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On the mean values of derivatives of entire functions of several complex variables represented by multiple Dirichlet series

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Abstract
In this paper we try to obtain lower bounds of $m_{2,k}(\sigma_1,\sigma_2;\, f^{(r)}), r = 1,2$ in terms of $m_{2,k}(\sigma_1,\sigma_2;\, f)$, where $I_2(\sigma_1,\sigma_2)$ and $m_{2,k}(\sigma_1,\sigma_2)$ are defined as mean values of an entire function of two complex variables.

Keywords: Entire functions, several complex variables represented, multiple Dirichlet series

Introduction
Let us consider

$$(1.1)\ f(s_1,s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(s_1\lambda_m + s_2\mu_n),$$

$\left((s_j = \sigma_j + it_j\right), j = 1,2)$ where $a_{m,n} \in C$, the field of complex numbers, $\lambda_m' s, \mu_n' s$ are real, and

$$0 < \lambda\_1 < \lambda\_2 < \cdots \lambda\_m \to \infty;$$

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty.$$

It has been proved [1] that if

$$(1.2)\ \lim_{m\to\infty} \frac{\log m}{\lambda_m} = 0,\ \lim_{n\to\infty} \frac{\log n}{\mu_n} = 0,$$

the domain of convergence of the series (1.1) coincides with its domain of absolute convergence.
Also, Sarkar [2, pp.99] has shown that the necessary and sufficient condition that the series

(1.1) satisfying (1.2) to be entire is that

$$(1.3)\ \lim_{(m,n)\to\infty} \frac{\log|a_{m,n}|}{\lambda_m+\mu_n} = -\infty$$

Let the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3) be denoted by F.
Then f $\in F$ Denotes an entire function over C^2. The results can be extended to several complex variables.
Corresponding to an f $\in F$, the maximum modulus M = M_f and the maximum term $\mu = \mu_f$

On R^2 are defined as [2, pp100]

$$M(\sigma) = M_f(\sigma_1,\sigma_2) = \max\{|f(s_1,s_2)|: s_1,s_2 \in C, \operatorname{Re} s_1 = \sigma_1, \operatorname{Re} s_2 = \sigma_2\}$$

$\mu(\sigma) = \mu_f(\sigma_1, \sigma_2) = \max_{(m,n)\in N^2} \{|a_{m,n}| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)\}$

where N is the set of natural numbers.

The mean value $I_2(\sigma_1, \sigma_2)$ of $|f(s_1, s_2)|$ is defined as

$$(1.4) I_2(\sigma_1, \sigma_2\,; f) = I_2(\sigma_1, \sigma_2) = \lim_{T_1 T_2 \to \infty} \frac{1}{4T_1T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^2 dt_1\, dt_2$$

Now we define the mean value

$m_{2,k}(\sigma_1, \sigma_2)$ of $|f(s_1, s_2)|$ as

$$(1.5)\; m_{2,k}(\sigma_1, \sigma_2; f) = m_{2,k}(\sigma_1, \sigma_2)$$

$$= \lim_{T_1,T_2\to\infty} \frac{1}{T_1 T_2 e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \{|f(x_1 + it_1, x_2 + jt_2)|^2 e^{kx_1} e^{kx_2}\} dx_1 dx_2 dt_1 dt_2$$

where k is any positive number.

These mean values defined by (1.4) and (1.5) are increasing functions of σ_1 and σ_2
For

$$|f(s_1, s_2)|^2 = f(s_1, s_2)\overline{f(s_1, s_2)}$$

$$= \sum_{m_1 n=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1 \lambda m + \sigma_2 \mu_n)\}$$

$$+ \sum_{m\neq M} \sum_{n\neq N} a_{m,n} \bar{a}_{M,N} \exp\{\sigma_1(\lambda_m + \lambda_M) + \sigma_2(\mu_n + \mu_N) + it_1(\lambda_m - \lambda_M) + it_2(\mu_n - \mu_N)\}$$

The series on the right being absolutely convergent, the resulting series is uniformly convergent for any finite t_1 and t_2 range, therefore integrating term by term, the terms for which m$\neq M$, n$\neq N$, vanish as $T_1, T_2 \to \infty$ and we get

$$(1.6)\; I_2(\sigma_1, \sigma_2) = \sum_{mn=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1 \lambda_m + \sigma_2 \mu_n)\}$$

Hence $I_2(\sigma_1, \sigma_2)$ is an increasing function of σ_1 and σ_2.
Also from (1.4) and (1.5)

$$(1.7)\; m_{2,k}(\sigma_1, \sigma_2) = \frac{4}{e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} I_2(x_1, x_2) e^{kx_1} e^{kx_2} dx_1\, dx_2$$

Using (1.6) we obtain

$$m_{2,k}(\sigma_1, \sigma_2) = \frac{4}{e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} \left[\sum_{m,n=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1 \lambda_m + \sigma_2 \mu_n)\}\, e^{kx_1} e^{kx_2} dx_1\, dx_2\right]$$

$$= 4 \sum_{m,n=1}^{\infty} \frac{|a_{m,n}|^2\, (e^{2\lambda_m \sigma_1} - e^{-k\sigma_1})\, (e^{2\mu_n \sigma_2} - e^{-k\sigma_2})}{(2\lambda_m + k)(2\mu_n + k)}$$

Thus $m_{2,k}(\sigma_1, \sigma_2)$ is also an increasing function of $\sigma_1 and\ \sigma_2$.

We now obtain a result on lower bound of $m_{2,k}(\sigma_1, \sigma_2;\ f^{(r)}), r = 1,2$ in terms of

$m_{2,k}(\sigma_1, \sigma_2;\ f)$, and any one of $\sigma_1 and\ \sigma_2$ while keeping the other fixed, where

$m_{2,k}(\sigma_1, \sigma_2;\ f^{(r)})$, is the mean value of $\left|\frac{\partial}{\partial s_r} f(s_1, s_2)\right|, r = 1,2$; that is

$$(2.1)\; m_{2,k}(\sigma_1, \sigma_2;\ f^{(1)}) = \lim_{T_1,T_2\to\infty} \frac{1}{T_1 T_2 e^{k\sigma_1} e^{k\sigma_2}} \times$$

$$\int_0^{\sigma_1} \int_0^{\sigma_2} \int_{-T1}^{T_1} \int_{-T2}^{T_2} \left\{\left|\frac{\partial}{\partial x_1} f(x_1+it_1, x_2+it_2)\right|^2\right\} \times$$

$$\times\, e^{kx_1} e^{kx_2} dx_1 dx_2 dt_1\, dt_2$$

$and\ similarly\ for\ m_{2,k}\left(\sigma_1, \sigma_2;\ f^{(2)}\right)$

3. Lemma 1: $e^{k\sigma_1} I_2(\sigma_1, \sigma_2)$ is a convex function of $e^{k\sigma_1} m_{2,k}(\sigma_1, \sigma_2)$ for a fixed σ_2

Proof
Using (1.7) we have

$$\frac{\partial\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\partial\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\}}$$

$$= \frac{\frac{\partial}{\partial\sigma_1}\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\frac{\partial}{\partial\sigma_1}\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\}}$$

$$= \frac{ke^{k\sigma_1} I_2(\sigma_1,\sigma_2) + e^{k\sigma_1}\frac{\partial}{\partial\sigma_1}\bar{I}_2(\sigma_{11}\sigma_2)}{\frac{\partial}{\partial\sigma_1}\left\{\frac{4}{e^{k\sigma_1} e^{k\sigma_2}} \cdot e^{k\sigma_1} \int_0^{\sigma_1} \int_0^{\sigma_2} I_2(x_1,x_2)\{e^{kx_1} e^{kx_2}\, dx_1\, dx_2\}\right.}$$

$$= \frac{ke^{k\sigma_1} \mathrm{I}_2(\sigma_1,\sigma_2) + e^{k\sigma_1}\frac{\partial}{\partial\sigma_1} I_2(\sigma_1,\sigma_2)}{\frac{4}{e^{k\sigma_2}} \cdot e^{k\sigma_1} \int_0^{\sigma_2} I_2(\sigma_1, x_2) e^{kx_2}\, dx_2}$$

Since $I_2(\sigma_1, x_2) e^{kx_2}$ is an increasing function of σ_1, x_2 and therefore applying the second mean value theorem, we have

$$\frac{\partial\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\partial\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\ \}} = \frac{ke^{k\sigma_1} I_2(\sigma_1,\sigma_2) + e^{k\sigma_1}\frac{\partial}{\partial\sigma_1} I_2(\sigma_1,\sigma_2)}{\frac{4}{e^{k\sigma_2}} e^{k\sigma_1} I_2(\sigma_1,\sigma_2) e^{k\sigma_2} \int_\xi^{\sigma_2} dx_2}$$

where $0 < \xi < \sigma_2$

$$= \frac{kI_2(\sigma_1,\sigma_2) + \frac{\partial}{\partial\sigma_1} I_2(\sigma_1,\sigma_2)}{4 I_2(\sigma_1,\sigma_2)(\sigma_2 - \xi)}$$

$$= \frac{1}{4(\sigma_2-\xi)}\left\{k + \frac{\left(\frac{\partial}{\partial\sigma_1}\right)(I_2(\sigma_1,\sigma_2))}{I_2(\sigma_1,\sigma_2)}\right\}$$

The right hand side increases with σ_1 for a fixed σ_2, from (1.6).
Therefore

$\frac{\partial^2\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\partial^2\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\}} > 0$, for a fixed σ_2. Hence the result

4. Lemma 2: $Log\, m_{2,k}(\sigma_1, \sigma_2)$ is a convex function of σ_1 for a fixed value of σ_2 and vice-versa

Proof: Using (1.7), we have

$$\frac{\partial\{log\, m_{2,k}(\sigma_1, \sigma_2)\}}{\partial\sigma_1} = \frac{\partial}{\partial\sigma_1}\{\log m_{2,k}\ (\sigma_1, \sigma_2)\}$$

$$= \frac{4}{m_{2,k}(\sigma_1,\sigma_2)}\left[\frac{1}{e^{k\sigma_2}}\{-ke^{-k\sigma_1}\int_0^{\sigma_1}\int_0^{\sigma_2} I_2(x_1,x_2) e^{kx_1+kx_2}\, dx_1\, dx_2 + \frac{1}{e^{k\sigma_1}} e^{k\sigma_1}\int_0^{\sigma_2} I_2(\sigma_1, x_2) e^{kx_2}\, dx_2\}\right]$$

$$= \frac{1}{m_{2,k}(\sigma_1,\sigma_2)}\{-km_{2,k}(\sigma_1,\sigma_2) + \frac{4}{e^{k\sigma_2}}\int_0^{\sigma_2} I_2(\sigma_1,x_2) e^{kx_2}\, dx_2\}$$

Since $I_2(\sigma_1, x_2) e^{kx_2}$ is an increasing function of σ_1, x_2 and therefore applying the second mean-value theorem, we get

$$\frac{\partial\{log\, m_{2,k}(\sigma_1, \sigma_2)\}}{\partial(\sigma_1)}$$

$=\frac{1}{m_{2,k}(\sigma_1,\sigma_2)}\{-km_{2,k}(\sigma_1,\sigma_2)+\frac{4I_2(\sigma_1,\sigma_2)e^{k\sigma_2}}{e^{k\sigma_2}}\int_{\xi}^{\sigma_2} dx_2\}$, $0<\xi<\sigma_2$

$=\frac{1}{m_{2,k}(\sigma_1,\sigma_2)}\{-km_{2,k}(\sigma_1,\sigma_2)+4(\sigma_2-\xi)I_2(\sigma_1,\sigma_2)\}$

$=\left\{4(\sigma_2-\xi)\frac{I_2(\sigma_1,\sigma_2)}{m_{2,k}(\sigma_1,\sigma_2)}-k\right\}$

The right-hand side increases with σ_1 for a fixed σ_2, follows from Lemma
1. Therefore,

$\frac{\partial^2\{\log m_{2,k}(\sigma_1,\sigma_2)\}}{\partial\sigma_1^2}>0$, for a fixed value of σ_2. Hence the result and vice-versa.

2. Theorem 1: If $m_{2,k}(\sigma_1,\sigma_2;f^{(1)})$ is the mean value of $|f^{(1)}(s_1,s_2)|$, where $f^{(1)}(s_1,s_2)=\frac{\partial}{\partial s_1}f(s_1,s_2)$, then for a fixed value σ_2,

(4.1) $m_{2,k}(\sigma_1,\sigma_2;f^{(1)})\geq\frac{m_{2,k}(\sigma_1,\sigma_2)}{2^2}\{\frac{\log m_{2,k}(\sigma_1,\sigma_2)-1}{\sigma_1^2}\}^2$

for $\sigma_1\geq\sigma_1^0$, where σ_1^0 is depending on the function of f.

Proof: We have

$m_{2,k}(\sigma_1,\sigma_2;f^{(1)})=$

$$\lim_{T_1,T_2\to\infty}\frac{1}{T_1T_2e^{k\sigma_1}e^{k\sigma_2}}\int_0^{\sigma_1}\int_0^{\sigma 2}\int_{-T_1}^{T_1}\int_{-T_2}^{T2}\left|f^{(1)}(x_1+it_1,x_2+it_2)\right|^2e^{kx_1}e^{kx_2}\,dx_1dx_2dt_1\,dt_2$$

$$=\lim_{T_1T_2\to\infty}\frac{1}{T_1T_2e^{k\sigma_1}e^{k\sigma_2}}\mathrm{x}$$

$$\mathrm{X}\int_0^{\sigma_1}\int_0^{\sigma 2}\int_{-T_1}^{T_1}\int_{-T_2}^{T2}\left|\lim_{\varepsilon\to0}\frac{f(x_1+it_1,x_2+it_2)-f\{x_1(1-\varepsilon)+it_1,x_2+it_2\}}{x_1\varepsilon}\right|^2\times$$

$$\times\ e^{kx_1}e^{kx_2}dx_1dx_2dt_1\,dt_2$$

$$\geq\lim_{\varepsilon\to0}\lim_{T_1,T_2\to\infty}\frac{1}{\varepsilon^2\sigma_1^2T_1T_2e^{k\sigma_1}e^{k\sigma_2}}\mathrm{x}$$

$$\mathrm{x}\int_0^{\sigma_1}\int_0^{\sigma 2}\int_{-T_1}^{T_1}\int_{-T_2}^{T2}\left[|f(x_1+it_1,x_2+it_2)|-\left|f\{x_1(1-\varepsilon)+it_1,x_2+it_2\}\right|\right]^2\mathrm{x}$$

$$\mathrm{x}\ e^{kx_1}e^{kx_2}dx_1dx_2dt_1\,dt_2$$

Applying Minkowski's inequality, we get

$$[\int_{-T_1}^{T_1}\int_{-T_2}^{T2}\{|f(x_1+it_1,x_2+it_2)|-|f(x_1(1-\varepsilon)+it_1,x_2+it_2)|\}^2dt_1\,dt_2]^2$$

$$\geq\{\int_{-T_1}^{T_1}\int_{-T_2}^{T2}[|f(x_1+it_1,x_2+it_2)|^2dt_1\,dt_2]\}^{1/2}$$

$$-\{\int_{-T_1}^{T_1}\int_{-T_2}^{T2}[|f(x_1(1-\varepsilon)+it_1,x_2+it_2)|^2dt_1\,dt_2]\}^{1/2}$$

Therefore,

$$m_{2,k}(\sigma_1,\sigma_2;f^{(1)})\geq\lim_{\varepsilon\to0}\lim_{T_1,T_2\to\infty}\frac{1}{\varepsilon^2\sigma_1^2T_1T_2\ e^{k\sigma_1}e^{k\sigma_2}}\mathrm{x}$$

$$\mathrm{x}\int_0^{\sigma_1}\int_0^{\sigma 2}[\{\int_{-T_1}^{T_1}\int_{-T_2}^{T2}|f(x_1+it_1,x_2+it_2|^2dt_1\,dt_2\}^{1/2}$$

$$-\{\int_{-T_1}^{T_1}\int_{-T_2}^{T2}|f(x_1(1-\varepsilon)+it_1,x_2+it_2)|^2dt_1\,dt_2\}^{1/2}]^2e^{kx_1}\,e^{kx_2}dx_1\,dx_2$$

$$=\lim_{\varepsilon\to0}\lim_{T_1,T_2\to\infty}\frac{1}{\varepsilon^2\sigma_1^2T_1T_2e^{k\sigma_2}\,e^{k\sigma_2}}\mathrm{x}$$

x$\int_0^{\sigma_1} \int_0^{\sigma 2}$[{ $e^{kx_1} e^{kx_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T2} |f(x_1+it_1,x_2+it_2)|^2 dt_1\, dt_2\}^{1/2}$

$-\{e^{kx_1} e^{kx_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T2} |f(x_1(1-\varepsilon)+it_1,x_2+it_2)|^2 dt_1\, dt_2\}^{1/2}]^2\, dx_1\, dx_2$

Again, using Minkowski's inequality, we obtain

$m_{2,k}(\sigma_1,\sigma_2;f^{(1)})$

$\geq \lim_{\varepsilon\to 0} \lim_{T_1,T_2\to\infty} \frac{1}{\varepsilon^2\sigma_1^2 T_1 T_2 e^{k\sigma_1} e^{k\sigma_2}}$ x

x[{ $\int_0^{\sigma_1} \int_0^{\sigma 2} e^{kx_1} e^{kx_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T2} |f(x_1+it_1\,,x_2+it_2)|^2 dx_1 dx_2 dt_1\, dt_2\}^{1/2}$

$-\{\int_0^{\sigma_1} \int_0^{\sigma 2} e^{kx_1} e^{kx_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T2} |f(x_1(1-\varepsilon)+it_1,x_2+it_2)|^2 dx_1 dx_2 dt_1\, dt_2\}^{1/2}]^2$

$\geq \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2\sigma_1^2}[\{m_{2,k}(\sigma_1,\sigma_2)\}^{1/2}\{\frac{e^{-\varepsilon k\sigma_1}}{(1-\varepsilon)} e^{k(\sigma_1-\sigma_1\varepsilon)(\frac{\varepsilon}{1-\varepsilon})} m_{2,k}(\sigma_1-\sigma_1\varepsilon,\sigma_2)\}^{1/2}]^2$

$= \lim_{\varepsilon\to 0} [\frac{\{m_{2,k}(\sigma_1,\sigma_2)\}^{1/2}-\{(1-\varepsilon)^{-1} m_{2,k}(\sigma_1-\sigma_1\varepsilon,\sigma_2)\}^{1/2}}{\sigma_1\varepsilon}]^2$

Let us now take, $g(\sigma_1,\sigma_2) = \frac{\log m_{2,k}(\sigma_1,\sigma_2)}{\sigma_1}$

Then from Lemma 2, $g(\sigma_1,\sigma_2)$ is an increasing function of σ_1 *for a fixed value of* σ_2 and so we have

$m_{2,k}(\sigma_1,\sigma_2;f^{(1)}) \geq \lim_{\varepsilon\to 0} [\frac{e^{\sigma_1 g(\sigma_1,\sigma_2)/2}-(1-\varepsilon)^{-1/2} e^{(\sigma_1-\sigma_1\varepsilon) g(\sigma_1-\sigma_1\varepsilon,\sigma_2)/2}}{\sigma_1\varepsilon}]^2$

$\geq \lim_{\varepsilon\to 0} [\frac{e^{\sigma_1 g(\sigma_1,\sigma_2)/2}-\left(1+\frac{\varepsilon}{2}+\frac{3}{8}\varepsilon^2+\cdots\right) e^{(\sigma_1-\sigma_1\varepsilon) g(\sigma_1,\sigma_2)/2}}{\sigma_1\varepsilon}]^2$

$=\{\frac{g(\sigma_1,\sigma_2)}{2} e^{\sigma_1 g(\sigma_1,\sigma_2)/2} - \frac{1}{2\sigma_1} e^{\sigma_1 g(\sigma_1,\sigma_2)/2}\}^2$

$=\frac{1}{2^2}\frac{\left[\{\sigma_1 g(\sigma_1,\sigma_2) e^{\sigma_1 g(\sigma_1,\sigma_2)/2}-e^{\sigma_1 g(\sigma_1,\sigma_2)/2}\}^2\right]}{\sigma_1^2}$

$=\frac{m_{2,k}(\sigma_1,\sigma_2)}{2^2}\left\{\frac{\log m_{2,k}(\sigma_1,\sigma_2)-1}{\sigma_1}\right\}^2$

5. Corollary 1: If $m_{2,k}(\sigma_1,\sigma_2;f^{(2)})$ is the mean value of $|f^{(2)}(s_1,s_2)|$

where

$f^{(2)}(s_1,s_2) = \frac{\partial}{\partial s_1} f(s_1,s_2)$ then for a fixed value of σ_1

(5.1) $m_{2,k}(\sigma_1,\sigma_2;f^{(2)}) \geq \frac{m_{2,k}(\sigma_1,\sigma_2)}{2^2}\{\frac{\log m_{2,k}(\sigma_1,\sigma_2)-1}{\sigma_2^2}\}^2$

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