ^[2, 3], Proved the following elegant result.

Theorem A: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is a polynomial of degree n , satisfying

$$a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 > 0$$

Then $P(z)$ has its zero's in $|z| \leq 1$.

Many attempts have been made by various researchers and scholars in the field of the distribution of the zero's of a polynomial and have extended and generalized the Enestrom – Kakeya theorem by relaxed the hypothesis in several ways Joyal *et al.* ^[1], extended theorem A to the polynomial whose coefficients are monotonic not necessarily non – negative in fact they proved the following result.

Theorem B: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is a polynomial of degree n , satisfying $a_n \geq a_{n-1} \geq \dots \geq a_1 \geq a_0 \geq 0$

Then $P(z)$ has its zero's lie in

$$|z| \leq \frac{1}{|a_n|} (|a_n| - a_0 + |a_0|)$$

Recently Dr. Mushtaq Ahmad Shah and H. G. Hyun [4], relaxed the hypothesis in various ways, in fact they proved.

Theorem C: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_p z^p + \dots + a_1 z + a_0$ is a polynomial of degree n , satisfying

$$a_n \geq a_{n-1} \geq \dots \geq a_p, 0 \leq p \leq n \text{ and } M_p = \sum_{j=0}^p |a_j - a_{j-1}|$$

Then $P(z)$ has its zero's lie in

$$|z| \leq \frac{a_n - a_p + M_p}{|a_n|}$$

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In this paper, we prove some extensions and generalizations of theorem A, B, C, in fact we prove the following results.

2: Main results

Theorem 1: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_p z^p + a_q z^q + \dots + a_1 z + a_0$ is a polynomial of degree n , satisfying

$$a_n \geq a_{n-1} \geq \dots \geq a_p, a_q \geq a_{q-1} \geq \dots \geq a_1 \geq a_0, p \geq q$$

$$\text{and } M_{(p,q)} = \sum_{j=q+1}^p |a_j - a_{j-1}|$$

Then $P(z)$ has all its zeroes in

$$|z| \leq \frac{[a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}]}{|a_n|}$$

Remark: If $P = q = n$ then theorem it reduced to theorem B.

Theorem 2: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_p z^p + \dots + a_q z^q + \dots + a_1 z + a_0$ is a polynomial of degree n , satisfying

$$ka_n \geq a_{n-1} \geq \dots \geq a_p, a_q \leq a_{q-1} \leq \dots \leq a_1 \leq \rho a_0 \text{ and } k \geq 1, 0 < \rho < 1$$

$$\text{and } M_{(p,q)} = \sum_{j=q+1}^p |a_j - a_{j-1}|$$

Then all the zeroes of $P(z)$ lie in

$$|z + k - 1| \leq \frac{ka_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)}}{|a_n|}$$

Theorem 3: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_p z^p + \dots + a_q z^q + \dots + a_1 z + a_0$ is a polynomial of degree n , satisfying

$$a_n \geq a_{n-1} \geq \dots \geq a_p, a_q \leq a_{q-1} \leq \dots \leq a_1 \leq a_0, p \geq q$$

$$\text{and } M_{(p,q)} = \sum_{j=q+1}^p |a_j - a_{j-1}|$$

Then $P(z)$ does not vanish in

$$|z| < \min \left(1, \frac{|a_0|}{|a_n| + a_n - a_0 + (a_q - a_p) + M_{(p,q)}} \right)$$

3: Proof of theorems

Proof of theorem 1: Consider the polynomial

$$F(z) = (1 - z)P(z)$$

$$= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_{p+1} z^{p+1} + a_p z^p + \dots + a_{q+1} z^{q+1} + a_q z^q + \dots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{p+1} - a_p)z^{p+1} + \dots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \dots + (a_1 - a_0)z + a_0$$

$$|F(z)| = \left| -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \dots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \dots + (a_1 - a_0)z + a_0 \right|$$

$$|F(z)| \geq |a_n||z|^{n+1} - \{ |a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \dots + |a_{p+1} - a_p||z|^{p+1} + |a_p - a_{p-1}||z|^p + \dots + |a_{q+1} - a_q||z|^{q+1} + |a_q - a_{q-1}||z|^q + \dots + |a_1 - a_0||z| + |a_0| \}$$

$$\geq |z|^n \left[|a_n||z| - \left\{ |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \frac{1}{|z|} + \dots + |a_{p+1} - a_p| \frac{1}{|z|^{n-p-1}} + |a_p - a_{p-1}| \frac{1}{|z|^{n-p}} \right\} \right. \\ \left. + \dots + |a_{q+1} - a_q| \frac{1}{|z|^{n-q-1}} + |a_q - a_{q-1}| \frac{1}{|z|^{n-q}} + \dots + |a_1 - a_0| \frac{1}{|z|^{n-1}} + |a_0| \frac{1}{|z|^n} \right]$$

For $|z| \geq 1$ so that $\frac{1}{|z|}$, we have,

$$|F(z)| \geq |z|^n \left[|a_n||z| - \left\{ |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_{p+1} - a_p| + |a_p - a_{p-1}| \right\} \right. \\ \left. + \dots + |a_{q+1} - a_q| + |a_q - a_{q-1}| + \dots + |a_1 - a_0| + |a_0| \right]$$

$$\geq |z|^n \left[|a_n||z| - \{a_n - a_{n-1} + a_{n-1} - a_{n-2} + \dots + a_{p+2} - a_{p+1} + a_{p+1} - a_p + |a_p - a_{p-1}| + \dots + |a_{q+1} - a_q| + a_q - a_{q-1} + a_{q-1} - a_{q-2} + \dots + a_2 - a_1 + a_1 - a_0 + |a_0|\} \right]$$

$$\geq |z|^n \left[|a_n||z| - \left\{ a_n - a_p + \sum_{j=q+1}^p |a_j - a_{j-1}| + a_q - a_0 + |a_0| \right\} \right]$$

$$\geq |z|^n \left[|a_n||z| - \{a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}\} \right]$$

$$\text{Where } M_{(p,q)} = \sum_{j=q+1}^p |a_j - a_{j-1}|$$

$$\geq |z|^n \left[|a_n||z| - \frac{1}{|a_n|} \{a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}\} \right] > 0$$

$$\text{If } |z| - \frac{1}{|a_n|} \{a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}\} > 0$$

$$\text{i.e. } |z| > \frac{1}{|a_n|} (a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)})$$

Thus all the zeros of $F(z)$, whose modulus is greater than 1 lie in

$$|z| \leq \frac{1}{|a_n|} (a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)}).$$

But those zeros of $F(z)$, whose modulus is less than or equal to 1 already satisfy above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$, hence it follows that all the zero's of $P(z)$ lie

$$|z| \leq \frac{1}{|a_n|} (a_n + (a_q - a_p) - a_0 + |a_0| + M_{(p,q)})$$

Hence it completes the proof of theorem 1.

Proof of theorem 2: Consider the polynomial

$$F(z) = (1 - z)P(z)$$

$$= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_{p+1} z^{p+1} + a_p z^p + \dots + a_{q+1} z^{q+1} + a_q z^q + \dots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \dots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \dots + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} - k a_n z^n + a_n z^n + (k a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \dots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \dots + (a_1 - \rho a_0)z + \rho a_0 z - a_0 z + a_0$$

$$= -z^n a_n (z + k - 1) + (k a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \dots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \dots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0$$

$$|F(z)| = |z^n a_n (z + k - 1) + (k a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \dots + (a_{p+1} - a_p) z^{p+1} + (a_p - a_{p-1}) z^p + \dots + (a_{q+1} - a_q) z^{q+1} + (a_q - a_{q-1}) z^q + \dots + (a_1 - \rho a_0) z + (\rho - 1) a_0 z + a_0|$$

$$\begin{aligned} |F(z)| &\geq [|z|^n |a_n| (z + k - 1) \\ &\quad - \{ |k a_n - a_{n-1}| |z|^n + |a_{n-1} - a_{n-2}| |z|^{n-1} + \dots + |a_{p+1} - a_p| |z|^{p+1} + |a_p - a_{p-1}| |z|^p + \dots \\ &\quad + |a_{q+1} - a_q| |z|^{q+1} + |a_q - a_{q-1}| |z|^q + \dots + |a_1 - \rho a_0| |z| + (\rho - 1) |a_0| |z| + |a_0| \}] \\ &\geq |z|^n |a_n| \left[|z + k - 1| - \frac{1}{|a_n|} \left\{ \begin{aligned} &|k a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \frac{1}{|z|} + \dots + |a_{p+1} - a_p| \frac{1}{|z|^{n-p+1}} + |a_p - a_{p-1}| \frac{1}{|z|^{n-p}} \\ &+ \dots + \\ &|a_{q+1} - a_q| \frac{1}{|z|^{n-q-1}} + |a_q - a_{q-1}| \frac{1}{|z|^{n-q}} + \dots + |a_1 - \rho a_0| \frac{1}{|z|^{n-1}} + (1 - \rho) |a_0| \frac{1}{|z|^{n-1}} + |a_0| \frac{1}{|z|^n} \end{aligned} \right\} \right] \end{aligned}$$

For $|z| \geq 1$, so that $\frac{1}{|z|} \leq 1$, we have

$$\begin{aligned} |F(z)| &\geq |z|^n |a_n| \left[|z + k - 1| - \frac{1}{|a_n|} \left\{ |k a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots + |a_{p+1} - a_p| + |a_p - a_{p-1}| \right\} \right. \\ &\quad \left. + \dots + |a_{q+1} - a_q| + |a_q - a_{q-1}| + \dots + |a_1 - \rho a_0| + (1 - \rho) |a_0| + |a_0| \right] \\ &\geq |z|^n |a_n| \left[|z + k - 1| - \frac{1}{|a_n|} \left\{ k a_n - a_{n-1} + a_{n-1} - a_{n-2} + \dots + a_{p+1} - a_p + |a_p - a_{p-1}| \right\} \right. \\ &\quad \left. + \dots + |a_{q+1} - a_q| + a_{q-1} - a_q + \dots + \rho a_0 - a_1 + (1 - \rho) |a_0| + |a_0| \right] \\ &\geq |z|^n |a_n| \left[|z + k - 1| - \frac{1}{|a_n|} \{ k a_n - a_p + M_{(p,q)} - a_q + \rho a_0 + (1 - \rho) |a_0| + |a_0| \} \right] \end{aligned}$$

Where $M_{(p,q)} = \sum_{j=q+1}^p |a_j - a_{j-1}|$

$$\geq |z|^n |a_n| \left[|z + k - 1| - \frac{1}{|a_n|} \{ k a_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)} \} \right] > 0$$

If $|z + k - 1| - \frac{1}{|a_n|} \{ k a_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)} \} > 0$

i.e $|z + k - 1| > \frac{1}{|a_n|} \{ k a_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)} \}$

Thus all the zeros of $F(z)$, whose modulus is greater than 1 lie in

$$|z + k - 1| \leq \frac{1}{|a_n|} \{ k a_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)} \}$$

But those zeros of (z) , whose modulus is less than or equal to 1 already satisfy above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$, hence it follows that all the zero's of $P(z)$ lie

$$|z + k - 1| \leq \frac{k a_n - (a_p + a_q) - \rho(|a_0| - a_0) + 2|a_0| + M_{(p,q)}}{|a_n|}$$

Hence it completes the proof of theorem 2.

Theorem 3: Consider the reciprocal polynomial

$$\begin{aligned} J(z) &= z^n P\left(\frac{1}{z}\right) \\ &= a_0 z^n + a_1 z^{n-1} + \dots + a_q z^{n-q} + a_{q+1} z^{n-q-1} + \dots + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \dots + a_{n-1} z + a_n \end{aligned}$$

$$\begin{aligned} \text{Let } R(z) &= (1 - z)J(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \dots + (a_{q-1} - a_q) z^{n-q+1} + (a_q - a_{q+1}) z^{n-q} + \dots + (a_{p-1} - a_p) z^{n-p+1} + \\ &\quad (a_p - a_{p+1}) z^{n-p} + \dots + (a_{n-1} - a_n) z + a_n \end{aligned}$$

This gives

$$|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \dots + |a_{q-1} - a_q||z|^{n-q+1} + |a_q - a_{q+1}||z|^{n-q} + \dots + |a_{p-1} - a_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \dots + |a_{n-1} - a_n||z| + |a_n| \}$$

$$|R(z)| \geq |z|^n \left\{ |a_0||z| - \left(|a_0 - a_1| + |a_1 - a_2| \frac{1}{|z|} + \dots + |a_{q-1} - a_q| \frac{1}{|z|^{q-1}} + |a_q - a_{q+1}| \frac{1}{|z|^q} + \dots + |a_{p-1} - a_p| \frac{1}{|z|^{p-1}} + |a_p - a_{p+1}| \frac{1}{|z|^p} + \dots + |a_{n-1} - a_n| \frac{1}{|z|^{n-1}} + |a_n| \frac{1}{|z|^n} \right) \right\}$$

For $|z| \geq 1$, so that $\frac{1}{|z|} \leq 1$, we have

$$\begin{aligned} |R(z)| &\geq |z|^n \{ |a_0||z| - (|a_0 - a_1| + |a_1 - a_2| + \dots + |a_{q-1} - a_q| + |a_q - a_{q+1}| + \dots + |a_{p-1} - a_p| + |a_p - a_{p+1}| + \dots + |a_{n-1} - a_n| + |a_n|) \} \\ &\geq |z|^n \{ |a_0||z| - (|a_1 - a_0| + |a_2 - a_1| + \dots + |a_q - a_{q-1}| + |a_{q+1} - a_q| + \dots + |a_p - a_{p-1}| + |a_{p+1} - a_p| + \dots + |a_n - a_{n-1}| + |a_n|) \} \\ &\geq |z|^n \{ |a_0||z| - (a_1 - a_0 + a_2 - a_1 + \dots + a_q - a_{q-1} + |a_{q+1} - a_q| + \dots + |a_p - a_{p-1}| + a_{p+1} - a_p + \dots + a_n - a_{n-1} + |a_n|) \} \\ &\geq |z|^n \{ |a_0||z| - (M_{(p,q)} - a_0 + a_q - a_p + a_n + |a_n|) \} > 0 \end{aligned}$$

$$\text{If } |z| > \frac{1}{|a_0|} \{ |a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)} \}$$

Thus all the zeros of $R(z)$, whose modulus is greater than 1 lie in

$$|z| \leq \frac{1}{|a_0|} \{ |a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)} \}$$

Hence all the zeroes of $R(z)$ and hence $J(z)$ lie in

$$|z| \leq \text{Max} \left\{ 1, \frac{1}{|a_0|} (|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)}) \right\}$$

Therefore zero's of $P(z)$ lie in

$$|z| \geq \text{Min} \left\{ 1, \frac{|a_0|}{(|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)})} \right\}$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < \text{Min} \left\{ 1, \frac{|a_0|}{(|a_n| + a_n + (a_q - a_p) - a_0 + M_{(p,q)})} \right\}$$

Hence it completes the proof of theorem 3.

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