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Vipin Vasishtha
Associate Professor,
Department of Mathematics
M.M.H. College, Ghaziabad,
Uttar Pradesh, India

Virendra Singh Yadav
Associate Professor,
Department of Mathematics
M.M.H. College, Ghaziabad,
Uttar Pradesh, India

Correspondence
Vipin Vasishtha
Associate Professor,
Department of Mathematics
M.M.H. College, Ghaziabad,
Uttar Pradesh, India

Exploration study of fixed point theorems for four mappings in metric spaces

Vipin Vasishtha and Virendra Singh Yadav

Abstract
The present study focused on to understand a common fixed point theorem for weak compatible mapping [2], [3] of type (A) for four mappings in metric space generalizes the result of Kang and Kim [6].

Keywords: Metric spaces, weak compatible mapping of type (A)

1. Introduction
In Jungck [1] and in all generalization of Jungck's theorems, families of commuting mappings have been considered. Rhoades, Sessa and Khan [8] improved the results by assuming weak commutativity. Further, Jungck, Murthy and Cho [4] introduced the concept of compatible mappings of type (A) in metric space and improved the results of various authors. We use the idea of weak compatible mappings of type (A) in metric space as used by Pathak, Kang and Beak [7] in Menger and 2-metric spaces respectively which is equivalent to the concept of compatible and compatible mappings of type (A) under some conditions. In this section, we present a common fixed point [5] theorem for weak compatible mapping of type (A) for four mappings in a metric space which generalizes the result of Kang and Kim [6].

2. Preliminaries
2.1 Definition
The pair (A, S) is said to be weak compatible of type (A) if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \le \lim_{n\to\infty} d(SAx_n, SSx_n)$$

And

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \le \lim_{n\to\infty} d(ASx_n, AAx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = t = \lim_{n\to\infty} Ax_n$ for some $t \in X$.

2.2 Preposition
Every pair of compatible mappings of type (A) is weak compatible of type (A).

2.3 Proof
Suppose that A and S are compatible mappings of type (A), therefore,

$$0 = \lim_{n\to\infty} d(ASx_n, SSx_n) \le \lim_{n\to\infty} d(SAx_n, AAx_n)$$

And

$$0 = \lim_{n\to\infty} d(SAx_n, AAx_n) \le \lim_{n\to\infty} d(ASx_n, AAx_n)$$

which shows that the pair {A, S} is weak compatible of type (A).

2.4 Preposition
Let A and S are continuous mappings of a metric space (X, d) into self. If A and S are weak compatible of type (A), then they are compatible of type (A).

2.5 Proof
Suppose that *A* and *S* are weak compatible mapping of type (A). Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sxn = \lim_{n\to\infty} Axn = t$ for some $t \in X$.
Since *A* and *S* are continuous mappings, then we have

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \le \lim_{n\to\infty} d(SAx_n, SSx_n) = d(St, St) = 0$$

And

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \le \lim_{n\to\infty} d(ASx_n, AAx_n) = d(At, At) = 0$$

Therefore *A* and *S* are compatible mappings of type (*A*) this completes the proof.

2.6 Preposition
Let A and S be weak compatible mappings of type (A) from metric space (X, d) into itself. If one of A and S is continuous, then A and S are compatible.

2.7 Proof
Without loss of generality, suppose that S is continuous. Let $\{x_n\}$ be a sequence in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \text{ for some } t \in X$$

Since *S* is continuous, we have

$$\lim_{n\to\infty} SAx_n = St = \lim_{n\to\infty} SSx_n.$$

Now $d(SAx_n, ASx_n) \le d(SAx_n, SSx_n) + d(SSx_n, ASx_n) \le 0 + d(SSx_n, ASx_n) = d(SSx_n, ASx_n)$

Since (*A*, *S*) are weak compatible of type (*A*). Therefore, we have

$$\lim_{n\to\infty} d(SAx_n, ASx_n) \le \lim_{n\to\infty} d(ASx_n, SSx_n) \le \lim_{n\to\infty} d(SAx_n, SSx_n) \le 0.$$

Therefore *A* and *S* are compatible.

2.8 Preposition
Let *A* and *S* be continuous mappings of (*X, d*) into itself. If *A* and *S* are compatible, then they are compatible of type (*A*).
As a direct consequence of Propositions 2.3 and 2.4, we have the following propositions.

2.9 Preposition
Let A and S be continuous mappings from a metric space (X, d) into itself. If A and S are compatible, then they are weak compatible of type (A).

Next we give some properties of weak compatible mappings of type (A).

2.10 Preposition
Let A and S be weak compatible maps of type (A) from metric space (X, d) into itself and let $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ then we have the following:
1. $\lim_{n\to\infty} ASx_n = St$ if *S* is continuous at *t*.
2. $\lim_{n\to\infty} SAx_n = At$ if *A* is continuous at *t*.
3. $S At = ASt$ and $At = St$ if *S* and *A* are continuous at *t*.

2.11 Preposition
Let A and S be continuous mappings from a metric space (X, d) into itself. Then
A and S are compatible of type A if and only if they are weak compatible of type (A).
A and S are compatible if and only if they are weak compatible of type (A).
Let A, B, S and T be self mappings from the metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subseteq T(X),\ B(X) \subset S(X) \tag{2.1}$$

$$D(Ax, By) \le h \max\{d(Ax, Sx), d(By, Ty),$$
$$\frac{1}{2}[d(Ax, Ty) + d(By, Sx)], d(Sx, Ty)\} \tag{2.2}$$

for all $x, y \in X$, where $0 \le h < 1$.
Then for any arbitrary point x_0 in *X* by (2.1), we choose a point $x_1 \in X$ such that $Tx_1 = Ax_0$ and for this point x_1, we can choose a point $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on inductively we can define a sequence $\{y_n\}$ in *X* such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1}. \tag{2.3}$$

3. Lemma [6]
Let *A*, *B*, *S* and *T* be mappings from a metric space (*X*, *d*) into itself satisfying the conditions (2.1) and (2.2). Then the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in *X*.

3.1 Lemma
Let *A*, *B*, *S* and *T* be mappings from a metric space (*X*, *d*) into itself satisfying the conditions (2.1), (2.2) and (2.3) A $(X) \cap B(X)$ is a complete subspace of *X*. Then pairs (*A*, *S*) and (*B*, *T*) have a coincidence point in *X*.

3.2 Proof
By Lemma 2.1, the sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in $A(X) \cap B(X)$. Since $A(X) \cap B(X)$ is a complete subspace of X, so $\{y_n\}$converges to a point w (say), in $A(X) \cap B(X)$.
On the other hand, since the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequence in $A(X) \cap B(X)$, so they also converges to the same limit *w*. Hence there exist two points $u, v \in X$ such that $A_u = w$ and $Bv = w$, respectively. By (2.2), we have

$$d(Su, y_{2n+1} = d(Su, Ax_{2n})$$
$$\le h \max\{ d(Su, Tu), d(Ax_{2n}, Bx_{2n+1}),$$
$$\frac{1}{2}[d(Su, Bx_{2n+1}) + d(Tx_{2n+1}, Au)], d(Au, Tx_{2n+1})\}.$$

Letting limit as n→ ∞, we have

$$d(Su,w) \le h\max\{d(Su,w), d(w,w), \frac{1}{2}[d(Su,w)+d(w,Au)], d(Au,w)\}$$
$$= hd(Su,w),$$

a contradiction. Hence $Au = w = Su$.
Similarly, we can show that v is also a coincidence point of B and T.

3.3 Lemma
Let A and S be weak compatible mappings of type (A) from a metric space (X, d) into itself. If $Au = Su$ for some $u \in X$, then $SAu = SSu = AAu = ASu$.

3.4 Proof
Let $\{x_n\}$ be a sequence in X defined by $x_n = u$, $n = 1, 2, 3,$ and $Su = Au$. Now, we have

$$\lim_{n\to\infty} Sx_n = Su = \lim_{n\to\infty} Ax_n.$$

Since S and A are weak compatible mappings of type (A), we have

$$d(S\,Au, AAu) = \lim_{n\to\infty} d(S\,Ax_n, AAx_n)$$
$$\le \lim_{n\to\infty} d(ASx_n, AAx_n) = 0.$$

Hence $S\,Au = AAu$. Therefore, $SAu = SSu = AAu = ASu$.
Kang and Kim [6] proved the following:

3.5 Theorem
Let A, B, S and T be self-mappings from a metric space (X, d) into itself satisfying the conditions (2.1), (2.2) and (2.3). Suppose that
One of A, B, S and T is continuous; (2.4)
Pairs [A, S] and [B, T] are compatible on X. (2.5)
Then A, B, S and T have a unique common fixed point in X.
Now, we prove the following theorem:

3.6 Theorem
Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (2.1),(2.2) and (2.3) and (B, T) are weak compatible of type (A). (2.6)
Then A, B, S and T have a unique common fixed point in X.

3.7 Proof
By Lemma 2.2, there exists two points u, v in X such that $Su = Au = w$ and $Tv = Bv = w$, respectively. Since A and S are weak compatible of type (A), then by Lemma 2.3, $SAu = SSu = AAu = ASu$, which implies that $Sw = Aw$.
Similarly, we have $Tw = Bw$.
Now, we prove that $Sw = w$. Suppose Sw □ w, then by (2.2), we have

$$d(Sw, y_{2n+1}) = d(Sw, Tx_{2n+1})$$
$$\le h\max\{d(Sw,Aw), d(Tx_{2n+1}, Bx_{2n+1}),$$
$$\frac{1}{2}[d(Sw, B_{x_{2n+1}}) + d(Tx_{2n+1}, Aw)], d(Aw, Tx_{2n+1})\}.|$$

Proceeding limit as n → ∞, we have

$$d(Sw,w) \le h\max\{d(Sw,w), d(w,w), \frac{1}{2}[d(Sw,w)+d(w,Aw)], d(Aw,w)\}$$
$$= hd(Sw,w),$$

a contradiction. Hence $Sw = w = Aw$.
Similarly, we have $Tw = w = Bw$. This means that w is a common fixed point of A, B, S and T.

3.8 Uniqueness
Let $z \ne w'$ be another common fixed point of A, B, S and T. Then we have

$$d(w', z) = d(Sw', Tz)$$
$$\le h\max\{d(Sw', Aw'), d(Tz, Bz, \frac{1}{2}[d(Sw', Bz)$$
$$+ d(Tz, Aw')], d(Aw', Tz)\}$$
$$= hd(Sw', Tz)$$
$$= hd(w', z),$$

a contradiction. Hence $w' = z$.
In support of our theorem, we give the following example.

3.9 Example
Let $X = [0, 1]$ with the Euclidean metric d. Define A, B, S and T: X → X by

$$Ax = x^3, Bx = x^2, Sx = 2x^6 - 1 \text{ and } Tx = 2x^4 - 1$$

for all x in X. Now A (X) = B (X) = S (X) = T (X) = X. Moreover, since

$$d(Ax_n, Sx_n) = |2x_n^3 + 1|\,|x_n - 1| \to 0 \text{ iff } x_n \to 1,$$
$$\lim_{n\to\infty} d(ASx_n, SAx_n) = \lim_{n\to\infty} 6x_n^6(x_n^6 - 1)^2 = 0 \text{ as } x_n \to 1.$$

Also, since

$$d(Bx_n, Tx_n) = 2x_n^2 + 1)|x_n^2 - 1| \to 0 \text{ iff } x_n \to 1,$$
$$\lim_{n\to\infty} (BTx_n, TBx_n) = \lim_{n\to\infty} 2(x_2^4 - 1)^2 = 0 \text{ as } x_n \to 1.$$

Clearly, (A, S) and (B, T) are weak compatible mappings of type (A). Further, we obtain

$$d(Ax, By) \le \frac{1}{4} d(Sx, Ty)$$
$$\le \frac{1}{4}\max\{d(Ax,Sx), d(By,Ty)\frac{1}{2}[d(Ax,Ty)|$$
$$+ d(By,Sx)], d(Sx,Ty)\},$$

Since

$$d(Sx, Ty) = 2|x^3 - y^2||x^3 + y^2|$$
$$\geq 4d(Ax, By) \qquad \text{for all x, y.}$$

Therefore all the conditions of Theorem 2.2 are satisfied and 0 is the unique common point fixed point.

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