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Correct solvability of nonlocal boundary value problem for one class of equations

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Abstract
The boundary value problem with nonlocal conditions for a system of partial differential equations of the second order is investigated. The parameterization method is used to establish coefficient sufficient conditions for the correct solvability of the problem under consideration in terms of the initial data. The algorithm for finding a solution is proposed.

Keywords: boundary value problem, nonlocal condition, parameterization method, correctness, solvability

Introduction

On the $\overline{\Omega}=\{(x,t): t\le x\le t+\omega,\ 0\le t\le T\},\ T>0,\ \omega>0$ we consider a non-local boundary value problem for the system of partial differential equations

$$D\left[\frac{\partial}{\partial x}u\right]=A(x,t)\frac{\partial u}{\partial x}+S(x,t)u+f(x,t), \tag{1}$$

$$B\frac{\partial u}{\partial x}(x,0)+C(x)\frac{\partial u}{\partial x}(x+T,T)=d(x), \quad x\in[0,\omega], \tag{2}$$

$$u(t,t)=\Psi_1(t), \quad t\in[0,T], \tag{3}$$

$$Du(t,t)=\Psi_2(t), \quad t\in[0,T]. \tag{4}$$

Here $u\in R^n$, $D=\frac{\partial}{\partial t}+\frac{\partial}{\partial x}$, $A(x,t)$, $S(x,t)$ are $(n\times n)$- matrices and n -vector-function $f(x,t)$ is continuous in x and t on $\overline{\Omega}$; $B(x)$, $C(x)$ are $(n\times n)$- matrices and n -vector-function $d(x)$ is continuous on $[0,\omega]$; the function $\Psi_1(t)$ is continuously differentiable and $\Psi_2(t)$ is continuous function on $[0, T]$.

We denote by $C(\overline{\Omega},R^n)$ the space of continuous in x and t functions $u:\overline{\Omega}\to R^n$ with the norm

$$\|u\|_0=\max_{(x,t)\in\overline{\Omega}}\|u(x,t)\|, \quad \|A\|=\max_{(x,t)\in\overline{\Omega}}\|A(x,t)\|=\max_{(x,t)\in\overline{\Omega}}\max_{i=\overline{1,n}}\sum_{j=1}^{n}|a_{ij}(x,t)|,$$

$$\|d\|_1=\max_{x\in[0,\omega]}\|d(x)\|, \quad \|\Psi_i\|_2=\max_{t\in[0,T]}\|\Psi_i(t)\|, \ i=1,2.$$

The aim of this work is to find coefficient sufficient conditions for the correct solvability of problem (1) - (4).

In the theory of boundary value problems for partial differential equations, problems with nonlocal constraints are of considerable interest. We note the papers [3], [7], where one can find a detailed survey and bibliography on these problems. To study boundary value problems for systems of hyperbolic equations with a mixed derivative, a method for introducing functional parameters was proposed in [2], which is a generalization of the parameterization method [6], developed for solving boundary value problems of ordinary differential equations.

Following [1], we introduce new unknown functions $v(x,t)=\frac{\partial u}{\partial x}(x,t)$, $w(x,t)=Du$ and then problem (1) - (4) is reduced to an equivalent problem for the system of first-order hyperbolic equations

$$Dv = A(x,t)v + S(x,t)u + f(x,t),\ (x,t)\in\overline{\Omega}, \tag{5}$$

$$B(x)v(x,0)+C(x)v(x+T,T)=d(x),\ \ x\in[0,\omega], \tag{6}$$

$$u(x,t)=\Psi_1(t)+\int_t^x v(\eta,t)d\eta,\ \ t\in[0,T], \tag{7}$$

$$w(x,t)=\Psi_2(t)+\int_t^x Dv(\eta,t)d\eta,\ \ t\in[0,T]. \tag{8}$$

If the continuous function *u*(*x*,*t*) is known, then by solving the two-point boundary value problem (5) - (6), we find *v*(*x*,*t*). If a continuous function *v*(*x*,*t*) is known, then from (7) - (8) we define the functions *u*(*x*,*t*) and *w*(*x*,*t*).
If the function *u*(*x*,*t*) is solution to problem (1)-(4), then the triple of continuous functions (*v*(*x*,*t*), *u*(*x*,*t*), *w*(*x*,*t*)) is solution to problem (5) - (8), where $v(x,t)=\frac{\partial u}{\partial x}$, $w(x,t)=Du$ and vice versa, if (*v*(*x*,*t*), *u*(*x*,*t*), *w*(*x*,*t*)) is solution to problem (5) - (8), then from (7) it follows that the function *u*(*x*,*t*) satisfies condition (3) and has continuous derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}$ and $D\left[\frac{\partial u}{\partial x}\right]$.
Substituting $v(x,t)=\frac{\partial u}{\partial x}$ into (5) - (6) and taking into account (7)-(8), *w*(*x*,*t*)=*Du*, we have that the function *u*(*x*,*t*) satisfies (1) and conditions (2) - (4) for all $(x,t)\in\overline{\Omega},\ x\in[0,\omega],\ t\in[0,T]$, that is the function *u*(*x*,*t*) is a solution to the nonlocal boundary value problem (1) - (4).
The triple of continuous on $\overline{\Omega}$ functions (*v*(*x*,*t*), *u*(*x*,*t*), *w*(*x*,*t*)) is called the solution of the boundary value problem for equation (5) under conditions (6) - (8) in the broad sense by Friedrichs [6], if the function *v*(*x*,*t*) is continuously differentiable with respect to the variable *t* along the characteristic and satisfies the family of ordinary differential equations, condition (6), where the function *v*(*x*,*t*) with functions *u*(*x*,*t*) and *w*(*x*,*t*) is related by relations (7) - (8).
The nonlocal boundary value problem (5) - (8) for a system of equations with the same principal part according to Courant is reduced to family of ordinary differential equations on $\overline{H}=\{(\xi,\tau): 0\le\xi\le\omega,\ 0\le\tau\le T\},\ T>0,\ \omega>0$:

$$\frac{\partial\tilde{v}}{\partial\tau}=\tilde{A}(\xi,\tau)\tilde{v}+\tilde{S}(\xi,\tau)\tilde{u}(\xi,\tau)+\tilde{f}(\xi,\tau),\ \ \tau\in[0,T], \tag{9}$$

with the boundary condition

$$\tilde{B}(\xi)\tilde{v}(\xi,0)+\tilde{C}(\xi)\tilde{v}(\xi,T)=\tilde{d}(\xi),\ \ \ \xi\in[0,\omega], \tag{10}$$

functional relations

$$\tilde{u}(\xi,\tau)=\Psi_1(\tau)+\int_\tau^{\tau+\xi}\tilde{v}(\varsigma,\tau)d\varsigma,\ \ \ \tau\in[0,T], \tag{11}$$

$$\tilde{w}(\xi,\tau)=\Psi_2(\tau)+\int_\tau^{\tau+\xi}\frac{\partial\tilde{v}(\varsigma,\tau)}{\partial\tau}d\varsigma,\ \ \ \tau\in[0,T], \tag{12}$$

Where

$\tilde{v}(\xi,\tau)=v(\xi+\tau,\tau)$, $\tilde{A}(\xi,\tau)=A(\xi+\tau,\tau)$, $\tilde{S}(\xi,\tau)=S(\xi+\tau,\tau)$, $\tilde{f}(\xi,\tau)=f(\xi+\tau,\tau)$, $\tilde{w}(\xi,\tau)=w(\xi+\tau,\tau)$, $\tilde{u}(\xi,\tau)=u(\xi+\tau,\tau)$; $\tilde{A}(\xi,\tau)$ and $\tilde{S}(\xi,\tau)$ are $(n\times n)$-matrices and the *n*-vector function $\tilde{f}(\xi,\tau)$ is continuous in τ and ξ on $\overline{H}$; $\tilde{C}(\xi)$ and $\tilde{B}(\xi)$ are $(n\times n)$-matrices and the *n*-vector function $\tilde{d}(\xi)$ is continuous on [0,ω]; the function $\Psi_1(\tau)$ is continuous and $\Psi_2(\tau)$ is continuously differentiable on [0,*T*].

By $C(\overline{H},R^n)$ we denote the space of functions $\tilde{v}:\overline{H}\to R^n$ continuous in ξ and τ with the norm $\|\tilde{v}\|_0=\max_{\xi\in[0,\omega]}\max_{t\in[0,T]}\|\tilde{v}(\xi,\tau)\|$.

A continuous function $\tilde{v}(\xi,\tau)$ is called the solution to the boundary value problem (9) - (12) with known continuous functions $\tilde{u}(\xi,\tau)$ and $\tilde{w}(\xi,\tau)$, if the function $\tilde{v}(\xi,\tau)\in C(\overline{H},R^n)$ has a continuous derivative with respect to variable τ and satisfies the family of ordinary differential equations (9), the boundary condition (10) for all $(\xi,\tau)\in\overline{H}$, $\xi\in[0,\omega]$, where the functions $\tilde{u}(\xi,\tau)$ and $\tilde{w}(\xi,\tau)$ are related to the function $\tilde{v}(\xi,\tau)$ by functional relations (11) - (12).

The continuous function $u(x,t)=\tilde{u}(x-t,t)$ on $\overline{\Omega}$ is called a solution of the boundary value problem for the system equations (1) with conditions (2) - (4) in the broad sense.

To find a solution to problem (9) - (12) algorithm is proposed.

Step 0: In (9), taking $\tilde{u}(\xi,\tau)=\Psi_1(\tau)$ and having solved the two-point boundary value problem (9) - (10), we determine the initial approximation $\tilde{v}^0(\xi,\tau)$. Using $\tilde{v}(\xi,\tau)=\tilde{v}^0(\xi,\tau)$, $\frac{\partial\tilde{v}}{\partial\tau}=\frac{\partial\tilde{v}^0}{\partial\tau}$ from relations (11) - (12) we find $\tilde{u}^0(\xi,\tau)$ and $\tilde{w}^0(\xi,\tau)$, respectively.

Step 1: Taking $\tilde{u}(\xi,\tau)=\tilde{u}^0(\xi,\tau)$ on the right-hand side of (9), solving the boundary value problem (9) - (10), we define the approximation $\tilde{v}^{(1)}(\xi,\tau)$. Substituting the found function $\tilde{v}^{(1)}(\xi,\tau)$ and $\frac{\partial\tilde{v}^{(1)}(\xi,\tau)}{\partial\tau}$ into (11) - (12), we find $\tilde{u}^{(1)}(\xi,\tau)$ and $\tilde{w}^{(1)}(\xi,\tau)$. Etc.

Continuing this process, at the *k*-th step we get $\left(\tilde{v}^k(\xi,\tau),\tilde{u}^k(\xi,\tau),\tilde{w}^k(\xi,\tau)\right)$.

At each step of the proposed algorithm, we apply the parameterization method to find a solution to the two-point boundary value problem.

One of the main conditions for the unique solvability of the problem posed is the invertibility of the matrix $Q_\nu(\xi,h)$, *h*>0: *Nh*=*T*, ν=1, 2,… composed of the sums of iterated integrals over variable τ of length *h* from the coefficient matrix of the system and the matrices of the boundary condition.

$$Q_\nu(\xi,h)=\begin{bmatrix} h\tilde{B}(\xi) & 0 & 0 & \cdots & 0 & h\tilde{C}(\xi)\left(I+L_{\nu N}(\xi,h)\right)\\ I+L_{\nu1}(\xi,h) & -I & 0 & \cdots & 0 & 0\\ 0 & I+L_{\nu2}(\xi,h) & -I & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & I+L_{\nu N-1}(\xi,h) & -I \end{bmatrix},$$

I is the identity matrix of dimension *n*,

$$L_{\nu r}(\xi,h)=\int_{(r-1)h}^{rh}\tilde{A}(\xi,\tau_1)d\tau_1+\int_{(r-1)h}^{rh}\tilde{A}(\xi,\tau_1)\int_{(r-1)h}^{r_1}\tilde{A}(\xi,\tau_2)d\tau_1 d\tau_2+\cdots+$$

$$+\int_{(r-1)h}^{rh}\tilde{A}(\xi,\tau_1)\cdots\int_{(r-1)h}^{r_{\nu-1}}\tilde{A}(\xi,\tau_\nu)d\tau_1\ldots d\tau_\nu .$$

Теорема 1. Пусть при некоторых h и N – матрица Q обратима при всех 0 и выполняются неравенства:
Тогда последовательные приближения v равномерно сходятся к v– единственному решению задачи (9)-(12).

При доказательстве теоремы используется схема доказательства теоремы [4, с.54], [1] и указанный алгоритм нахождения решения краевой задачи (9)-(12).

Теорема 2. Пусть выполнены условия теоремы 1. Тогда нелокальная краевая задача (1)-(4) имеет единственное решение u.
Из теоремы 1 вытекает, что задача (9)-(12) однозначно разрешима. Так как задача (9)-(12) эквивалентна задаче (5)-(8), а задача (5)-(8) эквивалентна задаче (1)-(4), то получим, что задача (1)-(4) имеет единственное решение u. Отметим, что при фиксированной функции u задача (9)-(10) является семейством двухточечных краевых задач для обыкновенных дифференциальных уравнений (11).

Theorem 1. Let for some $h>0$: $Nh=T$ and $\nu,\nu=1,2,\ldots$, $(nN\times nN)$-the matrix $Q_\nu(\xi,h)$ be invertible for all $\xi\in[0,\omega]$ and the following inequalities hold:

a) $\left\|\left[Q_\nu(\xi,h)\right]^{-1}\right\|\le\gamma_\nu(h)$;

b) $q_\nu(\xi,h)=\gamma_\nu(h)\max\left\{1,h\left\|\tilde{C}(\xi)\right\|\right\}\left[\ell^{\alpha(\xi)h}-1-\alpha(\xi)h-\ldots-\dfrac{(\alpha(\xi)h)^\nu}{\nu!}\right]\le\sigma<1$,

where $\alpha(\xi)=\max\limits_{\tau\in[0,T]}\left\|\tilde{A}(\xi,\tau)\right\|$, $\sigma=\text{const}$. Then successive approximations of $\left(\tilde{v}^k(\xi,\tau),\tilde{u}^k(\xi,\tau),\tilde{w}^k(\xi,\tau)\right)$ converge uniformly to $\left(\tilde{v}^*(\xi,\tau),\tilde{u}^*(\xi,\tau),\tilde{w}^*(\xi,\tau)\right)\in C\left(\overline{H},R^n\right)$- the unique solution of problem (9) - (12).
In the proof of the theorem, we use the scheme for proving the theorem [6, p. 54], [1] and the indicated algorithm for finding a solution to the boundary value problem (9) - (12).

Theorem 2. Let the conditions of Theorem 1 be satisfied. Then the nonlocal boundary value problem (1) - (4) has a unique solution $u^*(x,t)\in C\left(\overline{\Omega},R^n\right)$.
Theorem 1 implies that problem (9) - (12) is uniquely solvable. Since problem (9) - (12) is equivalent to problem (5) - (8), and problem (5) - (8) is equivalent to problem (1) - (4), we obtain that problem (1) - (4) has a unique solution $u^*(x,t)\in C\left(\overline{\Omega},R^n\right)$.
Note that for a fixed function $\tilde{u}(\xi,\tau)$, problem (9) - (10) is a family of two-point boundary value problems for ordinary differential equations.

$$\frac{\partial\tilde{v}}{\partial\tau}=\tilde{A}(\xi,\tau)\tilde{v}+\tilde{G}(\xi,\tau),\quad \tau\in[0,T], \tag{13}$$

with boundary condition (10).
The parameterization method is applied to the family of linear two-point boundary value problems (13), (10).

Definition 1. The two-point boundary value problem (13), (10) is called correctly solvable if for any $\tilde{G}(\xi,\tau)$ and $\tilde{d}(\xi)$ it has the unique solution $\tilde{v}(\xi,\tau)\in C\left(\overline{H},R^n\right)$ and for it the estimate

$$\max_{\tau\in[0,T]}\left\|\tilde{v}(\xi,\tau)\right\|\le K(\xi)\max\left(\left\|\tilde{d}\right\|_1,\ \max_{\tau\in[0,T]}\left\|\tilde{G}(\xi,\tau)\right\|\right),$$

where $K(\xi)$ is continuous on $[0,\omega]$ and function independent of $\tilde{G}(\xi,\tau)$, $\tilde{d}(\xi)$.

Definition 2. Boundary value problem (1) - (4) is called well-solvable in the broad sense if for any $f(x,t)$, $d(x)$, $\Psi_1(t)$ and $\Psi_2(t)$, it has a unique solution $u^*(x,t)\in C\left(\overline{\Omega},R^n\right)$ and it satisfies the estimate

$$\max\left(\|u\|_0,\ \left\|\frac{\partial u}{\partial x}\right\|_0\right)\le K\max\left(\|f\|_0,\ \|d\|_1,\ \|\Psi_1\|_2,\ \|\Psi_2\|_2\right),$$

where $K=const$ is independent of $f(x,t)$, $d(x)$, $\Psi_1(t)$ and $\Psi_2(t)$.

Theorem 3. If the boundary value problem (1) - (4) for any functions $f(x,t)$, $d(x)$, $\Psi_1(t)$ and $\Psi_2(t)$ has a unique solution, then it is correctly solvable.

If the constructed solution in the wide sense is continuously differentiable with respect to x and t, then function $u(x,t)$ with continuous partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, D\left[\frac{\partial u}{\partial x}\right]$ satisfying (1) for all $(x,t) \in \overline{\Omega}$ with conditions (2) - (4) is also a classical solution to the boundary value problem (1) -(4).

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