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The Moyal theory for generalized Wigner functions

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Abstract
In this article, we consider the Wigner function of the pair of functions defined on possible phase space with the Lie group actions on it. We introduce and establish properties of the generalized Wigner function for the group representation such that $\int_G |\langle \pi(g,\ \psi), \psi\rangle|^2\ d\mu(g) < \infty$ holds for all $g \in G$ and all vectors $\psi \in H$. We develop the Moyal theory for generalized Wigner functions and Wigner operators.

Keywords: Wigner function, Moyal identity, Weyl operator, Lieb theory, Lie group, Lie algebra

Introduction

The Wigner function is a quasi-probability distribution in the phase space, the Wigner function depends on two wavefunctions ψ and φ a real variable $z \in R^n$

$$W_\eta(\psi,\ \varphi)(x,\ p) = \left(\frac{1}{2\pi\eta}\right)^n \int_{R^n} exp\left(-\frac{i}{\eta} p \cdot z\right) \psi\left(x + \frac{1}{2} z\right) \bar{\varphi}\left(x - \frac{1}{2} z\right) dz.$$

Here $\boldsymbol{\eta}$ denotes a gauge parameter in quantum physics it is the Planck constant.
In the classical case, the Wigner function can be presented in the Wigner-Moyal form

$$W_\eta{}^M(\psi,\ \varphi)(x,\ p) = \frac{1}{(2\pi\eta)^n} \left\langle \hat{T}_R(x,\ p)\psi | \bar{\varphi} \right\rangle_{L^2}$$

where $\hat{\boldsymbol{T}}_{\boldsymbol{R}}(\boldsymbol{x},\ \boldsymbol{p}) :\ \boldsymbol{L}^2 \to \boldsymbol{L}^2$ is the Royer operator given by

$$\hat{T}_R(x_0,\ p_0)\psi(x) = exp\left(2\frac{i}{\eta} p_0 \cdot (x - x_0)\right)\psi(2x_0 - x).$$

This approach is fruitful in signal processing and quantum mechanics since the wavepacket transforms W_ϕ with window ϕ can be defined as

$$W_\phi(\psi)(x,\ p) = \left(\frac{2}{\pi\eta}\right)^n \left\langle \hat{T}_R(x,\ p)\psi, \phi \right\rangle_{L^2}.$$

The Wigner function allows describing the quantum optics processes and comparing classical and quantum approaches in phase space [1-5, 10-12]. Roughly speaking the result of classical mechanics can be considered a limiting case when the gauge constant approaches zero [9]. In [4], the authors introduced a Wigner function representation of the first-order electronic coherence in order to visualize excitations emitted by single electron sources and describe the experimental data.
In the descriptions of quantum systems, the observable quantities are the Lie groups generator's eigenvalues that encouraged a generalization of the idea of the Wigner function and the extension of the phase space conception. The possible phase spaces of the quantum mechanical system consist of the orbits of the coadjoint representation (the representation in the dual space of its Lie algebra) with their symplectic geometry. The wavepacket transform can be defined by employing the Moyal theory for the Wigner or ambiguity functions. The Moyal theory establishes that linear mapping $\psi \mapsto W(\psi, \varphi)(g) = \langle \pi(g)\ \Upsilon^{-1}\ \psi, \varphi \rangle$ defines, for each $\varphi \in L^2(G)$, an isometry of $L^2(G)$ in a close subspace of $L^2(G \oplus G)$.
As a corollary of the classical Moyal theory, we have the inversion formula

$$\psi(x) = 2^n \frac{1}{\langle\phi,\varphi\rangle_{L^2}} \int_{R^n \oplus R^n} W_\eta(\psi,\ \varphi)(x_0,\ p_0)\hat{T}_R(x_0,\ p_0)\phi(x)dp_0dx_0,$$

similarly, the ultimate results of the theory developed in the present article can be formulated in form of the formula

$$|\psi\rangle\langle\varphi| = \left(\frac{1}{2\pi}\right)^{-\frac{n}{2}} \int_{g^*}\int_{N_0} exp(-i\langle X^*, X\rangle)\, W(\rho, X^*)\pi(exp(X))\Upsilon^{-1}\big(m(X)\big)^{\frac{1}{2}}\big(\upsilon(X^*)\big)^{-\frac{1}{2}}dXdX^*$$

that holds for all $\psi,\ \varphi \in H$.

2. Definitions and Fourier transform

Let $\boldsymbol{G}$ be a locally compact group equipped with a Haar measure $\boldsymbol{\mu}$. A continuous homomorphism $\chi : G \to U(1)$ here $U(1)$ denotes the circle group, which is called a character χ of the group $\boldsymbol{G}$. We denote the dual group $\hat{G}$ of G consisting of all characters $\chi :\ G \to U(1)$ on G with its natural operation of multiplication.

Definition 1. The Fourier transform F of a function $\boldsymbol{\psi} \in \mathbf{L^1(G)}$ is defined by

$$F(\psi)(\chi) = \hat{\psi}(\chi) = \int_G \psi(g)\overline{\chi(g)}\ \ d\mu(g). \tag{1}$$

A dual measure $\hat{\mu}$ on $\hat{G}$ is uniquely defined by

$$\hat{\mu}(\chi) = \int_G \overline{\chi(g)}\ \ d\mu(g), \tag{2}$$

The mapping $\mu \mapsto \hat{\mu}$ is called the Fourier-Stieltjes transformation of the measure μ.

Definition 2. ***The Fourier inversion transform F^{-1} of a function $\hat{\psi} \in L^2(\hat{G}) \cap L^1(\hat{G})$ is defined by***

$$\psi(g) = F^{-1}(\hat{\psi})(g) = \left((\hat{\psi})\right)^{\vee}(g) = \int_{\hat{G}} \hat{\psi}(\chi)\chi(g)\ \ d\hat{\mu}(\chi).$$

We denote $\boldsymbol{M(G)}$ the associative Banach algebra of all measures on the $\boldsymbol{\sigma}$ -algebra of all Borel sets of $\boldsymbol{G}$. The convolution of pair of measures $\boldsymbol{\mu}, \boldsymbol{\eta} \in \boldsymbol{M(G)}$ is defined by

$$(\mu * \eta)(D) = \int_{G\times G} \phi_D(g,h)\ \ d\mu(g)d\eta(h), \tag{3}$$

where ϕ_D is an indicator of D, $\phi_D(g,h) = \begin{cases} 1 & if \quad g,h \in D \\ 0 & if \quad g,h \in G\backslash D \end{cases}$.

The convolution of functions $\psi,\ \varphi \in L^1(G)$ is defined by

$$(\psi * \varphi)(g) = \int_G \psi(h)\varphi(h^{-1}g)\ \ d\mu(h). \tag{4}$$

There is natural embedment $L^1(G) \subset M(G)$.

3. Wigner function

Let $\boldsymbol{G}$ be a unimodular locally compact Hausdorff group. Let $\boldsymbol{U(H)}$ be a group of all unitary operators on the separable Hilbert space H.

Definition 4. *A continuous homomorphism $G \mapsto U(H)$ from a unimodular locally compact Hausdorff group G to a group of all unitary operators $U(H)$ on the separable Hilbert space H is called a unitary representation of G in H, when this homomorphism $\pi :\ G \to U(H)$ satisfies the following equalities $\pi(gh) = \pi(g)\pi(h)$ and $\pi(g^{-1}) = \big(\pi(g)\big)^{-1} = \big(\pi(g)\big)^*$, and $g \to \pi(g)x$ is continuous mapping for each $x \in H$.*

Definition 5. ***The inverse Fourier transform is defined by $\zeta(g) = \int_G Tr\left(\hat{\pi}(g)\hat{\zeta}(\pi)\right) d\hat{\mu}(\pi)$for each $F(\zeta)$.***

Let $\pi :\ G \to U(H)$ be a unitary irreducible representation of G on H, representation π is called square-integrable if π satisfies the condition

$$\int_G |\langle\pi(g,\ \psi), \psi\rangle|^2\ \ d\mu(g) < \infty \tag{5}$$

for some $\psi \in H$. Since G is unimodular then (5) holds for all $\psi \in H$.

If we assume that the group G is nonunimodular then the set of intervening operators of the Plancherel theorem, Duflo-Moore operators become unbounded, however, assume G is nonunimodular and assume π_G is a type one regular representation of G, then there is a vector $\phi \in H$ such that

$$W(\phi,\ \psi)(g) = \langle \pi(g,\ \phi), \psi \rangle \tag{6}$$

is $W(\phi,\ )(g) :\ H \mapsto L^2(G)$ - isometry for each $\psi \in H$. Such an element $\phi \in H$ is called an admissible vector. For every square-integrable representation π_G, there is a unique positive invertible operator Υ with the domain $D(\Upsilon)$ of all admissible vectors. For a pair $\phi_1,\ \phi_2$ of admissible vectors, the operator Υ satisfies the condition

$$\int_G \langle \pi(g)\phi_1, \psi_1 \rangle \overline{\langle \pi(g)\phi_2, \psi_2 \rangle}\ d\mu(g) = \langle \Upsilon\phi_{2_1}, \Upsilon\phi \rangle \langle \psi_1, \psi_2 \rangle$$

for all $\psi_1, \psi_2 \in H$.

We assume that Lie algebra g generates a Lie group G. The exponential mapping $exp\colon\ g \mapsto G$ is a homeomorphism between a neighborhood N_0 of the origin 0 in g and a neighborhood B_e of the identity element e in G so that

$$exp(X) = g \in B_e \subset G \tag{7}$$

for all $X \in N_0 \subset\ g$. The inverse to exponential is a logarithmic mapping $log(g) = X \in N_0 \subset\ g$ for all $g \in B_e \subset G$.

The set of all linear functional on g composes a dual algebra g^* with the dual pairing $\langle X^*, X \rangle$ for $X \in g$ and $X^* \in g^*$.
The adjoint representation $Ad_{\tilde{g}}$ is a natural representation of the group G on the Lie algebra g generated by G. The adjoint representation Ad maps $G \mapsto Aut(g)$ by

$$exp\left(Ad_{\tilde{g}}(X)\right) = exp(Y) = \tilde{g}(exp(X))\tilde{g}^{-1}. \tag{8}$$

The coadjoint representation $Ad_{\tilde{g}}^{\#}$ is given by

$$\left\langle Ad_{\tilde{g}}^{\#}(X^*), X \right\rangle = \left\langle X^*, Ad_{\tilde{g}^{-1}}(X) \right\rangle. \tag{9}$$

The coadjoint representation $Ad_{\tilde{g}}^{\#}$ maps $G \mapsto Aut(g^*)$. The norms of adjoint and coadjoint actions are connected by

$$\|Ad_g\| \left\|Ad_g^{\#}\right\|^{-1} = 1.$$

The dual algebra g^* can be decomposed into the unity of coadjoint orbits O_λ^* so that $g^* = \bigcup_{\lambda \in I} O_\lambda^*$, where I is the index set and λ parameter that determines the orbit.

Let X^* be an element of the dual of the Lie algebra g^*, which components give the c -numbers of the Wigner function arguments so that

$$c(\psi_k) = \int_G |\langle \pi(g,\ \psi_k), \psi_k \rangle|^2\ \ d\mu(g) < \infty$$

in basis $\{\psi_k\}$.

Let $d\Omega_\lambda(X_\lambda^*)$ denote the Liouville measure on O_λ^* then we presume that there is a positive density $\upsilon_\lambda(X_\lambda^*)$ on the orbits O_λ^* such that

$$dX^* = \upsilon_\lambda(X_\lambda^*) d\Omega_\lambda(X_\lambda^*) d\eta(\lambda) \tag{10}$$

where $X_\lambda^* \in O_\lambda^*$ and η is a measure on the space I of parameter λ, and dX^* is a Lebesgue measure on g^*.

Definition 6. Let $\boldsymbol{\phi} \in$ H be an admissible vector and let $\boldsymbol{\psi} = \Upsilon\boldsymbol{\phi}$. Then, the Wigner transform is defined by

$$W(\phi,\ \psi)(X_\lambda^*) = \frac{\left(\upsilon_\lambda(X_\lambda^*)\right)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{N_0} exp(-i\langle X_\lambda^*, X \rangle) \langle \pi(exp(X)\ \ \Upsilon^{-1}\psi), \varphi \rangle \left(m(X)\right)^{\frac{1}{2}} dX, \tag{11}$$

Where $d\mu(g) \to m(X)dX$ is a representation of the Haar measure on the group G in the local coordinates via map $g = exp(X)$ and in the basis $\{X_k\}$, we have $X = \sum x^k X_k \in g$ so $g \to (x^1, \ldots, x^n) \in R^n$.

Wigner transform

Employing the notation of quantum physics, we denote the density matrix by $\boldsymbol{\rho} = |\boldsymbol{\psi}\rangle\langle\boldsymbol{\varphi}|$. Let $\boldsymbol{L^2(O_\lambda^*,\ d\Omega_\lambda) = H(\lambda)}$ be a Hilbert space composed of all square-integrable complex-valued functions on $\boldsymbol{O_\lambda^*}$, for each index $\boldsymbol{\lambda \in I}$.
For each index $\boldsymbol{\lambda \in I}$, the representation $\boldsymbol{\pi^\#(g,\lambda) : G \mapsto H(\lambda)}$ is given by

$$\pi^\#(g,\lambda,\ \hat{f}_\lambda)(X^*) \ = \hat{f}_\lambda\left(Ad^\#_{g^{-1}}(X^*)\right) \tag{12}$$

for all $X^* \in O_\lambda^*$.

The coadjoint representation $\pi^\#$ of the group G in $\int_{\oplus I} H(\lambda)d\eta(\lambda) = H_\oplus$ is given by

$$\pi^\#(g,\ f)(X_\lambda^*) \ = \pi_\lambda^\#(g,\ \hat{f}_\lambda)(X_\lambda^*) = \hat{f}_\lambda\left(Ad^\#_{g^{-1}}(X_\lambda^*)\right) \tag{13}$$

for all $X_\lambda^* \in g^* = \bigcup_{\lambda\in I} O_\lambda^*$.

Let ρ be a Hilbert-Schmidt operator given by $\rho = |\psi\rangle\langle\varphi|$ then the Wigner function can be written as

$$W(\rho,\ X^*) = Tr\big(\rho W(X^*)\big) \tag{14}$$

where the Wigner operator transform $W(X^*)$ is defined by

$$W(X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{N_0} exp(-i\langle X^*, X\rangle)\,\Upsilon^{-1}\pi(exp(-X)\ )\big(\upsilon(X^*)\big)^{\frac{1}{2}}\big(m(X)\big)^{\frac{1}{2}}dX \tag{15}$$

for dX -almost all $X^* \in g^*$. The Wigner function (14) can be presented in a more classical symmetric form $W(\phi,\ \psi,\ X^*) = \langle\phi|W(X^*)|\psi\rangle$. The Winer transform of the Hilvert-Schmidt operator $\rho = |\psi\rangle\langle\varphi|$ is

$$W(\rho, X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}}\big(\upsilon(X^*)\big)^{\frac{1}{2}}\int_{N_0} exp(-i\langle X^*, X\rangle)\,Tr(\pi(exp(-X)\ )\rho\ \Upsilon^{-1})\big(m(X)\big)^{\frac{1}{2}}dX \tag{16}$$

for any $X^* \in g^*$.

Definition 7. *The Wigner transform $W(\psi,\varphi)(g)$ is defined by $W(\psi,\varphi)(g) = \langle\pi(g)\ \Upsilon^{-1}\ \psi,\varphi\rangle$ for all $g \in G$.*

Let $X^* \in g^*$, us define the Wigner function by

$$W(\psi,\varphi)(X^*) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{N_0} exp(-i\langle X^*, X\rangle)\,\langle\pi(exp(X)\ )\ \Upsilon^{-1}\ \psi,\varphi\rangle\big(\upsilon(X^*)\big)^{\frac{1}{2}}\big(m(X)\big)^{\frac{1}{2}}dX\ . \tag{17}$$

Let $\{X_k\} \subset g$ and $\{X_k^*\} \subset g^*$ be bases in the Lie algebra g and its dual g^*, respectively. We denote $exp\big(-i\hat{X}_k\big) \ = \pi(exp(X_k)\ )$, where operators $\hat{X}_k$ correspond to the basis vectors $X_k \in g$. Any vector $X^* \in g^*$ can be decomposed $X^* = \sum_k \zeta_k X_k^*$ where $\vec{\zeta} = (\zeta_1, \ldots, \zeta_n)$ are coordinates of X^*. So, we rewrite the Wigner function as

$$W(\rho,\ \vec{\zeta}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\upsilon(\vec{\zeta})\right)^{\frac{1}{2}}\int_{\widetilde{N}_0} Tr\left(exp\left(-i\textstyle\sum_k\big(\zeta_k - \hat{X}_k\big)x_k\right)\rho\ \Upsilon^{-1}\right)\big(m(\vec{x})\big)^{\frac{1}{2}}d\vec{x} \tag{18}$$

for all $X^* \in g^*$.

5. The Moyal identity

Let $\phi_1,\ \phi_2$ be a pair of admissible vectors and let $\psi_1, \psi_2 \in H$, then the equality

$$\int_G \langle\pi(g)\phi_1, \psi_1\rangle\overline{\langle\pi(g)\phi_2, \psi_2\rangle}\ d\mu(g) = \langle\Upsilon\phi_2, \Upsilon\phi_1\rangle\langle\psi_1, \psi_2\rangle$$

is called the Moyal identity. If G is unimodular then the operator Υ is an identity operator with a scaling coefficient λ so that

$$\int_G \langle\pi(g)\phi_1, \psi_1\rangle\overline{\langle\pi(g)\phi_2, \psi_2\rangle}\ d\mu(g) = \lambda^2\langle\phi_2, \phi_1\rangle\langle\psi_1, \psi_2\rangle.$$

Let $\rho_1,\ \rho_2$ be a pair of Hilbert-Schmidt operators then the Wigner transforms are given by

$$W\big(\rho_j, X^*\big) = \frac{(\upsilon(X^*))^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{N_0} exp(-i\langle X^*, X\rangle)\,Tr\Big(\big(\pi(exp(X))\big)^*\rho_j\Upsilon^{-1}\Big)\big(m(X)\big)^{\frac{1}{2}}dX$$

for $j = 1,2$.

The Moyal identity for Wigner operator transform of Hilbert-Schmidt operators $\rho_1,\ \rho_2$ is taking a form

$$\int_{g^*} W(\rho_1, X^*)\overline{W(\rho_2, X^*)}\big(\upsilon(X^*)\big)^{-1}\ dX^* = Tr(\rho_1, \rho_2{}^*). \tag{19}$$

For the Wigner function, we formulate the following theorems.

Theorem (Moyal identity for Wigner function) 1. *Let g be a Lie algebra and g^* its dual, and let the Wigner functions $W(\psi_j,\ \varphi_j)(X^*)$, $j = 1,\ 2$ be defined by (17) for $X^* \in g^*$. Then, the Moyal identity*

$$\int_{g^*} W(\psi_1,\ \varphi_1)(X^*)\overline{W(\psi_2,\ \varphi_2)(X^*)}\big(\upsilon(X^*)\big)^{-1}\ dX^* = \langle\psi_1,\ \psi_2\rangle\langle\varphi_2,\ \varphi_1\rangle$$

holds for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in H$.

Theorem (Moyal identity for Wigner function) 2. *Let g be a Lie algebra and g^* its dual, let $\rho_j = |\psi_j\rangle\langle\varphi_j|$, $j = 1,\ 2$, and let the Wigner functions $W(\rho_j,\ \vec{\zeta})$ be defined by (18) for $X \in g$, $X^* \in g^*$ and $\rho_j = |\psi_j\rangle\langle\varphi_j|$, $j = 1,\ 2$. Then, the Moyal identity*

$$\int_{g^*} W(\psi_1,\ \varphi_1,\ \vec{\zeta})\overline{W(\psi_2,\ \varphi_2,\ \vec{\zeta})}\left(\upsilon(\vec{\zeta})\right)^{-1}\ d\vec{\zeta} = \langle\psi_1,\ \psi_2\rangle\langle\varphi_2,\ \varphi_1\rangle$$

holds for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in H$.

The proof of this theorem is straightforward if we remark that

$$\int_{g^*} exp(-i\langle X^*, X_1 - X_2\rangle)\ \ dX^* = (2\pi)^n\delta(X_1 - X_2)$$

holds for all $X_1,\ X_2 \in g$.

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