Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2022; 3(2): 69-77

www.mathematicaljournal.com
Received: 24-04-2022
Accepted: 06-06-2022

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On generating functions of the product of two polynomials defined by second-order linear recurrence relations

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Abstract
In the present paper we investigate polynomials and numbers satisfying a second-order recurrence relations. We give ordinary and exponential generating functions, Binet formulas and the generating function of the product of two polynomials. Our approach is based on geometric series summation, which is different of the analytic method of Boussayoud *et al.*

Keywords: Ordinary and exponential generating functions, geometric series, Binet formula

Introduction
Polynomials and numbers defined by second-order linear recurrence relations are widely studied in the literature. In Boughaba *et al.* 2021 [1] constructed generating functions of the product of Vieta polynomials with Gaussian numbers and polynomials. In [2] the authors constructed generating functions of the product of Gaussian numbers with Chebyshev polynomials of first and second kinds. In Boussayoud *et al.* 2015 [3] applied the operator $L_{b_1b_2}^{-k}$ in the series $\sum_{J=0}^{\infty} a_j b_1^j z^j$ and used the obtained result on Fibonacci numbers and Tchebychev polynomials of the first and second kind. In Boussayoud *et al.* 2017 [5] introduced the symmetrizing operator $\delta_{e_1e_2}^k$ for the determination of Fibonacci numbers F_{-n}, Lucas numbers L_{-n}, Pell numbers P_{-n} and Pell-Lucas numbers Q_{-1} at negative arguments. In with the same operator Boussayoud *et al.* 2018 [4] established some identities and generating functions of Mersenne numbers and polynomials to compute among other generating function of the product of Mersenne numbers with Tchebychev polynomials of first and second kinds. In [8] Saba and Boussayoud introduced complete homogeneous symmetric functions to give generating function to give generating functions for Gauss-Fibonacci, Gauss-Lucas, bivariate Fibonacci bivariate Lucas, bivariate Jacobsthal and bivariate Jacobsthal-Lucas polynomials. The two last polynomials are generalizations of Jacobsthal polynomials [7]. In this work we study the family $G_n(x,y) = G_{a,b,p,q,n}(x,y)$ defined by the second order recurrence relation.

$$G_{n+2}(x,y) = pxG_{n+1}(x,y) + qG_n(x,y)$$

With

$$G_0(x,y) = a \text{ and } G_1(x,y) = b, \tag{1}$$

Where a, b, p and q are polynomials over the field of real numbers. The corresponding univariate polynomials $(y = 1)$ and numbers $(x = y = 1)$ with a, b, p, q constants are given respectively by the second order recurrence relations.

$$G_{n+2}(x) = pxG_{n+1}(x) + qG_n(x)$$

with

$$G_0(x) = a \text{ and} G_1(x) = b, \tag{2}$$

and

$G_{n+2} = pG_{n+1} + qG_n$ with $G_0 = a$ and $G_1 = b$, (3)

We use geometric series summation to get the generating function and Binet formula. Thereafter we construct generating function of the product $G_n(x)G_n^*(y)$. The obtained result can be applied to infinitely many families of numbers and polynomials. We restrict our attention to polynomials and numbers studied in [1, 2, 3, 4, 5, 6, 8]; these are listed in Table 1. We attract attention to the fact that $GF_0 \neq GF_0(1)$ and $GL_0 \neq GL_0(1)$ because $GF_0(1) = GL_0(1) = a$ and $GF_0 = i$ and $GL_0 = 2 - i$. In fact the polynomials $GF_n(x)$ and $GL_n(x)$ respect the second-order linear recurrence (2) for $n \geq 1$. This phenomena directly influences the form of the generating functions and Binet formulae as we will see later.

Ordinary and exponential generating functions

Ordinary generating function

The characteristic equation of the polynomial $G_n(x, y)$ is $z^2 - pxz - q = 0$. So the roots are

$$r = r(x, y) = \frac{px+\sqrt{p^2x^2+4q}}{2} \text{ and } s = s(x, y) = \frac{px-\sqrt{p^2x^2+4q}}{2}, \qquad (4)$$

Which satisfy the symmetric relations $r + s = px$ and $ps = -q$. The ordinary generating function $G(z)$ of $G_n(x, y)$ is given by the following theorem.

Theorem 2.1. (For examples see Table 2)
The ordinary generating function $G(z)$ is given by the relation

$$G(z) = \frac{a-(apx-b)z}{1-pxz-qz^2} \qquad (5)$$

Proof. We have

$$q\sum_{n=0}^{\infty} G_n(x, y)z^n = \sum_{n=0}^{\infty}\big(G_{n+2}(x, y) - pxG_{n+1}(x, y)\big)z^n.$$

Then

$$q\sum_{n=0}^{\infty} G_n(x, y)z^n = (-a - bz + \sum_{n=0}^{\infty} G_n(x, y)z^n)z^{-2} - px(-a + \sum_{n=0}^{\infty} G_n(x, y)z^n)z^{-1},$$

and

$$(qz^2 + pxz - 1)G(z) = -a + (apx - b)z. \; \square$$

Exponential generating function

$G(z)$ is a rational function where the denominator is $qz^2 + pxz - 1$ which admits for roots the functions

$$u = u(x, y) = \frac{-px-\sqrt{p^2x^2+4q}}{2q} \text{ and } v = v(x, y) = \frac{-px+\sqrt{p^2x^2+4q}}{2q}. \qquad (6)$$

The symmetric relations of u and v are $u + v = -\frac{px}{q}$ and $uv = -\frac{1}{q}$. It follows that $u = -\frac{r}{q}$ and $v = -\frac{s}{q}$.

Theorem 2.2. (for examples see Table 3)
The following identity holds true

$$G_n(x, y) = (-q)^n \frac{\left(au+\frac{b}{q}\right)v^n - \left(av+\frac{b}{q}\right)u^n}{u-v} \qquad (7)$$

Proof. Since

$$qz^2 + pxz - 1 = -\left(1 - \frac{z}{u}\right)\left(1 - \frac{z}{v}\right).$$

For $|z| < min\{|u|, |v|\}$ we get

$$\frac{1}{1-z/u} = \sum_{n=0}^{\infty}(uz)^n \text{ and } \frac{1}{1-z/v} = \sum_{n=0}^{\infty}(vz)^n.$$

So by Cauchy product of generating functions we have

$$\frac{1}{qz^2+pxz-1} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\frac{1}{u}\right)^j\left(\frac{1}{v}\right)^{n-j} z^n.$$

Letting $c_n = \sum_{j=0}^{n} \left(\frac{1}{u}\right)^j \left(\frac{1}{v}\right)^{n-j}$, then we have

$$G(z) = -ac_0 + \sum_{n=1}^{\infty}\big((apx-b)c_{n-1} - ac_n\big)z^n.$$

Thereafter $G_0(x,y) = -ac_0 = a$ and $G_n(x,y) = (apx-b)c_{n-1} - ac_n$. It is easy to show that

$$c_n = -(-q)^n \frac{u^{n+1}-v^{n+1}}{u-v},$$

then

$$G_n(x,y) = (-q)^{n-1}(b-apx)\frac{u^n-v^n}{u-v} + (-q)^n a\frac{u^{n+1}-v^{n+1}}{u-v},$$

And

$$G_n(x,y) = (-q)^n \frac{\left(\frac{apx-b}{q}+au\right)u^n - \left(\frac{apx-b}{q}+av\right)v^n}{u-v}.$$

The result follows from the fact that $\frac{apx-b}{q} + au = -\frac{b}{q} - av$ and $\frac{apx-b}{q} + av = -\frac{b}{q} - au$.□

From the relation $uv = -\frac{1}{q}$ we can write

$$G_n(x,y) = (-q)^n \frac{(a-bv)uv^n - (a-bu)vu^n}{u-v} \tag{8}$$

Theorem 2.3. (For some examples see Table 4) We have

$$\sum_{n=0}^{\infty} G_n(x,y)\frac{z^n}{n!} = \frac{au+b/q}{u-v}e^{-qvz} - \frac{av+b/q}{u-v}e^{-quz} \tag{9}$$

Proof. It is obvious that

$$\sum_{n=0}^{\infty} G_n(x,y)\frac{z^n}{n!} = \frac{au+b/q}{u-v}\sum_{n=0}^{\infty}\frac{(-qvz)^n}{n!} - \frac{av+b/q}{u-v}\sum_{n=0}^{\infty}\frac{(-quz)^n}{n!}.$$

Thereafter the result follows.
In the case of the recurrence relation

$$G_{n+2}(x,z) = pxG_{n+1}(x,y) + qG_n(x,y),\ G_1(x,y) = b \text{ and } G_2(x,y) = c \tag{10}$$

we add the term $G_0^*(x,y) = a = \frac{c-pbx}{q}$ to obtain the second-order recurrence (1). Consequently the generating function is

$$G(z) = \frac{bz + (c-pbx)z^2}{1-pxz-qz^2} \tag{11}$$

The Binet formula is

$$G_n(x,y) = (-q)^n \frac{\left(\frac{c-pbx}{q}u + b/q\right)v^n - \left(\frac{c-pbx}{q}v + b/q\right)u^n}{u-v}. \tag{12}$$

Finally the exponential generating function is

$$\sum_{n=0}^{\infty} G_n(x,y)\frac{z^n}{n!} = \frac{\frac{c-pbx}{q}u + b/q}{u-v}e^{-qvz} - \frac{\frac{c-pbx}{q}v + b/q}{u-v}e^{-quz} - \frac{c-pbx}{q}. \tag{13}$$

For example Gauss-Fibonacci and Gauss-Lucas polynomials are of this kind and theorems 1 and 2 in [8] are a consequence of this remark.

Polynomial $G_{-n}(x,y)$

The polynomials $G_{-n}(x,y)$ are defined by the binet formula

$$G_{-n}(x,y) = (-q)^{-n}\frac{(au+b/q)v^{-n} - (av+b/q)u^{-n}}{u-v}. \tag{14}$$

Since $uv = -\frac{1}{q}$, we get

$G_{-n}(x,y) = \frac{(au+b/q)u^n-(av+b/q)v^n}{u-v}$, and the connection to $G_n(x,y)$ is

$G_{-n}(x,y) = -(-q)^n G_n(x,y) + a(u^n + v^n)$.

The generating function is

$$\sum_{n=0}^{\infty} G_{-n}(x,y)z^n = \frac{a+\frac{b}{q}z}{1+\frac{px}{q}z-\frac{1}{q}z^n}, \tag{15}$$

which leads to the following second-order linear recurrence relation

$$G_{-n-2}(x,y) = -\frac{p}{q}xG_{-n-1}(x,y) + \frac{1}{q}G_{-n}(x,y),\ G_0(x,y) = a \text{ and } G_{-1}(x,y) = \frac{b-apx}{q}. \tag{16}$$

It is obvious that

$$G_{a,b,p,q,-n}(x,y) = G_{a,\frac{b-apx}{q},\frac{-p}{q},\frac{1}{q},n}(x,y). \tag{17}$$

In the case of $a = 0$ we obtain $G_{-n}(x,y) = -(-q)^n G_n(x,y)$. For example we have

$$F_{-n} = (-1)^{n+1}F_n \text{ and } P_{-n} = (-1)^{n+1}P_n. \tag{18}$$

and

$$L_{-n} = (-1)^n L_n \text{ and } Q_{-n} = (-1)^n Q_n. \tag{19}$$

Letting $x = 1$ in the identity (15) to deduce that $\sum_{n=0}^{\infty} F_{-n}z^n = \frac{1}{z^2-z-1}$, $\sum_{n=0}^{\infty} P_{-n}z^n = \frac{z}{1+2z-z^2}$, $\sum_{n=0}^{\infty} L_{-n}z^n = \frac{2+z}{1+z-z^2}$

and

$\sum_{n=0}^{\infty} Q_{-n}z^n = \frac{2+2z}{1+2z-z^2}$, Results which are obtained in [5] by a different way.

Generating function of the product

Let $G_n^*(y) = G_{a',b',p',q',n}(y)$ be another family of polynomials defined by the second order linear recurrence relation (1), we consider the functions α_1, α_2, α_3 and α_4 given respectively by the relations

$$\alpha_1 = \frac{(qau(x)+b)(q'a'u'(y)+b')}{\sqrt{(p^2x^2+4q)(p'^2y^2+4q')}},\ \alpha_2 = -\frac{(qau(x)+b)(q'a'v'(y)+b')}{\sqrt{(p^2x^2+4q)(p'^2y^2+4q')}},$$

$$\alpha_3 = -\frac{(qav(x)+b)(q'a'u'(y)+b')}{\sqrt{(p^2x^2+4q)(p'^2y^2+4q')}},\ \alpha_4 = \frac{(qav(x)+b)(q'a'v'(y)+b')}{\sqrt{(p^2x^2+4q)(p'^2y^2+4q')}}.$$

and the functions

$$\lambda_1 = qq'v(x)v'(y),\ \lambda_2 = qq'v(x)u'(y),\ \lambda_3 = qq'u(x)v'(y) \text{ and } \lambda_4 = qq'u(x)u'(y).$$

Lemma 3.1. The generating function of the product $G_n(x)G_n^*(y)$ is given by the following relation

$$\sum_{n=0}^{\infty} G_n(x)G_n^*(y)z^n = \frac{\sum_{k=1}^{4}\alpha_k\prod_{j\neq k}(1-\lambda_j z)}{\prod_{j=1}^{4}(1-\lambda_j z)} \tag{20}$$

Proof.

The product of Binet formulae of $G_n(x)$ and $G_n^*(y)$ lead to

$$G_n(x)G_n^*(y) = \alpha_1(qq'v(x)v'(y))^n + \alpha_2(qq'v(x)u'(y))^n + \alpha_3(qq'u(x)v'(y))^n + \alpha_4(qq'u(x)u'(y))^n,$$

and

$$\sum_{n=0}^{\infty} G_n(x)G_n^*(y)z^n = \frac{\alpha_1}{1-qq'v(x)v'(y)} + \frac{\alpha_2}{1-qq'v(x)u'(y)} + \frac{\alpha_3}{1-qq'u(x)v'(y)} + \frac{\alpha_4}{1-qq'u(x)u'(y)}.$$

Thereafter

$\prod_{j=0}^{4}(1-\lambda_j z)\sum_{n=0}^{\infty} G_n(x)G_n^*(y)z^n = \sum_{k=0}^{4}\alpha_k \prod_{j\neq k}(1-\lambda_j z),$

and the desired result follows.

Theorem 3.1.

Let $g(z)$ the ordinary generating function of the product $G_n(x)G_n^*(y)$. Then we have

$$g(z) = \frac{P(x,y,z)}{1-pp'xyz-(q'p^2x^2+qp'^2y^2+2qq')z^2-pqp'q'xyz^3+q^2q'^2z^4}, \quad (21)$$

where

$$P(x,y,z) = aa' + (bb' - apa'p'xy)z + \left(a'q'bpx + aqb'p'y - aa'(q'p^2x^2 + qp'^2y^2 + qq')\right)z^2 - qq'(b' - a'p'y)(b - apx)z^3. \quad (22)$$

Proof. The Theorem 3.1 follows from Lemma 3.1 and the fact that

$\prod_{j=1}^{4}(1-\lambda_j z) = 1 - pp'xyz - (q'p^2x^2 + qp'y^2 + 2qq')z^2 - pqp'q'xyz^3 + q^2q'^2z^4$ and

$\sum_{k=0}^{4}\alpha_k \prod_{j\neq k}(1-\lambda_j z) = aa' + (bb' - apa'p'xy)z + \left(a'q'bpx + aqb'p'y - aa'(q'p^2x^2 + qp'^2y^2 + qq')\right)z^2 - qq'(b' - a'p'y)(b - apx)z^3$. □

Corollary 3.1
Let $m(z)$ be the ordinary generating function of the product $G_n(x)G_n(y)$. Then we have

$$m(z) = \frac{P'(x,y,z)}{1-p^2xyz-(qp^2y^2+qp^2x^2+2q^2)z^2-p^2q^2xyz^3+q^4z^4}, \quad (23)$$

Where

$$P'(x,y,z) = aa' + (bb' - aa'p^2xy)z + \left(a'bpqx + ab'pqy - aa'(qp^2x^2 + qp^2y^2 + q^2)\right)z^2 - q^2(b' - a'py)(b - apx)z^3. \quad (24)$$

Corollary 3.1 is a main tool to reproduce the generating function of the product of two polynomials investigated in [1, 2, 6], we resume the results in Table 5.

Illustrative Tables

Table 1: Examples of $G_n(x,y)$

a	b	p	q	$G_n(x,y)$	Name
0	1	$2y$	y	$P_n(x,y)$	Bivariate Pell polynomials
2	$2xy$	$2y$	y	$Q_n(x,y)$	Bivariate Pell-Lucas polynomials
0	1	y	$2y$	$J_n(x,y)$	Bivariate Jacobsthal polynomials
2	xy	y	$2y$	$j_n(x,y)$	Bivariate Jacobsthal-Lucas polynomials
0	1	1	y	$F_n(x,y)$	Bivariate Fibonacci polynomials
2	x	1	y	$L_n(x,y)$	Bivariate Lucas polynomials
/	1	1	1	$GF_n(x,y)$	Gauss-Fibonacci polynomials
/	$x+2i$	1	1	$GL_n(x,y)$	Gauss-Lucas polynomials
0	1	2	1	$P_n(x)$	Pell polynomials
2	$2x$	2	1	$Q_n(x)$	Pell-Lucas polynomials
0	1	1	-1	$V_n(x)$	Vieta-Fibonacci polynomials
2	x	1	-1	$v_n(x)$	Vieta-Lucas polynomials
0	1	2	-1	$t_n(x)$	Vieta-Pell polynomials
2	$2x$	2	-1	$s_n(x)$	Vieta-Pell-Lucas polynomials
1	x	2	-1	$T_n(x)$	Chebyshev polynomials of first kind
1	$2x$	2	-1	$U_n(x)$	Chebyshev polynomials of second kind
0	1	2	1	P_n	Pell numbers
2	2	2	1	Q_n	Pell-Lucas numbers
0	1	3	-2	M_n	Mersenne numbers
i	1	1	1	GF_n	Gauss-Fibonacci numbers
$2-i$	$1+2i$	1	1	GL_n	Gauss-Lucas numbers
$i/2$	1	1	2	GJ_n	Gauss-Jacobsthal numbers
$2-i/2$	$1+2i$	1	2	Gj_n	Gauss-Jacobsthal -Lucas numbers
i	1	2	1	GP_n	Gauss-Pell numbers
$2-2i$	$2+2i$	2	1	GQ_n	Gauss-Pell-Lucas numbers

Table 2: Ordinary generating functions

$G_n(x,y)$	Ordinary generating function
$P_n(x,y)$	$\frac{z}{1-2xyz-yz^2}$
$Q_n(x,y)$	$\frac{2-2xyz}{1-2xyz-yz^2}$
$J_n(x,y)$	$\frac{z}{1-xyz-2yz^2}$
$j_n(x,y)$	$\frac{2-xyz}{1-xyz-2yz^2}$
$F_n(x,y)$	$\frac{z}{1-xz-yz^2}$
$L_n(x,y)$	$\frac{2-xz}{1-xz-yz^2}$
$GF_n(x)$	$\frac{z+iz^2}{1-xz-z^2}$
$GL_n(x)$	$\frac{(2i+x)z+(2-ix)z^2}{1-xz-z^2}$
$P_n(x)$	$\frac{z}{1-2xz-z^2}$
$Q_n(x)$	$\frac{2-2xz}{1-2xz-z^2}$
$V_n(x)$	$\frac{z}{1-xz+z^2}$
$v_n(x)$	$\frac{2-xz}{1-xz+z^2}$
$t_n(x)$	$\frac{z}{1-2xz+z^2}$
$s_n(x)$	$\frac{2-2xz}{1-2xz+z^2}$
$T_n(x)$	$\frac{1-xz}{1-2xz+z^2}$
$U_n(x)$	$\frac{1}{1-2xz+z^2}$
P_n	$\frac{z}{1-2z-z^2}$
Q_n	$\frac{2-2z}{1-2z-z^2}$
M_n	$\frac{z}{1-3z+2z^2}$
GF_n	$\frac{i-(i-1)z}{1-z-z^2}$
GL_n	$\frac{2-i-(1-3i)z}{1-z-z^2}$
GJ_n	$\frac{i-(i-2)z}{2(1-z-2z^2)}$
Gj_n	$\frac{4-i-(2-5i)z}{2(1-z-2z^2)}$
GP_n	$\frac{i-(2i-1)z}{1-2z-z^2}$
GQ_n	$\frac{2-2i-(2-6i)z}{1-2z-z^2}$

Table 3: Binet formulae (BF)

J	$G_n(x,y)$	u_j	v_j	BF
1	$P_n(x,y)$	$\left(-xy-\sqrt{x^2y^2+y}\right)/y$	$\left(-xy+\sqrt{x^2y^2+y}\right)/y$	$((-y)^{n-1}(u_1^n-v_1^n))/(u_1-v_1)$
2	$Q_n(x,y)$	$\left(-xy-\sqrt{x^2y^2+y}\right)/y$	$\left(-xy+\sqrt{x^2y^2+y}\right)/y$	$2(-y)^n((u_2+x)v_2^n-(v_2+x)u_2^n)/(u_2-v_2)$
3	$J_n(x,y)$	$\left(-xy-\sqrt{x^2y^2+8y}\right)/4y$	$\left(-xy+\sqrt{x^2y^2+8y}\right)/4y$	$(-2y)^{n-1}(v_3^n-u_3^n)/(v_3-u_3)$
4	$j_n(x,y)$	$\left(-xy-\sqrt{x^2y^2+8y}\right)/4y$	$\left(-xy+\sqrt{x^2y^2+8y}\right)/4y$	$2^{n-1}(-y)^n((4u_4+x)v_4^n-(4v_4+x)u_4^n)/(u_4-v_4)$
5	$F_n(x,y)$	$\left(-x-\sqrt{x^2+4y}\right)/2y$	$\left(-x+\sqrt{x^2+4y}\right)/2y$	$(-y)^{n-1}(u_5^n-v_5^n)/(u_5-v_5)$
6	$L_n(x,y)$	$\left(-x-\sqrt{x^2+4y}\right)/2y$	$\left(-x+\sqrt{x^2+4y}\right)/2y$	$(-y)^{n-1}((2yv+x)u_6^n-(2yu+x)v_6^n)/(u_6-v_6)$
7	$GF_n(x)$	$\left(-x-\sqrt{x^2+4}\right)/2$	$\left(-x+\sqrt{x^2+4}\right)/2$	$(-1)^n((iu_7+1)v_7^n-(iv_7+1)u_7^n)/(u_7-v_7)$
8	$GL_n(x)$	$\left(-x-\sqrt{x^2+4}\right)/2$	$\left(-x+\sqrt{x^2+4}\right)/2$	$\left(((2-ix)u_8+2i+x)v_8^n-((2-ix)v_8+2i+x)u_8^n\right)/(-1)^n(u_8-v_8)$
9	$P_n(x)$	$-x-\sqrt{x^2+1}$	$-x+\sqrt{x^2+1}$	$(-1)^{n-1}(u_9^n-v_9^n)/(u_9-v_9)$
10	$Q_n(x)$	$-x-\sqrt{x^2+1}$	$-x+\sqrt{x^2+1}$	$2(-1)^n((u_{10}+x)v_{10}^n-(v_{10}+x)u_{10}^n)/(u_{10}-v_{10})$

11	$V_n(x)$	$\left(x+\sqrt{x^2-4}\right)/2$	$\left(x-\sqrt{x^2-4}\right)/2$	$(u_{11}^n-v_{11}^n)/(u_{11}-v_{11})$
12	$v_n(x)$	$\left(x+\sqrt{x^2-4}\right)/2$	$\left(x-\sqrt{x^2-4}\right)/2$	$\left((2u_{12}-x)v_{12}^n-(2v_{12}-x)u_{12}^n\right)/(u_{12}-v_{12})$
13	$t_n(x)$	$x+\sqrt{x^2-1}$	$x-\sqrt{x^2-1}$	$(u_{13}^n-v_{13}^n)/(u_{13}-v_{13})$
14	$s_n(x)$	$x+\sqrt{x^2-1}$	$x-\sqrt{x^2-1}$	$(2(u_{14}-x)v_{14}^n-2(v_{14}-x)u_{14}^n)/(u_{14}-v_{14})$
15	$T_n(x)$	$x+\sqrt{x^2-1}$	$x-\sqrt{x^2-1}$	$\left((u_{15}-x)v_{15}^n-(v_{15}-x)u_{15}^n\right)/(u_{15}-v_{15})$
16	$U_n(x)$	$x+\sqrt{x^2-1}$	$x-\sqrt{x^2-1}$	$\left((u_{16}-2x)v_{16}^n-(v_{16}-2x)u_{16}^n\right)/(u_{16}-v_{16})$
17	P_n	$-1-\sqrt{2}$	$-1+\sqrt{2}$	$(-1)^{n-1}(u_{17}^n-v_{17}^n)/(u_{17}-v_{17})$
18	Q_n	$-1-\sqrt{2}$	$-1+\sqrt{2}$	$2(-1)^n\left((u_{18}+1)v_{18}^n-(v_{18}+1)u_{18}^n\right)/(u_{18}-v_{18})$
19	M_n	1	1/2	2^n-1
20	GF_n	$(-1-\sqrt{5})/2$	$(-1+\sqrt{5})/2$	$(-1)^n\left((iu_{20}+1)v_{20}^n-(iv_{20}+1)u_{20}^n\right)/(u_{20}-v_{20})$
21	GL_n	$(-1-\sqrt{5})/2$	$(-1+\sqrt{5})/2$	A
22	GJ_n	-1	1/2	$(-2)^{n-1}\left((iu_{22}+1)v_{22}^n-(iv_{22}+1)u_{22}^n\right)/(u_{22}-v_{22})$
23	Gj_n	-1	1/2	B
24	GP_n	$-1-\sqrt{2}$	$-1+\sqrt{2}$	$(-1)^n\left((iu_{24}+1)v_{24}^n-(iv_{24}+1)u_{24}^n\right)/(u_{24}-v_{24})$
25	GQ_n	$-1-\sqrt{2}$	$-1+\sqrt{2}$	C

Table 4: Exponential generating functions

$G_n(x,y)$	Ordinary generating function
$P_n(x,y)$	$\frac{1}{y(u_1-v_1)}(e^{-yv_1z}-e^{-yu_1z})$
$Q_n(x,y)$	$\frac{2}{u_2-v_2}\left((u_2+x)e^{-yv_2z}-(v_2+x)e^{-yu_2z}\right)$
$J_n(x,y)$	$\frac{1}{2y(u_3-v_3)}(e^{-2yvz}-e^{-2yuz})$
$j_n(x,y)$	$\frac{1}{u_4-v_3}\left((4u_4+x)e^{-2yv_4z}-(4v_4+x)e^{-2yu_4z}\right)$
$F_n(x,y)$	$\frac{1}{y(u_4-v_4)}(e^{-yv_5z}-e^{-yu_5z})$
$L_n(x,y)$	$\frac{1}{y(u_6-v_6)}\left((2yu_6+x)e^{-v_6z}-(2yv_6+x)e^{-yu_6z}\right)$
$GF_n(x)$	$\frac{i}{u_7-v_7}(u_7e^{-v_7z}-v_7e^{-u_7z})$
$GL_n(x)$	$\frac{1}{u_8-v_8}\left(\left((2-ix)u_8+x+2i\right)e^{-v_8z}-\left((2-ix)v_8+x+2i\right)e^{-u_8z}\right)-2-ix$
$P_n(x)$	$\frac{1}{u_9-v_9}(e^{-v_9z}-e^{-u_9z})$
$Q_n(x)$	$\frac{2}{u_{10}-v_{10}}\left((u_{10}+x)e^{-v_{10}z}-(v_{10}+x)e^{-u_{10}z}\right)$
$V_n(x)$	$\frac{1}{u_{11}-v_{11}}(e^{u_{11}z}-e^{v_{11}z})$
$v_n(x)$	$\frac{1}{u_{12}-v_{12}}\left((2u_{12}-x)e^{v_{12}z}-(2v_{12}-x)e^{u_{12}z}\right)$
$t_n(x)$	$\frac{1}{u_{13}-v_{13}}(e^{u_{13}z}-e^{v_{13}z})$
$s_n(x)$	$\frac{2}{u_{14}-v_{14}}\left((u_{14}-x)e^{v_{14}z}-(v_{14}-x)e^{u_{14}z}\right)$
$T_n(x)$	$\frac{1}{u_{15}-v_{15}}\left((u_{15}-x)e^{v_{15}z}-(v_{15}-x)e^{u_{15}z}\right)$
$U_n(x)$	$\frac{1}{u_{16}-v_{16}}\left((u_{16}-2x)e^{v_{16}z}-(v_{16}-2x)e^{u_{16}z}\right)$
P_n	$\frac{e^{-v_{17}z}-e^{-u_{17}z}}{u_{17}-v_{17}}$
Q_n	$\frac{2}{u_{18}-v_{18}}\left((u_{18}+1)e^{-v_{18}z}-(v_{18}+1)e^{-u_{18}}\right)$
M_n	$e^{2z}-e^z$
GF_n	$\frac{1}{u_{20}-v_{20}}\left((iu_{20}+1)e^{-v_{20}z}-(iv_{20}+1)e^{-u_{20}z}\right)$
GL_n	$\frac{1}{u_{21}-v_{21}}\left(\left((2-i)u_{21}+1+2i\right)e^{-v_{21}z}-\left((2-i)v_{21}+1+2i\right)e^{-u_{21}z}\right)$

GJ_n	$\frac{1}{2(u_{22}-v_{22})}\left((iu_{22}+1)e^{-2v_{22}z}-(iv_{22}+1)e^{-u_{22}z}\right)$
Gj_n	$\frac{1}{2(u_{23}-u_{23})}\left(\left((4-i)u_{23}+1+2i\right)e^{-2v_{23}z}-\left((4-i)v_{23}+1+2i\right)e^{-2uz}\right)$
GP_n	$\frac{1}{u_{24}-v_{24}}\left((iu_{24}+1)e^{-v_{24}z}-(iv_{24}+1)e^{-u_{24}z}\right)$
GQ_n	$\frac{2}{u_{25}-v_{25}}\left((1-i)u_{25}+1+i\right)e^{-v_{25}z}-\left((1-i)v_{25}+1+1\right)e^{-u_{25}z})$

Table 5: Ordinary generating function of the products

$G_n(x)G_n(y)$	Generating function
$V_n(x)V_n(y)$	$\frac{z-z^3}{1-xyz+(x^2+y^2-2)z^2-xyz^3+z^4}$
$v_n(x)v_n(y)$	$\frac{4-3xyz+2(x^2+y^2-2)z^2-xyz^3}{1-xyz+(x^2+y^2-2)z^2-xyz^3+z^4}$
$t_n(x)t_n(y)$	$\frac{z-z^3}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}$
$s_n(x)s_n(y)$	$\frac{4-12xyz+4(2x^2+2y^2-1)z^2-4xyz^3}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}$
$T_n(x)T_n(y)$	$\frac{1-3xyz+(2x^2+2y^2-1)z^2-xyz^3}{1-4xyz+(4x^2+4x^2-2)z^2-4xyz^3+z^4}$
$U_n(x)U_n(y)$	$\frac{1-z^2}{1-4xyz+(4x^2+4y^2-2)z^2-4xyz^3+z^4}$
$GF_nT_n(x)$	$\frac{i+(1-2i)xz-(2i-1-2ix^2)z^2+(i-1)xz^3}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}$
$GL_nU_n(x)$	$\frac{2-i+2(3i-1)xz+(3-4i)z^2}{1-2xz+(3-4x^2)z^2+2xz^3+z^4}$
$GJ_nT_n(x)$	$\frac{\frac{i}{2}+(1+i)xz-\left(1+\frac{i}{2}+2ix^2\right)z^2-(2+3i)xz^3}{1-2xz+(5-8x^2)z^2+4xz^3+4z^4}$
$GJ_nU_n(x)$	$\frac{\frac{i}{2}+(2+3i)xz-(1+i/2)z^2}{1-2xz+(5-8x^2)z^2+4xz^3+4z^4}$
$GP_nT_n(x)$	$\frac{i+(1-2i)xz+(2-i+2ix^2)z^2-xz^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$
$GQ_nT_n(x)$	$\frac{2-2i+(10i-6)xz+(6-14i-4(1-i)x^2)z^2+(2-6i)z^3}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$
$GQ_nU_n(x)$	$\frac{2-2i+(12i-4)xz+(6-14i)z^2}{1-4xz+(6-4x^2)z^2+4xz^3+z^4}$
$M_nU_n(x)$	$\frac{2xz-3z^2}{1-6xz+(5+8x)z^2-12xz^3+4z^4}$
$M_nT_n(x)$	$\frac{xz-3z^2+2xz^3}{1-6xz+(5+8x^2)z^2-12xz^3+4z^4}$
M_n^2	$\frac{z-4z^3}{1-9z+28z^2-362z^3+16z^4}$

The values of A, B and C in Table 3 are

$$A=(-1)^n\frac{((2-i)u_{21}+1+2i)v_{21}^n-((2-i)v_{21}+1+2i)u_{21}^n}{u_{21}-v_{21}},$$

$$B=(-2)^n\frac{((4-i)u_{23}+1+2i)v_{23}^n-((4-i)v_{23}+1+2i)u_{23}^n}{u_{23}-v_{23}}$$

and

$$C=2(-1)^n\frac{((1-i)u_{25}+1+i)v_{25}^n-((1-i)v_{25}+1+i)u_{25}^n}{u_{25}-v_{25}}.$$

Conclusion

The polynomials satisfying second-order linear recurrence relation and their products are widely studied in the literature. In this work we revisited the generating functions and associated Binet formulae. To end with the general rule to follow to determine generating function of a product of any tow polynomials. In the light of this method based on geometric series summation, we come back to most of the polynomials studied by Boussayoud *et al*. in their different works. We presented the obtained results in tables 1, 2, 3, 4 and 5.

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