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Formulae of special numbers and polynomials by algebraic method

Mouloud Goubi

Abstract
It is a new tradition that several authors defined and studied families of special numbers and polynomials. Our goal in this work is to give an extension of numbers and polynomials introduced and investigated by Simsek *et al*. We give explicit formulae, recurrence relations and other interesting identities. Our demarche uses algebraic method and the obtained results give new statements on the works of Simsek *et al*., Kim *et al*. and other works in the literature.

Keywords: Exponential partial bell polynomials, stirling numbers of second kind, special numbers and polynomials, cauchy product of generating functions

Introduction
Many authors are interested in special families of numbers and polynomials, which are in fact extensions of families already studied in the literature. Kim *et al*. introduced and investigated in [11] numbers $s_n(a;q)$ and polynomials $s_n(x;a,q)$ defined respectively by the generating functions

$$\left(\frac{\log a}{\log q}z+1\right)\frac{q(q-1)}{qa^z-1}=\sum_{n=0}^{\infty}s_n(a;q)\frac{z^n}{n!}, \tag{1}$$

and

$$a^{xz}\left(\frac{\log a}{\log q}z+1\right)\frac{q(q-1)}{qa^z-1}=\sum_{n=0}^{\infty}s_n(x;a,q)\frac{z^n}{n!} \tag{2}$$

Simsek *et al*. revisited in [14] this family to give numbers and polynomials of higher order, They gave the following generalizations

$$\left(\left(\frac{\log a}{\log q}z+1\right)\frac{q(q-1)}{qa^z-1}\right)^k=\sum_{n=0}^{\infty}s_n^{(k)}(a;q)\frac{z^n}{n!} \tag{3}$$

and

$$a^{xz}\left(\left(\frac{\log a}{\log q}z+1\right)\frac{q(q-1)}{qa^z-1}\right)^k=\sum_{n=0}^{\infty}s_n^{(k)}(x;a,q)\frac{z^n}{n!} \tag{4}$$

For k=1 we obtain the first family. In this work we introduce and investigate more general family of numbers and polynomials, which are defined by the generating functions (with $b\neq 0$ and $c=0$ if $q=1$ and $a=e$).

$$\left(\frac{bz+c}{qa^z-1}\right)^{\alpha}=\sum_{n=0}^{\infty}s_n^{(\alpha)}(a,b,c,q)\frac{z^n}{n!}, \tag{5}$$

and

$$a^{xz}\left(\frac{bz+c}{qa^z-1}\right)^{\alpha}=\sum_{n=0}^{\infty}s_n^{(\alpha)}(x;a,b,c,q)\frac{z^n}{n!} \tag{6}$$

We give the explicit formulae by means of Stirling numbers of second kind, which are special case of the exponential partial Bell polynomials.

The obtained results are used to compute explicit formula of numbers $s_n^{(k)}(a;q)$, polynomials $s_n^{(k)}(x;a,q)$ and other special numbers and polynomials in the literature. We reserve the notations α to designate a complex number and k to designate a positive integer. In addition to numbers and polynomials investigated by Simsek *et al*. and Kim *et al*., this family is a generalization of numerous families of numbers and polynomials. We give some examples in the Table 1.

Table 1: special cases of $s_n^{(\alpha)}(a,b,c,q)$ and $s_n^{(\alpha)}(x;a,b,c,q)$

$s_n^{(\alpha)}(a,b,c,q)$	$s_n^{(\alpha)}(x;a,b,c,q)$	a	b	c	q	α
B_n [12]	$B_n(x)$	e	1	0	1	1
$B_n^{(k)}(\gamma)$ [10, 12]	$B_n^{(k)}(x;\gamma),\gamma\neq 1$	e	1	0	γ	k
$s_n(a;q)$ [11]	$s_n(x;a,q)$	a	$\frac{q(q-1)\log a}{\log q}$	$q(q-1)$	q	1
$s_n^{(k)}(a;q)$ [14]	$s_n^{(k)}(x;a,q)$	a	$\frac{q(q-1)\log a}{\log q}$	$q(q-1)$	q	k
$\mathcal{G}_n^{(k)}(\gamma)$ [10, 13]	$\mathcal{G}_n^{(k)}(x;\gamma),\gamma\neq 1$	e	-2	0	γ	k
$k!\,y_1(n,k;\gamma),\gamma\neq 1$ [15]	$k!\,y_1(x;n,k;\gamma)$	e	0	-1	γ	$-\alpha$
$\mathcal{E}_n^{(\alpha)}(\gamma)$ [18]	$\mathcal{E}_n^{(\alpha)}(x;\gamma),\gamma\neq 1$	e	0	-2	γ	α
$H_n^{(k)}(u)$ [16]	$H_n^{(k)}(u,x)$	e	0	$\frac{1-u}{u}$	$\frac{1}{u}$	k
$\mathfrak{B}_n(\lambda)$ [17]	$\mathfrak{B}_n(\lambda;x)$	e	1	$\log\lambda$	λ	1

To better understand Table 1 we give Table 2 concerning generating functions of these numbers and polynomials.

Table 2: Generating functions of numbers and polynomials in Table 1

$s_n^{(\alpha)}(a,b,c,q)$	**Generating function**	$s_n^{(\alpha)}(x;a,b,c,q)$	**Generating function**
B_n	$\frac{z}{e^z-1}$	$B_n(x)$	$\frac{ze^{xz}}{e^z-1}$
$B_n^{(k)}(\gamma)$	$\left(\frac{z}{\gamma e^z-1}\right)^k$	$B_n^{(k)}(x;\gamma)$	$\left(\frac{z}{\gamma e^z-1}\right)^k e^{xz}$
$\mathcal{G}_n^{(k)}(\gamma)$	$\left(\frac{2z}{\gamma e^z+1}\right)^k$	$\mathcal{G}_n^{(k)}(x;\gamma)$	$\left(\frac{2z}{\gamma e^z+1}\right)^k e^{xz}$
$y_1(n,k;\gamma)$	$\frac{1}{k!}(\gamma e^z+1)^k$	$y_1(x;n,k;\gamma)$	$\frac{1}{k!}(\gamma e^z+1)^k e^{xz}$
$\mathcal{E}_n^{(\alpha)}(\gamma)$	$\left(\frac{2}{\gamma e^z+1}\right)^\alpha$	$\mathcal{E}_n^{(\alpha)}(x;\gamma)$	$\left(\frac{2}{\gamma e^z+1}\right)^\alpha e^{xz}$
$H_n^{(k)}(u)$	$\left(\frac{1-u}{e^z-u}\right)^k$	$H_n^{(k)}(x;u)$	$\left(\frac{1-u}{e^z-u}\right)^k e^{xz}$
$\mathfrak{B}_n(\lambda)$	$\frac{z+\log\lambda}{\lambda e^z-1}$	$\mathfrak{B}_n(x;\lambda)$	$\frac{z+\log\lambda}{\lambda e^z-1}e^{xz}$

Exponential partial bell polynomials and stiriling numbers of second kind

In our algebraic approach we use exponential partial Bell polynomials firstly introduced by Bell [1], for that we give this overview on these polynomials. The exponential partial Bell polynomials $B_{n,k}(x_1,x_2,\dots,x_n)$ are defined by the combinatorial formula [2]:

$$B_{n,k}(x_1,x_2,\dots,x_n)=\frac{n!}{k!}\sum_{\substack{k_1+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}}\binom{k}{k_1,\dots k_{n-k+1}}\prod_{r=1}^{n-k+1}\left(\frac{x_r}{r!}\right)^{k_r} \tag{7}$$

where $\binom{k}{k_1,\dots,k_n}=\frac{k!}{k_1!\dots k_n!}$ is the multinomial coefficient. Their generating function is

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{z^m}{m!}\right)^k=\sum_{n=k}^{\infty}B_{n,k}\left(x_1,x_2,\dots,x_n\right)\frac{z^n}{n!}. \tag{8}$$

Numerous works are down on Bell polynomials, especially to evaluate them for certain particular sequences $(x_n)_{n\in\mathbb{N}}$ as in [9]. In [3-7] we explained how to use them to evaluate numbers and polynomials arising from Discrete Mathematics and Number Theory. It is well known that the Stirling numbers of second kind $S_n(n,k)$ are related to exponential partial Bell polynomials by the following relation

$$B_{n,k}(1,1,\dots,1)=S(n,k). \tag{9}$$

We recall that Stirling numbers of second kind are defined by the generating function

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{z^m}{m!}\right)^k=\sum_{n=k}^{\infty}S\left(n,k\right)\frac{z^n}{n!}, \tag{10}$$

for which the explicit formula is

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n. \tag{11}$$

Among others, we meet exponential partial Bell polynomials in the formal calculus to compute a power of a given generating function. We announce this result as follows:

$$\left(\sum_{n=0}^{\infty} b_n\frac{z^n}{n!}\right)^{\alpha}=b_0^{\alpha}+\sum_{n=1}^{\infty}\sum_{j=1}^{n}(\alpha)_j b_0^{\alpha-j}B_{n,j}(b_1,\dots,b_n)\frac{z^n}{n!},\ b_0\neq 0. \tag{12}$$

For the proof see [8]. This result is important it is with it that we compute the explicit formula of numbers $s_n^{(\alpha)}(a,b,c,q)$ and other examples in the literature.

Explicit Formula of $s_n^{(\alpha)}(x;a,b,c,q)$

Let $f(z)$ be the generating function of the numbers $s_n^{(\alpha)}(a,b,c,q)$. Then $f(z)=\left(\frac{bz+c}{qa^z-1}\right)^{\alpha}$. We take $g(z)=bz+c$ and $h(z)=qa^z-1$ to get $f(z)=g^{\alpha}(z)h^{-\alpha}(z)$. Thus the algebraic structure of f is made up of product of two power functions, which lets the Cauchy product and Exponential partial Bell polynomials intervene. The Cauchy product of two exponential generating functions $\sum_{n=0}^{\infty}a_n\frac{z^n}{n!}$ and $\sum_{n=0}^{\infty}b_n\frac{z^n}{n!}$ is given by the following relation

$$\left(\sum_{n=0}^{\infty}a_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_n\frac{z^n}{n!}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\frac{z^n}{n!}. \tag{13}$$

It is easy to notice that the series expansions of $g^{\alpha}(z)$ and $h(z)$ are

$$g^{\alpha}(z)=\sum_{n=0}^{\infty}(\alpha)_n b^n c^{\alpha-n}\frac{z^n}{n!} \text{ and } h(z)=q-1+q\sum_{n=1}^{\infty}(\log a)^n\frac{z^n}{n!},$$

where $(\alpha)_n$ is a falling number, defined by the expression

$$(\alpha)_n=\alpha(\alpha-1)\dots(\alpha-n+1).$$

First we evaluate the special case $s_n^{(\alpha)}(a,0,1,q)$ generated by the function $h^{-\alpha}(z)$:

$$h^{-\alpha}(z)=\left(\frac{1}{qa^z-1}\right)^{\alpha}=\sum_{n=0}^{\infty}s_n^{(\alpha)}(a,0,1,q)\frac{z^n}{n!}.$$

Proposition 3.1. For $q\neq 1$ we have $s_0^{(\alpha)}(a,0,1,q)=(q-1)^{-\alpha}$ and

$$s_n^{(\alpha)}(a,0,1,q)=\sum_{j=1}^{n}(-\alpha)_j(q-1)^{-\alpha-j}q^j(\log a)^n S(n,j) \tag{14}$$

Thereafter

$$s_n^{(\alpha)}(e,0,1,q)=\sum_{j=1}^{n}(-\alpha)_j(q-1)^{-\alpha-j}q^j S(n,j). \tag{15}$$

Proof. In one hand we have $h^{-\alpha}(z)=\sum_{n=0}^{\infty}s_n^{(\alpha)}(a,0,1,q)\frac{z^n}{n!}$, in another hand we have

$$h^{-\alpha}(z)=(q-1)^{-\alpha}+\sum_{n=1}^{\infty}\sum_{j=1}^{n}(-\alpha)_j(q-1)^{-\alpha-j}B_{n,j}(q\log a,\dots,q(\log a)^n)\frac{z^n}{n!},$$

which follows from identity (12). Since

$$B_{n,j}(q\log a,\dots,q(\log a)^n)=q^j(\log a)^n B_{n,j}(1,\dots,1)=q^j(\log a)^n S(n,j),$$

then

$$h^{-\alpha}(z)=(q-1)^{-\alpha}+\sum_{n=1}^{\infty}\sum_{j=1}^{n}(-\alpha)_j(q-1)^{-\alpha-j}q^j(\log a)^n S(n,j)\frac{z^n}{n!}.$$

The comparison between the two expressions of $h^{-\alpha}(z)$ leads to the desired result.□

In the case $a=e$ and $q=1$, $h^{-\alpha}(z)$ is not a generating function because $h(0)=0$. Thus if $b\neq 0$ and $c=0$, $f(z)$ is a generating function. And we have $f(z)=\sum_{n=0}^{\alpha}s_n^{(\alpha)}(e,b,0,1)\frac{z^n}{n!}$.

In this case we have $f(z)=b^{\alpha}\left(\frac{z}{e^z-1}\right)^{\alpha}=b^{\alpha}\left(\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{z^n}{n!}\right)^{-\alpha}$, According to identity (12) we get $s_n^{(\alpha)}(e,b,0,1)=b^{\alpha}$ and

$s_n^{(\alpha)}(e,b,0,1)=b^{\alpha}\sum_{j=1}^{n}(-\alpha)_j B_{n,j}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n+1}\right)$

We quote from [9] the identity

$$B_{n,j}\left(\frac{1}{2},\dots,\frac{1}{l+1}\right)=\frac{n!}{(n+j)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{n+j}{j-k}S(n+k,k).$$

Thereafter

$$s_n^{(\alpha)}(e,b,0,1)=b^{\alpha}\sum_{j=1}^{n}(-\alpha)_j\frac{n!}{(n+j)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{n+j}{j-k}S(n+k,k).$$

Consequently

$$s_n(e,1,0,1)=\sum_{j=1}^{n}\frac{n!j!}{(n+j)!}\sum_{k=0}^{j}(-1)^{k}\binom{n+j}{j-k}S(n+k,k).$$

It is obvious to remark that $s_n(e,1,0,1)$ is the Bernoulli number B_n. Thus this method is another way to prove the identity (6) Theorem 1 of [9]:

$$B_n=\sum_{l=0}^{n}(-1)^{l}\binom{n+1}{l+1}\binom{n+l}{l}^{-1}S(n+l,l).$$

Certainly the proof of Guo and Qi uses exponential partial Bell polynomials, but it is analytic. It involves the derivative and the integral at the same time, which made it a bit long compared to our algebraic method.
According to numbers $s_n^{(\alpha)}(a,0,1,q)$, the expressions of $s_n^{(\alpha)}(a,b,c,q)$ and $s_n^{(\alpha)}(x;a,b,c,q)$ are established in the following theorem.

Theorem 3.1. For $q\neq 1$ we have $s_0^{(\alpha)}(a,b,c,q)=\left(\frac{c}{q-1}\right)^{\alpha}$,

$$s_n^{(\alpha)}(a,b,c,q)=\sum_{l=0}^{n}\binom{n}{l}(\alpha)_l b^l c^{\alpha-l}s_{n-l}^{(\alpha)}(a,0,1,q), \tag{16}$$

and

$$s_n^{(\alpha)}(x;a,b,c,q)=\sum_{l=0}^{n}\binom{n}{l}(\log a)^l s_{n-l}^{(\alpha)}(a,b,c,q)\,x^l, \tag{17}$$

Proof
The Cauchy product of the exponential generating functions $g^{\alpha}(z)$ and $h^{-\alpha}(z)$ makes it possible to obtain the result (16). We apply again the Cauchy product on the two functions $a^{xz}=e^{xz\log a}=\sum_{n=0}^{\alpha}(x\log a)^n\frac{z^n}{n!}$ and $f(z)$ to get the expression (17). □
According to Proposition 3.1 and Theorem 3.1 the explicit formula of $s_n^{(\alpha)}(a,b,c,q)$ is
$s_n^{(\alpha)}(a,b,c,q)=\sum_{l=0}^{n}\binom{n}{l}(\alpha)_l b^l c^{\alpha-l}\sum_{j=1}^{n-l}(-\alpha)_j(q-1)^{-\alpha-j}q^j(\log a)^{n-l}S(n-l,j)$.
For $\alpha=1$, the numerator is the polynomial $g(z)$ of degree one, the identities (16) and (17) become as in the following corollary.

Corollary 3.1. Letting $s_n(a,b,c,q)=s_n^{(1)}(a,b,c,q)$ and $s_n(x;a,b,c,q)=s_n^{(1)}(x;a,b,c,q)$. then we have

$$s_n(a,b,c,q)=cs_n(a,0,1,q)+bns_{n-1}(a,0,1,q) \tag{18}$$

And

$$s_n(x;a,b,c,q)=\sum_{l=0}^{n}\binom{n}{l}(\log a)^l s_n(a,b,c,q)\,x^l \tag{19}$$

This completes the works of Kim *et al.* [11] and Simsek *et al.* [14] by finding simple explicit formulas of the concerned numbers and polynomials. In the case $a=e$ Simsek *et al.* [14] (Theorem 9) showed the following result

$$s_n^{(k)}(e;q)=q^k(q-1)^k\sum_{j=0}^{k}\binom{k}{j}\frac{1}{(\log q)^j}\frac{1}{(k-1)^j}\times\sum_{m=0}^{n}\binom{n}{m}\binom{n+k-j}{k-j}^{-1}B_n^{(j)}(q)\,B_{n-m}^{(k-j)}(q), \tag{20}$$

which can be written under the form

$$s_n^{(k)}(e;q)=q^k(q-1)^k\sum_{j=0}^{k}\binom{k}{j}\frac{1}{(\log q)^j}\frac{1}{(k-1)^j}\times\sum_{m=0}^{n}\binom{n}{m}\binom{n+k-j}{k-j}^{-1}s_n^{(j)}(e,1,0,q)\,s_{n-m}^{(k-j)}(e,1,0,q).$$

But in general case we have the following theorem

Theorem 3.2. For $n \geq 1$ we have

$$s_n(a;q) = q(q-1)s_n(a,0,1,q) + \frac{q(q-1)\log a}{\log q} n s_{n-1}(a,0,1,q) \tag{21}$$

and

$$s_n^{(k)}(a;q) = \sum_{l=0}^{n} \binom{n}{l}\binom{k}{l} l! \left(\frac{\log a}{\log q}\right)^l \left(q(q-1)\right)^k s_{n-l}^{(k)}(a,0,1,q)\ , \tag{22}$$

Proof. We take $\alpha = 1$, $b = q(q-1)\frac{\log a}{\log q}$ and $c = q(q-1)$ to obtain the first identity and for $\alpha = k$ we get the second identity.

If $a = e$, we have $s_n^{(k)}(e;q) = \sum_{l=0}^{n} \binom{n}{l}(k)_l \left(\frac{1}{\log q}\right)^l \left(q(q-1)\right)^k s_{n-l}^{(k)}(e,0,1,q)$, and then

$$s_n^{(k)}(e;q) = q^k \sum_{l=0}^{n-1} \binom{n}{l}\binom{k}{l} l! \left(\frac{1}{\log q}\right)^l \sum_{j=1}^{l} (-k)_j \left(\frac{q}{q-1}\right)^j S(l,j)\ , \tag{23}$$

which is an improvement of the identity (19). The corresponding polynomials are of the form

$s_n(x;a,q) = \sum_{l=0}^{n} \binom{n}{l} (\log a)^l s_{n-l}(a;q)\, x^l$ (24)

and

$s_n^{(k)}(x;a,q) = \sum_{l=0}^{n} \binom{n}{l} (\log a)^l s_{n-l}^{(k)}(a;q)\, x^l$. (25)

In addition to Theorem 3.2, the explicit formulas of other numbers in Table 1 are given in the Table 3.

Table 3: Explicit formulas of numbers in Table 1

$s_n^{(\alpha)}(a,b,c,q)$	**Explicit formula**
B_n	$s_n(e,1,0,1)$
$B_n^{(k)}(\gamma)$	$(n)_k s_{n-k}^{(k)}(e,0,1,\gamma)$ and 0 if $n < k$
$\mathcal{G}_n^{(k)}(\gamma)$	$(n)_k s_{n-k}^{(k)}(e,0,1,-\gamma)$ and 0 if $n < k$
$k!\,y_1(n,k;\gamma)$	$(-1)^k s_n^{(k)}(e,0,1,-\gamma)$
$\mathcal{E}_n^{(k)}(\gamma)$	$(-2)^k s_n^{(k)}(e,0,1,-\gamma)$
$H_n^{(k)}(u)$	$\left(\frac{1-u}{u}\right)^k s_n^{(k)}\left(e,0,1,\frac{1}{u}\right)$
$\mathfrak{B}_n(\lambda)$	$(\log\lambda)s_n(e,0,1,\lambda) + n s_{n-1}(e,0,1,\lambda)$

For example from Table 2 we obtain the identity $\mathcal{E}_n^{(k)}(\gamma) = 2^k k!\, y_1(n,k;\gamma)$ which correspond to identity (1.12) given in [10]. page 2454.

Recurrence Relations

The algebraic manipulation of the function $f(z)$ allows to obtain interesting recurrence relations on the numbers $s_n^{(\alpha)}(a,b,c,q)$.

Theorem 4.1. The following two recurrence relations hold true.

$$(\alpha)_n b^n c^{\alpha-n} = \sum_{l=0}^{n} \binom{n}{l} s_{n-l}^{(-\alpha)}(a,0,1,q) s_l^{(\alpha)}(a,b,c,q), \tag{26}$$

and

$$s_n^{(\alpha)}(a,0,1,q) = \sum_{l=0}^{n} \binom{n}{l} (-\alpha)_{n-l} b^{n-l} c^{-\alpha-n+l} s_l^{(\alpha)}(a,b,c,q), \tag{27}$$

Proof.
In one hand we have $(qa^z-1)^\alpha f(z) = (bz+c)^\alpha$, since

$$(qa^z-1)^\alpha = \left(\frac{1}{qa^z-1}\right)^{-\alpha} = \sum_{n=0}^{\infty} s_n^{(-\alpha)}(a,0,1,q)\frac{z^n}{n!}\ .$$

Then the Cauchy product of $f(z)$ and $(qa^z-1)^\alpha$ conducts to

$(qa^z-1)^\alpha f(z)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} s_{n-l}^{(-\alpha)}(a,0,1,q)s_l^{(\alpha)}(a,b,c,q)\frac{z^n}{n!}$,

and the identity (26) follows. In another hand we have $(bz+c)^{-\alpha}f(z)=(qa^z-1)^{-\alpha}$. By Cauchy product of the generating functions $(bz+c)^{-\alpha}$ and $f(z)$ we get

$\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(-\alpha)_{n-l}b^{n-l}c^{-\alpha-n+l}s_l^{(\alpha)}(a,b,c,q)\frac{z^n}{n!}=\sum_{n=0}^{\infty}s_n^{(\alpha)}(a,0,1,q)\frac{z^n}{n!}$,

and the identity (27) follows.

Theorem 4.1 conducts to the following iterations which compute successively the values of $s_n^{(\alpha)}(a,b,c,q)$.

$$(q-1)^\alpha s_n^{(\alpha)}(a,b,c,q)=(\alpha)_n b^n c^{\alpha-n}-\sum_{l=0}^{n-1}\binom{n}{l}s_{n-l}^{(-\alpha)}(a,0,1,q)s_l^{(\alpha)}(a,b,c,q) \quad (28)$$

and

$$c^{-\alpha}s_n^{(\alpha)}(a,b,c,q)=s_n^{(\alpha)}(a,0,1,q)-\sum_{l=0}^{n-1}\binom{n}{l}(-\alpha)_{n-l}b^{n-l}c^{-\alpha-n+l}s_l^{(\alpha)}(a,b,c,q). \quad (29)$$

In addition to relationships

$$s_n^{(v+k)}(a;q)=\sum_{j=0}^{n}\binom{n}{j}s_j^{(v)}(a;q)s_{n-j}^{(k)}(a;q), \quad (30)$$

and

$$s_n^{(2)}(a;q)=\sum_{j=0}^{n}\binom{n}{j}s_j(a;q)s_{n-j}(a;q). \quad (31)$$

given in [14], we obtain the following new identity which combine between $s_n^{(k)}(a;q)$ and $s_n^{(-k)}(a,0,1,q)$.

$$(q-1)^k s_n^{(k)}(a;q)=(k)_n\left(\frac{\log a}{\log q}\right)^n\big(q(q-1)\big)^k-\sum_{l=0}^{n-1}\binom{n}{l}s_{n-l}^{(-k)}(a,0,1,q)s_l^{(k)}(a;q), \quad (32)$$

and the identity

$$\big(q(q-1)\big)^{-k}s_n^{(k)}(a;q)=s_n^{(k)}(a,0,1,q)-\sum_{l=0}^{n-1}\binom{n}{l}(-k)_{n-l}\left(\frac{\log a}{\log q}\right)^{n-l}\big(q(q-1)\big)^{-k}s_l^{(k)}(a;q). \quad (33)$$

Theorem 4.2. For $n\geq 1$ we have

$$\left(\frac{q-1}{c}\right)^{\alpha} s_n^{(\alpha)}(a,b,c,q)=-\sum_{l=0}^{n-1}\binom{n}{l}G_{n-l}^{b,c}\,s_l^{(\alpha)}(a,b,c,q), \quad (34)$$

where

$$G_n^{b,c}=\sum_{j=0}^{n}\binom{n}{j}(-\alpha)_j b^j c^{-\alpha-j}s_{n-j}^{(-\alpha)}(a,0,1,q).$$

Proof. We have $\left(\frac{qa^z-1}{bz+c}\right)^\alpha f(z)=1$ and

$$\left(\frac{qa^z-1}{bz+c}\right)^\alpha=\left(\sum_{n=0}^{\infty}s_n^{(-\alpha)}(a,0,1,q)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-\alpha)_n b^n c^{-\alpha-n}\frac{z^n}{n!}\right)=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-\alpha)_j b^j c^{-\alpha-j}s_{n-j}^{(-\alpha)}(a,0,1,q)\frac{z^n}{n!}.$$

By the Cauchy product of $\left(\frac{qa^z-1}{bz+c}\right)^\alpha$ and $f(z)$ we get $1=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}G_{n-l}^{b,c}\,s_l^{(\alpha)}(a,b,c,q)\frac{z^n}{n!}$. Thus $\sum_{l=0}^{n}\binom{n}{l}G_{n-l}^{b,c}\,s_l^{(\alpha)}(a,b,c,q)=0$ for $n\geq 1$, and the desired result follows.

Further Identities

The analytic approach leads to other recurrence relations. Essentially we use partial derivation operator, firstly we have

$$\frac{\partial f(z)}{\partial z}=\alpha\left(\frac{bz+c}{qa^z-1}\right)^{\alpha-1}\frac{b(qa^z-1)-qa^z(bz+c)\log a}{(qa^z-1)^2},$$

by taking a count that $\frac{da^z}{dz} = \frac{de^{z\log a}}{dz} = a^z \log a$.

Letting the sequence $v_n^{a,b,c,q}$ to be the numbers defined by the exponential generating function $b(qa^z-1) - qa^z(bz+c)\log a$. Since we have

$$(bz+c)e^{z\log a} = c + \sum_{n=1}^{\infty}(c(\log a)^n + bn(\log a)^{n-1})\frac{z^n}{n!},$$

and

$$b(qa^z-1) = b(q-1) + bq\sum_{n=1}^{\infty}(\log a)^n \frac{z^n}{n!},$$

then

$$v_n^{a,b,c,q} = bq(1-n)(\log a)^n - qc(\log a)^{n+1}.$$

Thereafter we deduce that

$$\frac{b(qa^z-1)-qa^z(bz+c)\log a}{(qa^z-1)^2} = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} v_{n-l}^{a,b,c,q} s_l^{(2)}(a,0,1,q)\frac{z^n}{n!}.$$

Theorem 5.1. For $q \neq 1$ we have

$$s_{m+1}^{(\alpha)}(a,b,c,q) = \alpha\sum_{n=0}^{m}\binom{m}{n}\sum_{l=0}^{n}\binom{n}{l} v_{n-l}^{a,b,c,q} s_l^{(2)}(a,0,1,q)\, s_{m-n}^{(\alpha-1)}(a,b,c,q). \tag{35}$$

Proof. The identities

$$\left(\frac{bz+c}{qa^z-1}\right)^{\alpha-1} = \sum_{n=0}^{\infty} s_n^{(\alpha-1)}(a,b,c,q)\frac{z^n}{n!},\ \frac{\partial f(z)}{\partial z} = \sum_{n=0}^{\infty} s_{n+1}^{(\alpha)}(a,b,c,q)\frac{z^n}{n!}$$

and the form of $\frac{\partial f(z)}{\partial z}$ lead to the result (35). □

Secondly it is possible to consider a, b, c and $q \neq 1$ variables too. One can easily proof that

$$\frac{\partial}{\partial b}\left(\frac{bz+c}{qa^z-1}\right) = \frac{z}{qa^z-1},\ \frac{\partial}{\partial c}\left(\frac{bz+c}{qa^z-1}\right) = \frac{1}{qa^z-1}, \tag{36}$$

and

$$\frac{\partial}{\partial q}\left(\frac{bz+c}{qa^z-1}\right) = -\frac{a^z(bz+c)}{(qa^z-1)^2},\ \frac{\partial}{\partial a}\left(\frac{bz+c}{qa^z-1}\right) = -\frac{qz}{a}\frac{a^z(bz+c)}{(qa^z-1)^2}. \tag{37}$$

According to

$$\frac{z}{qa^z-1} = \sum_{n=0}^{\infty} s_n(a,1,0,q)\frac{z^n}{n!},$$

$$\frac{1}{qa^z-1} = \sum_{n=0}^{\infty} s_n(a,0,1,q)\frac{z^n}{n!},$$

$$\frac{a^z(bz+c)}{(qa^z-1)^2} = \frac{a^z}{qa^z-1}\sum_{n=0}^{\infty} s_n(a,b,c,q)\frac{z^n}{n!}$$

and

$$\frac{qz}{a}\frac{a^z(bz+c)}{(qa^z-1)^2} = \frac{a^z}{qa^z-1}\sum_{n=1}^{\infty} n s_{n-1}\left(a,\frac{qb}{a},\frac{qc}{a},q\right)\frac{z^n}{n!},$$

with

$$\frac{a^z}{qa^z-1} = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} s_l(a,0,1,q)(\log a)^{n-l}\frac{z^n}{n!}.$$

We traduce identities (36) and (37) to identities on numbers $s_m(a,b,c,q)$ by the following theorem.

Theorem 5.2. We have

$$\frac{\partial}{\partial a} s_m(a,b,c,q) = -\sum_{n=0}^{m-1}\binom{m}{n}\sum_{l=0}^{n}\binom{n}{l} s_l(a,0,1,q)(\log a)^{n-l}(m-n)s_{m-n-1}\left(a,\frac{q}{a},0,q\right) \tag{38}$$

$$\frac{\partial}{\partial b} s_m(a,b,c,q) = s_m(a,1,0,q) \tag{39}$$

$$\frac{\partial}{\partial b} s_m(a,b,c,q) = s_m(a,0,1,q) \tag{40}$$

$$\frac{\partial}{\partial q} s_m(a,b,c,q) = -\sum_{n=0}^{m}\binom{m}{n}\sum_{l=0}^{n}\binom{n}{l} s_l(a,0,1,q)(\log a)^{n-l}s_{m-n}(a,b,c,q). \tag{41}$$

These identities are compatible with the identities (18) and (19) of Corollary 3.1. Which lead us to deduce several relations, for example we have

$$c\frac{\partial}{\partial q} s_m(a,0,1,q) + bm\frac{\partial}{\partial q} s_{m-1}(a,0,1,q) = -\sum_{n=0}^{m}\binom{m}{n}\sum_{l=0}^{n}\binom{n}{l} s_l(a,0,1,q)(\log a)^{n-l}s_{m-n}(a,b,c,q).$$

Conclusion

In this work we have introduced a generalization of the special family of numbers and polynomials defined and studied by Simsek *et al*. By hint of exponential partial Bell polynomials and Cauchy product of generating functions, we initialized an algebraic method for computing the explicit formulae of these numbers and polynomials. The obtained results are applied to numerous numbers and polynomials studied in the literature.

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