Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2022; 3(1): 21-24

www.mathematicaljournal.com
Received: 14-11-2021
Accepted: 19-12-2021

Mustafoev Nemat
Senior Lecturer of the Department of Higher Mathematics, Bukhara Institute of Engineering and Technology, Uzbekistan

Corresponding Author:
Mustafoev Nemat
Senior Lecturer of the Department of Higher Mathematics, Bukhara Institute of Engineering and Technology, Uzbekistan

On application of "fractional calculation" to mechanical problems

Mustafoev Nemat

Abstract
The application of derivatives and integrals of fractional order in mechanical problems is discussed. An informal conclusion is made for the "fractional calculus" formulas. The problem of continuous models of "force" mechanical systems is discussed.

Keywords: Fractional order, fractional integral, fractional derivative, fractional calculus

Introduction

In traditional courses in differential calculus, the concept of a derivative is usually preceded by the introduction of the concept of a continuous function. The presentation of mathematical courses is built in such a way that a picture arises when the specificity of discrete systems becomes secondary in relation to the main continuous and analytical objects. Discrete (or other "discontinuous") systems become more convenient to consider as limiting properties of continuous systems.

On the other hand, in integral calculus, which traditionally follows differential calculus in teaching, discrete sums are still the primary objects of analysis. And integrals are the limits of the corresponding integral sums.

It is well known that professional mathematicians should not get carried away with the content side of the formal mathematical apparatus (and who in our time would not want to be considered a professional?) Mathematical literature is cleared of illustrations, images and interpretations of their own text. Including from natural associations. As for the “primacy” of the continuous, which follows only from the logic of teaching mathematical analysis, then this “primacy” becomes unobtrusively concrete, and in the future may manifest itself in some special physical text. Continuous is very seriously interpreted in naive physical pictures of the world, projecting only the language of presentation of mathematical disciplines on the explanation of nature. There have arisen, and will continue to arise, heated debates about discreteness or continuity, both in certain areas of physical knowledge and in areas thematically related to the fundamental principles of the universe. Meanwhile, a much stronger proposition is that neither the world nor any phenomenon can be inherently discrete or continuous. It makes sense to calmly relate to the pragmatics of cognitive actions (including your own), so that, while remaining scientifically honest with the phenomena of nature, do not diminish the value of various points of view.

Informal derivation of formulas for fractional calculus

Let us consider the concept of a derivative for a certain function as a special case of the calculus of finite differences, and, therefore, from a certain discrete construction. Even the English mathematician Brook Taylor (1685-1731) came to the "Taylor theorem" starting from finite differences. The value of any quantity is easier to interpret as the average value of instrumentally measured over a certain, albeit very small, but finite interval (time, space). Following the modernized Taylor's scheme, we construct a table in which all possible differences are placed for a uniform partition of x into Δx

x		$x+\Delta x$		$x+2\Delta x$		$x+3\Delta x$	…	…
y		y_1		y_2		y_3	…	…
	Δy		Δy_1		Δy_2		…	…
		$\Delta^2 y$		$\Delta^2 y_1$		…	…	…
			$\Delta^3 y$		$\Delta^3 y_1$	…	…	…
				…	…	…	…	…

For differences of different orders, the expressions are valid, which can be seen from the table:

$$\Delta^1 y_i = y_{i+1} - y_i$$
$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i = y_{i+2} - 2y_{i+1} + y_i$$
$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

………………………………………………………

The general formula for the difference of order n is as follows:

$$\Delta^n y_i = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot y(x_i + n \cdot \Delta x - k \cdot \Delta x) \qquad (1)$$

Here

$$\binom{n}{k} = \frac{n!}{r!(n-r)!}$$

is the well-known formula for binomial coefficients, expressed in terms of factorials.
Using formula (1), we can calculate the limits when □x tends to zero of the following ratios:

$$\frac{\Delta^1 y_i}{\Delta x^1}; \; \frac{\Delta^2 y_i}{\Delta x^2}; \; \frac{\Delta^3 y_i}{\Delta x^3}; \ldots, \qquad (2)$$

and, which should define derivatives of different orders at the point x_i (if, of course, they exist).
The "differential calculus" of the great mathematician and member of the St. Petersburg Academy of Sciences Leonard Euler also began with the calculus of finite differences. He also introduced the so-called gamma function of the complex variable Γ(z), which for Re (z)> 0 is determined by the integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad (3)$$

In its particular properties, the gamma function is a generalization of the factorial function. It is important that the gamma function can be analytically extended to the entire complex plane, with the exception of the points $z = 0, -1, -2, \ldots$, at which it has first-order poles.
Actually, here we use the "factorial" properties of Γ(z) for positive and negative real numbers:

$$\Gamma(1+z) = z \cdot \Gamma(z); \qquad \Gamma(n+1) = n!. \qquad (4)$$

Since the gamma function admits the above generalizations, it is possible to construct a generalization of formula (1) for a difference of arbitrary order:

$$\Delta^\alpha y_i = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \cdot y(x_i + \alpha \cdot \Delta x - k \cdot \Delta x) \qquad (5)$$

where α is an optional integer and positive number. In this case, instead of the traditional binomial coefficients, the expressions are used

$$\binom{\alpha}{k} = \frac{\Gamma(1+\alpha)}{\Gamma(1+k) \cdot \Gamma(\alpha - k + 1)} \qquad (6)$$

Consider special cases of the values of α.

1) If α = m is a non-negative integer, then due to the poles of the gamma function, the expression
2)

$$\binom{\alpha}{k} = \begin{cases} \dfrac{m!}{k!(m-k)!}, & k \le m \\ 0, & k > m \end{cases} \qquad (7)$$

coincides completely with the binomial coefficients in (1). Consequently, the expressions

$$\frac{\Delta^1 y_i}{\Delta x^1}; \; \frac{\Delta^2 y_i}{\Delta x^2}; \; \frac{\Delta^3 y_i}{\Delta x^3}; \ldots,$$

defined by the new formula will allow calculating the usual derivatives of various integer orders.

1) If α = 0, then
2)

$$\binom{\alpha}{k} = \binom{0}{k} = \frac{\Gamma(1)}{k!\Gamma(1-k)} = \begin{cases} 1, & k = 0 \\ 0, & k > 0 \end{cases} \qquad (8)$$

Therefore,

$$\frac{\Delta^0 y_i}{\Delta x^0} = y(x_i)$$

1) If $\alpha = -1$, then after substitution the expression

$$\binom{-1}{k} = \frac{\Gamma(0)}{k!\Gamma(-k)}$$

contains an undefined ratio of simple poles. Given the property

$$\Gamma(1+z) = z \cdot \Gamma(z),$$

we obtain

$$\Gamma(0) = (-1)^k k! \Gamma(k)$$

And

$$\binom{-1}{k} = (-1)^k$$

Thus

$$\frac{\Delta^{-1} y_i}{\Delta x^{-1}} = \sum_{k=0}^{\infty} (-1)^{2k} \cdot y(x_i - (k+1)\Delta x) = -\sum_{k=0}^{\infty} y(x_i - k \cdot \Delta x) \tag{9}$$

Under conditions that ensure mathematical correctness, this expression can be considered as an integral sum, which, as $\Delta x \to 0$, will give an expression for a definite integral with a variable upper limit

$$\int_{-\infty}^{x} y(x)dx \tag{10}$$

The relationship between definite and indefinite integrals is known.

$$\text{«} \int \Phi(x)dx = \int_{a}^{x} \Phi(t)dt + C$$

They read this wonderful formula: an indefinite integral is the sum of a definite integral with a variable upper limit and a constant lower limit, plus an arbitrary surplus constant.».

If α is a fractional, positive or negative number, then the convergence of series (5) as $\Delta x \to 0$ will determine the value of the fractional "differential integral" at the point x_i.

On the nature of the application of fractional calculus in mechanics

The first difference ratio, when the independent variable is time, has a transparent physical interpretation. This is usually represented as the "average speed" of a point object over some finite time interval. That is, the speed calculated when the object moves from the beginning of the time interval to the end. Accordingly, the ratio of the second order difference is considered "average acceleration" over a certain interval. And the ratio of zero-order differences is the exact coordinate function of a point object. Fractional differences do not give similar interpretations that can be attributed to point properties of an object or a small interval - rather, these are properties of the entire path-trajectory as a whole. That is why fractional derivatives (or rather, differential equations with fractional derivatives) often find their semantic interpretations in the descriptions of diffusion processes, in which the object of modeling is considered to be distributed in principle, and all possible density distributions are considered. In classical mechanics, however, two methods of calculating the motion of systems represented as a point of some, sometimes even very abstract space, were embodied. In one, originating from Newton, motion is considered to be uniquely defined if all forces and moments of forces are known. Another method, derived from Leibniz, is called "analytical mechanics". Here the motion is calculated from two scalar quantities - potential and kinetic energy, which, with their clear "definiteness", can determine the direction and trajectories of movements of a material point object (dissipative processes naturally interfere with "determinability"). Historically, the successes of "analytical mechanics" have neutralized the variety of possible forces acting on material objects. For example, the elementary behavior under viscous friction - Newton's viscosity - is considered outside of the "classical analytical" mechanics: in the mechanics of continuous media, rheology. Rheology, in turn, is forced to return to the questions of constructing elementary models of the actual diversity of forces. The models of Kelvin, Maxwell, Voigt, Saint-Venant, Burgers, Shvedov, and others can be considered elementary, since they are uniaxial and linear, rather simple, and cannot be reduced to each other only by a linear combination.

It is curious that classical mechanics, formulating the axioms of its analytical procedures, relies on the experimental material of physical knowledge. "Newton's principle of determinism.

The initial state of a mechanical system (a set of positions and velocities of points of the system at some point in time) uniquely determines all of its movements.

We do not have time to be surprised at this fact, since we learn it very early. One can imagine a world in which, in order to determine the future of the system, one must also know accelerations at the initial moment. Experience shows that our world is not like that "[3]. However, in rheology one can "imagine a world" in which there are not only coordinates, velocities and accelerations, but also "intermediate" values. "The creep of many materials is described by the Abel kernel $k = \lambda \cdot I_\alpha$. In this case, the amount of deformation from above is not limited by anything, but the rate of deformation decreases all the time. There are experiments on the creep of plastics, lasting 100,000 hours (about 12 years). The dependence $e(t)$ (deformation) throughout the test was power-law without any tendency to reach the horizontal asymptote. "

We may recall that the word "dynamics" comes from the Greek δυναμισ with the meaning of strength, power, opportunity. But the "real" description of the variety of (dynamic) behaviors appeared in rheology. "The term" rheology "brings to mind the expression" παντα ρει "(everything flows). With the same reason, we can say: "any body is solid." There is a quantitative rather than qualitative difference between liquids and solids ... we, for example, considered concrete to be a liquid with a relaxation time of $\approx 10^6$ sec, and air as a solid with a relaxation time of $\approx 10^{-10}$ sec. If we consider concrete as a solid and air as a liquid, then such a consideration is not of interest to a rheologist. "

It should be admitted that there is a rather substantial distance between the two above-mentioned areas of mechanics. If in the first one introduces the Galilean relativity of space-time coordinates, then in the second one introduces the relativity of the scale of dynamic action and relaxation. In the first, "classical" one, the exact coordinates, as well as the first and second derivatives of the motion, are essential. In the second, the phenomenon in the general case must inevitably be represented in the form of spectra or combinations of relaxation processes, and, therefore, the integrity of the phenomenon contains the uncertainty associated with "extra-large" and "ultra-small" measurements. If, according to certain rules, uncertainty is allowed in analytical procedures, then it is justified to use fractional calculus, which is "nonlocal", in modeling dynamic systems. Even an approximate calculation of fractional derivatives and integrals must involve a practically impossible summation of infinite series. Oldham & Spanier introduced a variant of the fractional calculus that includes a "watch window" between x and x_0:

$$D^s_{x_0,x} = \lim_{m\to\infty} \left(\frac{x - x_0}{m}\right)^{-s} \sum_{k=0}^{m-1} (-1)^k \binom{s}{k} y\left(x - k\frac{x - x_0}{m}\right) \tag{11}$$

However, the quality of nonlocality remains in this case as well.

Interestingly, nonlocality can manifest itself in structural diagrams in the form of infinite chains. An element specific for highly elastic deformations, independent of Hooke and Newtonian, and expressed through the operation of fractional integration, was proposed back in 1961 by Slonimsky. Currently, there are works in which such elements are built in the form of hierarchical and endless chains of the simplest Hooke elements and Newton's elements (for example, Schiessel & Blumen). In the OSENS laboratory, it is believed that various types of algebraic expressions from differentiation operators, including fractional powers of operators, are obtained by introducing a self-similarity structure into the description of special elements. In this case, the structural formula takes the form of an infinite chain tree, and in particular, the form of infinite continued fractions, as in the works of Schiessel & Blumen *et al*. Authors. However, in principle, all elements with fractional differential integrals are not linear combinations of elements with zero (Hooke) and first (Newton) derivatives, since the combinatorial scheme includes both serial and parallel connections. Special sums (and even integrals) with operators in the form of fractional differentiation should be considered as natural models of elastic, viscoelastic, and other soft-rigid phenomena. There is a transition from discrete structural models of mechanical systems to models with continuity properties. The characteristics of the phenomenon, and, consequently, the summation parameters, can be estimated only in relation to certain scale classes of dynamic effects. In this case, the meaning of some concepts (for example, softness) becomes parameterized from the value of the scale factor.

References

1. Mamatov T. Composition of mixed Riemann-Liouville fractional integral and mixed fractional derivative. “Journal of Global Research in Mathematical Archives”. India. 2019;6(11): 23-32.
2. Letnikov AV. An explanation of fundamental notions of the theory of differentiation of fractional order. Mat. Sb. 1872;6:413-445.
3. Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York, 1993.
4. Mamatov T. Mixed Fractional Integration Operators in Mixed Weighted Hölder Spaces. Laplambert Academic Publishing, 2018, 78.
5. Mamatov T. Mapping Properties of Mixed Fractional Integro-Differentiation in Hölder Spaces. JCAAM. 2014;12(3):272-290.
6. Mamatov T, Homidov F, Rayimov D. On Isomorphism Implemented by Mixed Fractional Integrals In Hölder Spaces, International Journal of Development Research. 2019;9(5):27720-27730. (journal style)
7. Mamatov T. Mapping Properties of Mixed Fractional Integro-Differentiation in Hölder Spaces. Journal of Concrete and Applicable Mathematics (JCAAM). 2014;12(3-4):272-290.
8. Mamatov T. Mixed Fractional Integro-Differentiation Operators in Hölder Spaces. «The latest research in modern science: experience, traditions and innovations» Proceedings of the VII International Scientific Conference North Charleston, SC, USA, 2018, 6-9p.
9. Mamatov T. Fractional integration operators in mixed weighted generalized Hölder spaces of function of two variables defined by mixed modulus of continuity. Journal of Mathematical Methods in Engineering. Auctores Publishing. 2019;1(1)-004:1-16.
10. Mamatov T. Mapping Properties of Mixed Fractional Differentiation Operators in Hölder Spaces Defined by Usual Hölder Condition. JCSCM. 2019;9(2). DOI: 10.20967/.2019
11. Маматов Т. Гельдер фазосида аралаш каср тартибли интеграл операторлар. «НамДУ илмий ахборотномаси», 2-сон, Наманган, 2020, 32-39. бетлар.
12. Mamatov T, Raimov D. Some properties of mixed fractional integro-differentation operators in Hölder spaces. Journal of Global Research in Mathematical Archives, Available online at http://www.jgrma.info. Issn 2320-5822. India, 2019 November;6:(11):13-22.
13. Mamatov T, Mustafoev N. Non-Weighted Zygmund Type Estimates for The Volterra Convolution Type. Impact Factor 3.582 Case Studies Journal ISSN (2305-509X). 2019 Nov1;18:119-122. http://www.casestudiesjournal.com/volume-8-issue-11.