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Mathematical analysis of non integer orders in comparison with classical

Nemat Mustafoev

Abstract
At present, fractional calculus is in the process of rapid development both in theoretical terms and in its applications. Composition rules for integro-differential operators play an important role in solving the corresponding differential equations and constructing mathematical and computer models of dynamical systems.

Keywords: Function, fractional derivative, differential, fractional integral, mathematical analysis

Introduction
Mathematical analysis using integro-differential operators of non-integer orders or fractional calculus (Fractional Calculus) has more than three centuries of history. The first mention of derivatives of non-integer order is contained in the correspondence between J. Bernoulli and G. Leibniz. The latter, in particular, in a letter to G. Lopital dated 1695, discussing the possibilities of differentials of the order of ½, prophetically stated that this is an apparent paradox, from which useful results one day will follow. In the 18th century, fractional calculus did not receive much attention. Only a few publications are known related to the names of Euler and Lagrange. The entire XIX and the first half of the XX century was a period of accumulation of results, and the formation of fractional calculus as an independent branch of mathematical analysis. There are known publications on this issue by famous mathematicians, mechanics and physicists: Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Heaviside, Hardy, Zygmund, Courant, and others. A great contribution to the development of mathematical analysis of non-integer orders was made by the famous Russian mathematician, President of the Moscow Mathematical society A.V. Letnikov. His doctoral dissertation and a series of works published in the Mathematical Collection are devoted to the theory of differentiation of fractional order, the historical development of this direction of mathematics, the application of the theory of fractional calculus to integral calculus and the solution of differential equations. The first publications of A.V. Letnikov on fractional calculus date back to 1868-1872 [2]. A new surge of interest of the scientific community to fractional calculus occurred after the publication of the book "Fractional calculus" (K.B. Oldham, J.Spanier) in 1974 [3]. In this book, the theory of fractional calculus is systematically presented, as well as the areas of its application. A number of scientific and technical conferences and seminars are devoted to fractional calculus, special journals are organized. There are thematic issues of various journals devoted to the applications of fractional calculus in various fields of science, technology, natural science [5]. In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann - Liouville integral was studied in [6], [7].
At present, fractional calculus is in the process of rapid development both in theoretical terms and in its applications. We can say that this section of mathematical analysis has turned into a tool for mathematical modeling of the most complex dynamic processes in ordinary and fractal environments, which makes it possible to solve on a new basis a variety of problems of analysis, synthesis, identification, diagnostics, and creation of new control systems.

Comparison of classical and fractional mathematical analysis
In fractional mathematical analysis, there are often functions that are a generalization of the widely known and used in classical mathematical analysis, in particular, the exponential function and factorial. Let's start the comparison by looking at these functions.

A) Euler's gamma function and related functions

The gamma function is defined as follows

$$\Gamma(x)=\begin{cases}\int_0^\infty e^{-t}t^{x-1}dt, & \operatorname{Re}(x)\\ \lim_{n\to\infty}\dfrac{n!\,n^{x-1}}{x(x+1)(x+2)\cdots(x+n-1)}, & (x-\text{every})\end{cases}$$

The arguments of the Gamma function can be any number (real and complex, integer and non-integer). The gamma function for positive integers $x=n$ is related to the factorial as follows:

$\Gamma(n)=(n-1)!,\ \ n>0-\text{integer}$.

In fig. 1 and 2 show the view of the module Gamma - functions of complex and Gamma - functions of real arguments.

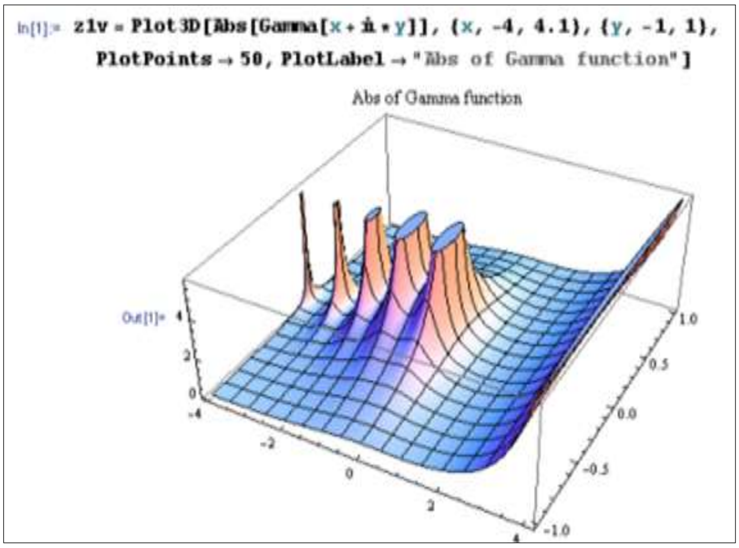


Fig 1: Gamma function module of complex argument

In fig. 2 that the Gamma function undergoes discontinuities of the form $\pm\infty$ at zero and for negative integer values of the argument.

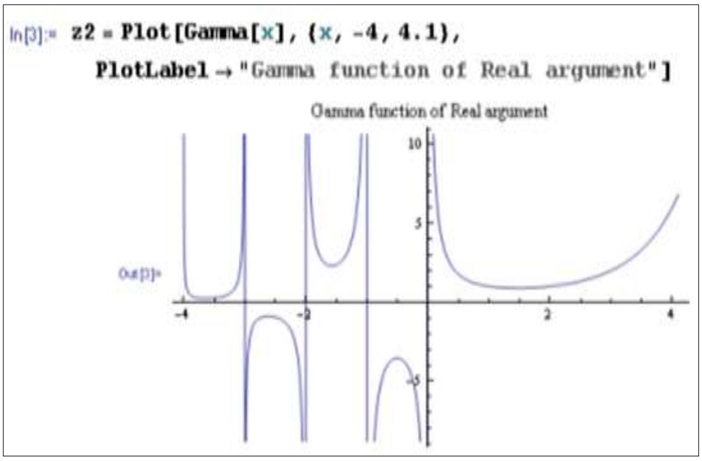


Fig 2: The gamma function of a real argument

However, in fractional analysis, the most common values are the Gamma function to the power of -1 or the Gamma ratio of functions of various arguments, which are continuous functions on the set of considered values of the arguments. For example, in Fig. 3 shows the form of the function $\dfrac{1}{\Gamma(x)}$, which is a continuous function.

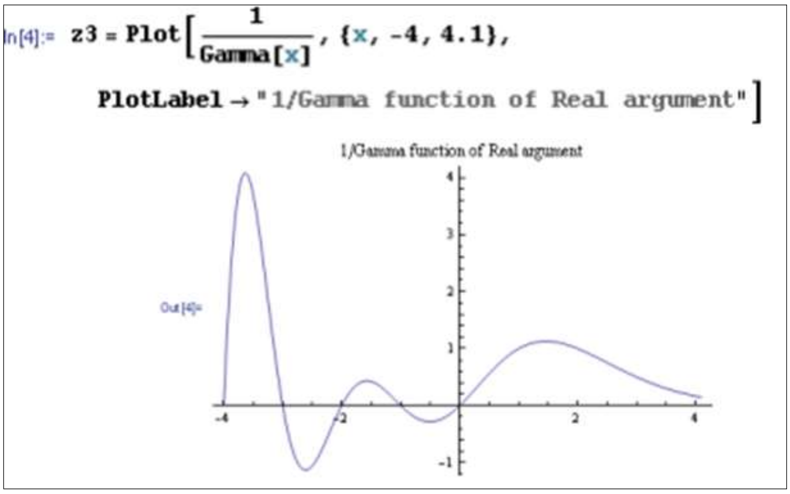


Fig 3: Graph of the function 1 / Γ (x)

Along with the considered Gamma function, several functions closely related to it find application. These include, in particular, the incomplete Gamma function and the Beta function.

B) An incomplete Gamma function is defined by the following expressions

$$\gamma(c,x)=\frac{c^{-x}}{\Gamma(x)}\int_0^x y^{x-1}\exp(-y)dy=\exp(-x)\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)}.$$

The visualization program for the incomplete Gamma function and its appearance are shown below.

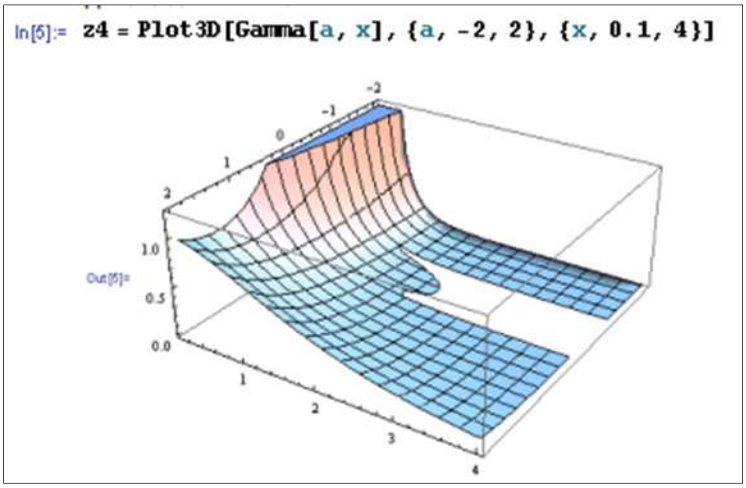


Fig 4: Incomplete Gamma is a function of the real arguments *a* and *x*

C) Beta function is expressed in terms of Gamma functions as follows: $\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Beta renderer - functions and its graphical image in the range of $(-2\le p\le 2,\ -2\le q\le 2)$ arguments are shown below.

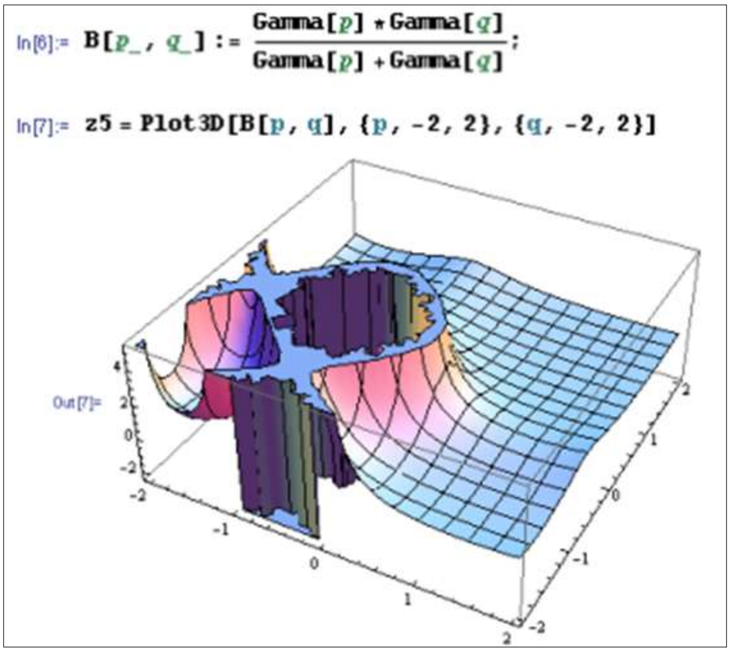


Fig 5: Beta function of real arguments *p* and *q*

Differentiation with non-integer order of some elementary functions

Power functions. Let us write for the power function $x(t)=t^k$ the well-known differentiation formulas with orders 1, 2,…, *n*:

$$\begin{aligned} &\frac{dx(t)}{dt}=kt^{k-1}, \\ &\frac{d^2x(t)}{dt^2}=k(k-1)t^{k-2}, \\ &\dots \\ &\frac{d^nx(t)}{dt^n}=k(k-1)(k-2)\dots(k-n+1)t^{k-n}=\frac{k!}{(k-n)!}t^{k-n}. \end{aligned} \tag{1}$$

Analysis of the formula for the derivative of order n (the last line of expressions (1)) shows that there are no obstacles for the order of differentiation to be different from an integer. To do this, you need to use the Gamma function, which generalizes the factorial function for the case of non integer arguments. Replacing the integer order of differentiation n by a fractional β and introducing the Gamma function, we obtain the following formula for the differentiation of a power function with fractional order:

$$\frac{d^\beta x(t)}{dt^\beta}=\frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}t^{k-\beta}. \tag{2}$$

Expression (2) can be considered as a function of three arguments: t, k and β and, thus, gives a more detailed description of the function and all its derivatives and integrals (both integer and fractional orders).

As an illustration, we give graphical image of expression (2) for the case $k=0$, when the order of the differential operator varies from -1 to +1 (Fig. 9).

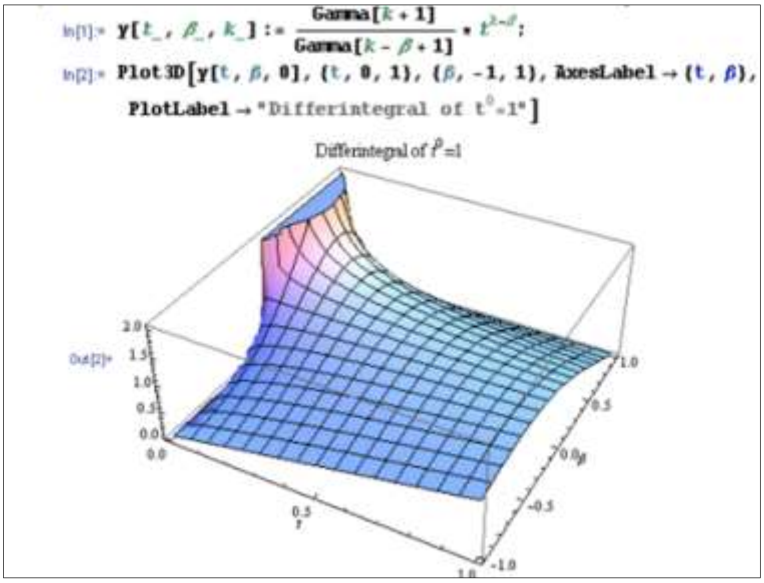


Fig 6: Differential integrals of constants 1 of orders β $(-1<\beta<1)$

Exponential functions: Let the exponential function $y(t)=e^{kt}$ be given. Differentiating it with orders 1, 2,…, *n*, we get:

$$\frac{dy(t)}{dt}=ke^{kt},\ \frac{d^2y(t)}{dt^2}=k^2e^{kt},\ldots,\frac{d^ny(t)}{dt^n}=k^ne^{kt}. \tag{3}$$

The expression for the derivative of order n is generalized to fractional orders by simply replacing n with β:

$$\frac{d^\beta y(t)}{dt^\beta}=k^\beta e^{kt}. \tag{4}$$

It should be noted that the diff-integrals of fractional orders of an exponential function with negative exponents are functions of a complex variable; the real and imaginary parts of such diff-integrals are shown in Fig. 7-8.

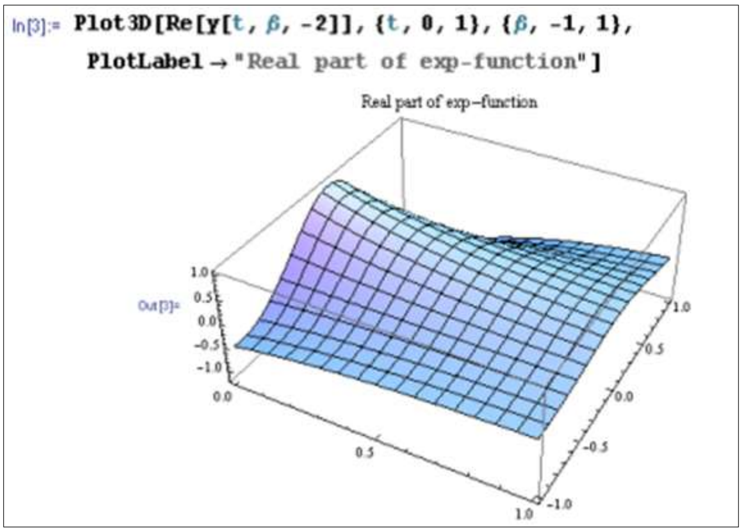


Fig 7: The real part of the differential integral of the exponential function $\left(\mathrm{Re}\left((-2)^k e^{-2t}\right)\right)$.

```
In[4]:= Plot3D[Im[y[t, β, -2]], {t, 0, 1}, {β, -1, 1},
         PlotLabel → "Imaginery part of exp-function"]
```

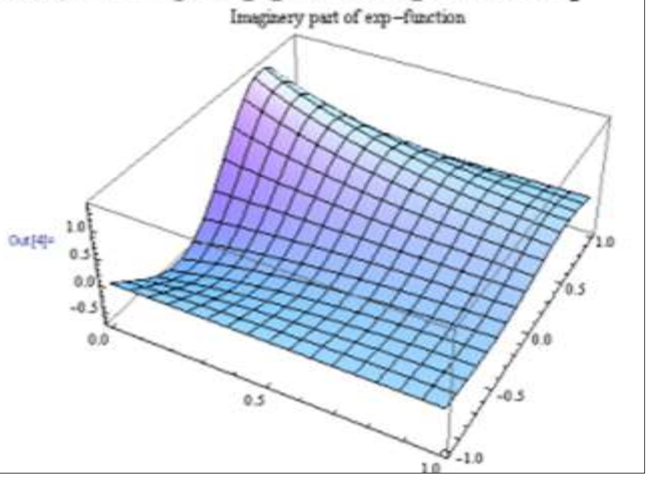
Out[4]=


Fig 8: The imaginary part of the differential integral of the exponential function $\left(\mathrm{Im}\left((-2)^k e^{-2t}\right)\right)$

Integral representations of differential integrals of non-integer orders

Another widespread definition of derivatives and integrals of non-integer orders is the Riemann-Liouville definition [3, 4], which is a generalization to non-integer orders of the Cauchy integral formula known from classical mathematical analysis [1]:

$$\underbrace{\int_a^x \int_a^x \cdots \int_a^x \int_a^x}_{n} f(t) \underbrace{dt dt \ldots dt}_{n} = \frac{1}{(n-1)!} \int_a^x \frac{f(t)dt}{(x-t)^{n-1}} . \tag{5}$$

As you can see, with the introduction of derivatives and integrals of non-integer orders, the sharp boundary between derivatives and integrals is erased: integrals can be interpreted as derivatives of negative order, and derivatives as integrals of negative order. A new term has appeared in the mathematical analysis of non-integer orders: the differential integral [4]. A generalization of the Cauchy integral formula to noninteger orders of integro-differential operators leads to the following definitions of diff-integrals of fractional (non integer) orders:

$$\begin{aligned} \left(I_a^\alpha \varphi\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} , \\ \left(D_a^\alpha \varphi\right)(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^x \frac{\varphi(t)dt}{(x-t)^{\alpha-n+1}} , \end{aligned} \tag{6}$$

where $\alpha,\ a \in \mathrm{R},\ n-1 < \alpha < n$.

I_a^α - integral operator of order α ; D_a^α - differential operator of order α .

Let us show by examples the application of the Riemann-Liouville definition of differential integrals of non integer orders.

Definition of a power function: `In[1]:= f[t_, k_] := t^k;`

The definition of the Riemann-Liouville integral:

```
In[2]:= dβf[a_, n_, β_, k_, t_] := 1/Gamma[n - β] * D[∫_a^t f[τ, k]/(t - τ)^(β-n+1) dτ, {t, n}];
```

Finding the derivative of order 1.5 of a power function t^3:

```
In[3]:= y1 = dβf[0, 3, 1.5, 3, t]
Out[3]= 4.51352 t^(3/2)
```

Finding the derivative of order 0.5 of a power function t^0:

```
In[4]:= y2 = dβf[0, 1, 0.5, 0, t]
```

$$\text{Out[4]= } \frac{0.56419}{\sqrt{t}}$$

Finding the Integral of Order 1.5 of a power function t^3:

```
In[5]:= y3 = Simplify[dβf[0, 0, -1.5, 3, t]]
```

$$\text{Out[5]= } 0. + 0.114629\, t^{9/2}$$

Properties of differential integrals of non-integer orders

A. *Linearity*. Integro-differential operators of non-integer orders are linear operators, as are their integer counterparts. For operations of differentiation with integer order, as is known, the following formula is valid:

$$\frac{d^m}{dt^m}\left(\sum_{k=1}^{n} b_k f_k(t)\right) = \sum_{k=1}^{n} b_k \frac{d^m}{dt^m}\left(f_k(t)\right), \tag{7}$$

where m is an integer. For derivatives of fractional order, we have a similar relation (8), noting that α is fractional and D_a^{α} is the symbol of fractional derivative of order α:

$$D_a^{\alpha}\left(\sum_{k=1}^{n} b_k f_k(t)\right) = \sum_{k=1}^{n} b_k D_a^{\alpha}\left(f_k(t)\right), \tag{8}$$

The linearity of the integration operations is expressed by similar relations.
So, for multiple integrals of integer order m:

$$\underbrace{\int_a^x \int_a^x \cdots \int_a^x \int_a^x}_{n} \left(\sum_{k=1}^{n} b_k f_k(t)\right) \underbrace{dtdt\ldots dt}_{n} = \sum_{k=1}^{n} b_k \underbrace{\int_a^x \int_a^x \cdots \int_a^x \int_a^x}_{n} f_k(t) \underbrace{dtdt\ldots dt}_{n} . \tag{9}$$

For integrals of fractional order, formula (8) is in fact valid, taking into account that the order of differentiation is taken negative ($\alpha < 0$):

$$D_a^{-\alpha}\left(\sum_{k=1}^{n} b_k f_k(t)\right) = \sum_{k=1}^{n} b_k D_a^{-\alpha}\left(f_k(t)\right), \tag{10}$$

Formulas (7-10) combine two properties of linearity of the operators of differentiation and integration: the constant (b_k) can be taken outside the differentiation (integration) sign, and the derivative (integral) of the sum of functions is equal to the sum of derivatives (integrals) of functions of the same order.

B. Rules for differentiation (integration) of the product of two functions

In classical mathematical analysis, the rule for taking the first derivative of the product of two functions is well known:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{df(x)}{dx}\cdot g(x) + f(x)\cdot\frac{dg(x)}{dx} . \tag{11}$$

A generalization of this rule to arbitrary integer orders of differentiation is also well known as the Leibniz rule [3]:

$$\frac{d^n}{dx^n}\left(f(x)\cdot g(x)\right) = \sum_{i=0}^{n}\binom{n}{i}\frac{d^{n-i}f(x)}{dx^{n-i}}\cdot\frac{d^i g(x)}{dx^i} . \tag{12}$$

In passing to multiple integrals, the corresponding expression for the multiple integral of the product of two functions, obtained using the integration by parts formula, has the form [3]:

$$\frac{d^{-n}}{(d(x-a))^{-n}}\left(f(x)\cdot g(x)\right)=\sum_{i=0}^{n}\binom{-n}{i}\frac{d^{-n-i}f(x)}{(d(x-a))^{-n-i}}\cdot\frac{d^{i}g(x)}{(d(x-a))^{i}}. \quad (13)$$

In formula (13), the integral of multiplicity n is denoted as the derivative of the corresponding negative order, and the integration is carried out in the range from a to x.

Formulas (12) and (13) are generalized to arbitrary non-integer orders of integration. Since under $i>n$ $\binom{n}{i}\equiv 0$, the upper limit of the sum in formula (12) can be replaced by ∞. Thus, the differential integral of arbitrary order α is calculated by the formula:

$$\frac{d^{\beta}}{(d(x-a))^{\beta}}\left(f(x)\cdot g(x)\right)=\sum_{i=0}^{n}\binom{\beta}{i}\frac{d^{\beta-i}f(x)}{(d(x-a))^{\beta-i}}\cdot\frac{d^{i}g(x)}{(d(x-a))^{i}}. \quad (14)$$

C. Rules for the composition of integro-differential operations

The composition rules for integro-differential operations provide for the relations between $\frac{d^{\beta}}{dx^{\beta}}\left(\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right)$, $\frac{d^{\alpha}}{dx^{\alpha}}\left(\frac{d^{\beta}f(x)}{dx^{\beta}}\right)$ and $\frac{d^{\alpha+\beta}f(x)}{dx^{\alpha+\beta}}$ for various combinations of the orders of the operators α and β (integer and fractional, positive and negative).

The functions are understood to be capable of differentiation and integration of the respective orders. In addition, for fractional values of the order of the operator, it is also necessary to take into account the values of the lower limit of integration (previously denoted by *a*).

The classical mathematical analysis of integer orders of integro-differential operators gives the following rules for the composition of operators.

- Both orders are integers and one sign $(\alpha=\pm m,\ \beta=\pm n,\ m>0,\ n>0)$:

$$\begin{aligned}&\frac{d^{+m}}{dx^{+m}}\left(\frac{d^{+n}f(x)}{dx^{+n}}\right)=\frac{d^{+n}}{dx^{+n}}\left(\frac{d^{+m}f(x)}{dx^{+m}}\right)=\frac{d^{+n+m}f(x)}{dx^{+n+m}},\\&\frac{d^{-m}}{dx^{-m}}\left(\frac{d^{-n}f(x)}{dx^{-n}}\right)=\frac{d^{-n}}{dx^{-n}}\left(\frac{d^{-m}f(x)}{dx^{-m}}\right)=\frac{d^{-n-m}f(x)}{dx^{-n-m}}.\end{aligned} \quad (15)$$

In expressions (15), in order to simplify the notation, the lower limit of integration a is omitted, which is essential for negative values of the orders of the operators.

- Integration is performed first (order $-n$), and then differentiation (order $+m$):

$$\frac{d^{m}}{dx^{m}}\left(\frac{d^{-n}f(x)}{dx^{-n}}\right)=\frac{d^{m-n}f(x)}{dx^{m-n}}. \quad (16)$$

- Differentiation is performed first (order $+n$) and then integration (order $-m$):

$$\frac{d^{-m}}{(d(x-a))^{-m}}\left(\frac{d^{+n}f(x)}{(d(x-a))^{+n}}\right)=\frac{d^{n-m}f(x)}{(d(x-a))^{n-m}}-\sum_{i=m-n}^{m-1}\frac{(x-a)^{i}}{i!}f^{(i+n-m)}(a). \quad (17)$$

where $f^{(i+n-m)}(a)$ is the value of the derivative of the corresponding order of the function f at the lower limit a of the change in the argument x.

Thus, in the latter case, the initial values of the derivatives must be taken into account.

We now turn to the composition rules for integro-differential operators of arbitrary orders. Here we restrict ourselves to the cases when these operators are defined according to Riemann-Liouville. Investigations and proofs of composition rules for integro-differential operators are contained in the published papers of many researchers [3, 4]. We present them without proofs, in most cases, following the works of A.V Letnikov [2].

Let's first assume that one of the operators is integer.

a) The integer derivative of order r of the derivative of the function $f(t)$ of order α:

$$\frac{d^{r}}{dt^{r}}\left(D_{a}^{\alpha}(f(t))\right)=D_{a}^{\alpha+r}(f(t)). \quad (18)$$

b) Derivative of order α of the integer derivative of the function $f(t)$ of order r:

$$D_a^{\alpha}\left(\frac{d^r f(t)}{dt^r}\right)=\frac{d^r}{dt^r}\left(D_a^{\alpha}\left(f(t)\right)\right)-\sum_{i=0}^{i=r-1}\frac{f^{(i)}(a)(t-a)^{-\alpha-r+i}}{\Gamma(-\alpha-r+i+1)}. \tag{19}$$

Analysis of expressions (18) and (19) shows that, in the general case, these two operators are not commutative and only under the following conditions:

$$f^{(i)}(a)=0,\ i=\overline{0,r-1} \tag{20}$$

These operations are equivalent:

$$D_a^{\alpha}\left(\frac{d^r f(t)}{dt^r}\right)=\frac{d^r}{dt^r}\left(D_a^{\alpha}\left(f(t)\right)\right)=D_a^{\alpha+r}\left(f(t)\right). \tag{21}$$

Let us now consider variants of the rules for the composition of operators when both of them are non-integer (including complex ones).

c) The order of the first operator (α) satisfies the condition $\mathrm{Re}(\alpha)<0$, the order of the second operator (β) has no restrictions:

$$D_a^{\beta}\left(D_a^{\alpha} f(t)\right)=D_a^{\alpha+\beta}\left(f(t)\right). \tag{22}$$

d) The order of the first operator (α) satisfies the condition $0\le m-1<\mathrm{Re}(\alpha)<m$. In this case, the result of applying the second operator (β) will be finite only under the following conditions:

$$f^{(i)}(a)=0,\ i=\overline{0,m-1} \tag{23}$$

e) If the orders of the operators satisfy the constraints:

$$0\le m-1<\mathrm{Re}(\alpha)<m;\ \ 0\le n-1<\mathrm{Re}(\beta)<n, \tag{24}$$

and the values of the function and its derivatives on the left boundary of the interval of variation of the argument (a) satisfy the conditions:

$$f^{(i)}(a)=0,\ i=\overline{0,\max(m,n)-1} \tag{25}$$

Then the following composition rules are valid for operators:

$$D_a^{\beta}\left(D_a^{\alpha} f(t)\right)=D_a^{\alpha}\left(D_a^{\beta} f(t)\right)=D_a^{\alpha+\beta}\left(f(t)\right). \tag{26}$$

Composition rules for integro-differential operators play an important role in solving the corresponding differential equations and constructing mathematical and computer models of dynamical systems.

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