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Application of the main theorems of differential calculus to functional equations and inequalities

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Abstract
This article provides some applications of the basic terms of differential calculus.

Keywords: Function, equation, system, transformation, variable

Introduction

Examples and problem solving are at the core of mathematics education. A number of reformatory changes were introduced into the country's education system, the main purpose of which is to identify, open and create conditions and opportunities for the development of the abilities (giftedness) and talents of students.

From the science of mathematical analysis, we know such concepts as the continuity of functions, the derivative of functions, the differential of functions, as well as their geometric and mechanical values.

In this article, we present an application of the main theorems of differential calculus to functional equations and inequalities. These results can be widely used to solve equations, inequalities, and many other important problems in the future.

There are several applications of these theorems, and many problems can be easily solved with their introduction.

Results

We apply them to the following questions.

Example 1. Prove that if a non-linear function $f(x)$ is defined and continuous in the closed interval $[a;b]$ and there is a finite derivative $f'(x)$ in the open interval $(a;b)$. Then between a and b there is a point c $(a<c<b)$ such that the inequality

$$|f'(c)|>\left|\frac{f(b)-f(a)}{b-a}\right| \tag{1}$$

holds for it.

Solution: Let's divide the segment $[a;b]$ into n parts in an arbitrary way: $a=x_0<x_1<x_2<...<x_{n-1}<x_n=b$. In each $[x_k;\ x_{k+1}],\ k=\overline{0,n-1}$ segment, the function $f(x)$ satisfies the condition of the Lagrange theorem. Then between x_k and x_{k+1} there is a point ξ_k $\left(x_k<\xi_k<x_{k+1},\ k=\overline{0,n-1}\right)$ such that equality $f'(\xi_k)=\dfrac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}$ holds for it. Introduce the notation $x_{k+1}-x_k=\Delta x_k$ and we have

$$f(b)-f(a)=\sum_{k=0}^{n-1}[f(x_{k+1})-f(x_k)]=\sum_{k=0}^{n-1}f'(\xi_k)\Delta x_k \tag{2}$$

Now we introduce the designation $|f'(c)|=\max\limits_k\{f'(\xi_k)\},\ c\in(a;\ b)$ and from (2), we obtain.

$$|f(b)-f(a)|=\left|\sum_{k=0}^{n-1} f'(\xi_k)\Delta x_k\right| \le \sum_{k=0}^{n-1}|f'(\xi_k)|\Delta x_k \le |f'(c)|\sum_{k=0}^{n-1}\Delta x_k = |f'(c)|\cdot|b-a|$$

From the latter follows

$$|f'(c)| \ge \left|\frac{f(b)-f(a)}{b-a}\right|$$

Example 2. Let 1) $f(x)$ function have a second order derivative in a closed interval $[a;\, b]$; 2) at the ends of the interval, the function takes on equal values and equal to zero, i.e. $f'(a)=f'(b)=0$. Then prove that between a and b there is a point c ($a<c<b$) such that

$$|f''(c)| > \frac{4}{(b-a)^2}|f(b)-f(a)| \tag{3}$$

holds for it.

Solution: Let $f(x)$ function have a second-order derivative in $[a;\, b]$ and $f'(a)=f'(b)=0$. Prove that between a and b there is a point c ($a<c<b$) such that the following relation is true:
1) Let $f(x)=\text{const}$. In this case, equality (3) holds for any c from $(a;\, b)$.
2) Let $f(x)$ be a linear function. In this case, the condition $f'(a)=f'(b)=0$ is unrealizable.
3) Divide the interval $[a;\, b]$ in half by point $\frac{a+b}{2}$. And we get two gaps: $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. Accordingly, in each interval, consider the auxiliary functions $\varphi(x)=\frac{(x-a)^2}{2}$ and $\psi(x)=\frac{(x-b)^2}{2}$. Functions $\varphi(x)$ and $f(x)$ satisfy all conditions of the Cauchy theorem in $\left[a, \frac{a+b}{2}\right]$, and functions $f(x)$ and $\psi(x)$ satisfy in $\left[\frac{a+b}{2}, b\right]$. That is, there is a point $\xi_1 \in \left(a, \frac{a+b}{2}\right)$ such that equality

$$\frac{f'(\xi_1)}{\varphi'(\xi_1)} = \frac{f\left(\frac{a+b}{2}\right)-f(a)}{\varphi\left(\frac{a+b}{2}\right)-\varphi(a)} = \frac{8\left[f\left(\frac{a+b}{2}\right)-f(a)\right]}{(b-a)^2}.$$

holds for it. And also there is such a point $\xi_2 \in \left(\frac{a+b}{2}, b\right)$, then

$$\frac{f'(\xi_2)}{\psi'(\xi_2)} = \frac{f(b)-f\left(\frac{a+b}{2}\right)}{\psi(b)-\psi\left(\frac{a+b}{2}\right)} = \frac{8\left[f(b)-f\left(\frac{a+b}{2}\right)\right]}{(b-a)^2}.$$

We set the values $\varphi'(\xi_1)$ and $\psi'(\xi_2)$ and we obtain

$$\begin{cases} \dfrac{f'(\xi_1)}{\varphi'(\xi_1)} = \dfrac{8\left[f\left(\frac{a+b}{2}\right)-f(a)\right]}{(b-a)^2} \\ \dfrac{f'(\xi_2)}{\psi'(\xi_2)} = \dfrac{8\left[f(b)-f\left(\frac{a+b}{2}\right)\right]}{(b-a)^2} \end{cases} \Rightarrow \begin{cases} \dfrac{f'(\xi_1)}{\xi_1-a} = \dfrac{8\left[f\left(\frac{a+b}{2}\right)-f(a)\right]}{(b-a)^2} \\ \dfrac{f'(\xi_2)}{\xi_2-b} = \dfrac{8\left[f(b)-f\left(\frac{a+b}{2}\right)\right]}{(b-a)^2} \end{cases} + \Rightarrow$$

$$\frac{f'(\xi_1)}{\xi_1-a}+\frac{f'(\xi_2)}{\xi_2-b} = \frac{8[f(b)-f(a)]}{(b-a)^2}$$

$$\frac{8[f(b)-f(a)]}{(b-a)^2} = \frac{f'(\xi_1)-f'(a)}{\xi_1-a}+\frac{f'(\xi_2)-f'(b)}{\xi_2-b}$$

Taking into account that $a < \xi_1 < \frac{b+a}{2}$ $\left(\text{and } \frac{b+a}{2} < \xi_2 < b\right)$ in these expressions, we see that the function $f'(x)$ satisfies Lagrange's theorem on $[a, \xi_1]$ $(\text{and } [\xi_2; b])$. Then between a and ξ_1 there is a point c_1 ($a < c_1 < \xi_1$) such that the equality

$$f''(c_1) = \frac{f'(\xi_1) - f'(a)}{\xi_1 - a}$$

holds for it. And also between ξ_2 and b there is a point c_2 ($\xi_2 < c_2 < b$) such that the equality

$$f''(c_2) = \frac{f'(\xi_2) - f'(b)}{\xi_2 - b}$$ holds for it.

If we take that $f''(c) = \max\{f''(c_1), f''(c_2)\}$, then

$$\left|\frac{8[f(b) - f(a)]}{(b-a)^2}\right| = |f''(c_1) + f''(c_2)| \le |f''(c_1)| + |f''(c_2)| \le 2|f''(c)|$$

$$|f''(c)| \ge \frac{4}{(b-a)^2}|f(b) - f(a)|$$.

Example 3. Let the function $f(x)$ satisfy the following conditions:

1. $f(x) \in C^{n-1}([x_0; x_n])$;
2. $f(x)$ have n-th order derivative in $(x_0; x_n)$.
3. The equality $f(x_0) = f(x_1) = ... = f(x_n)$ holds for $x_0 < x_1 < x_2 < ... < x_{n-1} < x_n$.

Then prove that between x_0 and x_n there is a point ε ($x_0 < c < x_n$) such that $f^{(n)}(\varepsilon) = 0$.

Solution: On $[x_k; x_{k+1}]$, $\forall k = \overline{0, n-1}$, the function $f(x)$ satisfies Roll's theorem, so there is such $\exists \xi_k \in (x_k; x_{k+1})$ and $f'(\xi_k) = 0$. This confirmation is relevant for each segment. Those $f'(\xi_1) = f'(\xi_2) = ... = f'(\xi_k) = 0$. Now, for each $[\xi_0; \xi_1], [\xi_1; \xi_2], ..., [\xi_i; \xi_{i+1}]...$ $i = \overline{0, k-1}$, the function $f'(x)$ satisfies the condition of Roll's theorem. Then $\exists \eta_i \in (\xi_i; \xi_{i+1})$ and $f''(\eta_i) = 0$. Using mathematical induction, it is easy to see that the function $f^{(n-1)}(x)$ satisfies Roll's theorem on the segment $[\varsigma_1; \varsigma_2]$ after step $n-2$. Then $f^{(n-1)}(\varsigma_1) = f^{(n-1)}(\varsigma_2) = 0$. Hence there is $\exists\, \varepsilon \in (\varsigma_1; \varsigma_2)$ and $f^{(n)}(\varepsilon) = 0$.

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