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On the closed formula of special partition polynomials

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Abstract
Partition polynomials are widely studied in the literature. In this work we are interested by a special case to give some recurrence relations and theirs connection to Bernoulli polynomials and numbers. By hint of Bell polynomials; we compute the explicit formula in two different ways. As a perspective for this study; we ask some questions not yet answered.

Keywords: Bell polynomials, partition polynomials, Stirling numbers, Cauchy product

Introduction

The partition polynomials $P_n(z)$ are defined by the generating function

$$f(t,z,C,a) = \prod_1 (1 - z_1 t^{n_1})^{a_1} \prod_2 (1 - z_2 t^{n_2})^{a_2} \dots \prod_s (1 - z_s t^{n_s})^{a_s} = \sum_n P_n(z) t^n \,;$$

Where
$C = \{C_1, C_2, \dots, C_s\}$, C_j denote a set of distinct integers, $z = (z_1, \dots, z_s)$ and $a = (a_1, \dots, a_s)$. The explicit formula of $P_n(z)$ is given by the relation

$$P_n(z) = \sum_{\pi(n)} \prod_{j=1}^{n} \left[\frac{1}{k_j!} \left(\frac{\varphi_{j(z)}}{j} \right)^{k_j} \right] \qquad (1)$$

Where

$$\pi(n) = \{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n;\ k_1 + 2k_2 + \dots + nk_n = n\},\ \varphi_n(z) = -\sum_{j=1}^{s} a_j \theta_n^{(j)}(z_j)$$

And

$$\theta_n^{(j)}(z_j) = \sum_{d \in C_j d | n} d z_j^{n/d}.$$

For the proof see the works [1, 10]. The partition polynomials $F_n(z)$ and $Q_n(z)$ defined respectively by the generating functions

$$\prod_{n=0}^{+\infty} \frac{1}{1 - zt^n} = 1 + \sum_{n \geq 1} F_n(z) t^n$$

And

$$\prod_{n=1}^{+\infty} \frac{1}{(1 - zt^n)^n} = 1 + \sum_{n \geq 1} Q_n(z) t^x$$

appear in the theory of plane partitions [2, 3, 4]. These polynomials are in fact special case of polynomials $P_n(z)$. Her we focus on the sequence $a_n(m), m \geq 1$ of numbers defined by $a_n(1) = 1$ and the recurrence relation:

$$a_n(m+1) = \sum_{k=0}^{n} a_k(m). \qquad (2)$$

We will see later that $a_n(m)$ belongs to a particular family of partition polynomials. Since $a_n(1) = 1$ we can see that $a_n(m)$ is a polynomial on the integer variable n of degree $n - 1$. But the explicit formula of the coefficients still unknown. The following table gives the few first polynomials.

Table 1: B_n and $B_n(x)$ are respectively the well-known Bernoulli numbers and polynomials, B_n are defined by the generating function

m	$a_n(m)$
1	1
2	$n+1$
3	$(1/2)n^2+(3/2)n+1$
4	$(1/6)(B_3(n+1)-B_3)+(3/4)n^2+(7/4)n+1$

$\frac{t}{e^t-1}=\sum_{n\geq 0} B_n \frac{t^n}{n!}$.

Polynomials $B_n(x)$ take the form

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

The first Bernoulli numbers are $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$ and $B_3=0$. So the first few Bernoulli polynomials are

$$B_0(x)=1$$

$$B_1(x)=x+B_1$$

$$B_2(x)=x^2-x+B_2$$

$$B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x.$$

In the table 1 we used the well-known identity

$$1^k+2^k+\cdots+n^k=\frac{1}{k+1}(B_{k+1}(n+1)-B_k),$$

To deduce that

$$1^2+2^2+\cdots+n^2=\frac{1}{6}(2n^3+3n^2+n)=\frac{B_3(n+1)-B_3}{3}.$$

In this work we give another recurrence relation and show that $a_n(m)$ is a polynomial of m. finally we compute the explicit formula in two different ways and ask some questions as a perspective for this study.

Generating function and recurrence relations

From the definition of the sequence $a_n(m)$; we deduce that $f_m(t)=\frac{1}{(1-t)^m}$ is the corresponding generating function and we write

$$f_m(t)=\sum_{n\geq 0} a_n(m)t^n.$$

The reason is that

$$f_{m+1}(t)=\frac{f_m(t)}{1-t}=\left(\sum_{n\geq 0} t^n\right)\left(\sum_{n\geq 0} a_n(m)t^n\right)=\sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k(m)\right)t^m.$$

$a_n(m)$ Correspond to $P_n(z)$ for $C=C_1=\{1\}$, $z=z_1=1$ and $a=a_1=-m$. After sample computations we conclude that

$$a_n(m)=\sum_{\pi(n)}\prod_{j=1}^{n}\left[\frac{1}{k_j!}\binom{m}{j}^{k_j}\right]. \quad (3)$$

For a,b two positive integers, it follows that

$$a_n(a+b)=\sum_{k=0}^{n} a_k(a)a_{n-k}(b). \quad (4)$$

In section 3 we explain how to obtain another formulation of the expression (3). Let $\binom{m}{k}$ the binomial coefficient given by $\binom{m}{k}=\frac{m!}{k!(m-k)!}$, with the convention that $\binom{m}{k}=0$ for $k>m$. The natural extension to complex numbers α is defined by $\binom{\alpha}{k}=\frac{(\alpha)_k}{k}$, where $(\alpha)_k$ is the falling number $(\alpha)_k=\alpha(\alpha-1)\ldots(\alpha-k+1)$. Consequently, the series expansion of $(1+t)^\alpha$ is

$(1+t)^{\alpha} = \sum_{n\geq 0}\binom{\alpha}{n}t^n, |t| < 1.$

Theorem 3.2: For m fixed and $n \geq 1$, the sequence $a_n(m)$ satisfies the recurrence relation

$$a_n(m) = \sum_{k=0}^{n-1}\binom{m}{n-k}(-1)^{n-k-1}\, a_k(m) \quad (5)$$

Proof: In one hand we have

$$(1-t)^m = \sum_{n\geq 0}\binom{m}{n}(-1)^n\, t^n,$$

and in other hand we have $(1-t)^m f_m(t) = 1$, then

$$\left(\sum_{n\geq 0}\binom{m}{n}(-1)^n\, t^n\right)\left(\sum_{n\geq 0} a_n(m)\, t^n\right) = 1.$$

By Cauchy product of generating functions we get

$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{m}{n-k}(-1)^{n-k} a_k(m)\right)t^n = 1$. Thus $a_0(m) = 1$ and for $n \geq 1$;

$$\sum_{k=0}^{n}\binom{m}{n-k}(-1)^{n-k}\, a_k(m) = 0.$$

Table 2: It follows from the table 2 that $a_n(m)$ is a polynomial of degree n on the variable m. One can write $a_n(m) = \sum_{k=0}^{n} \alpha_k m^k$; in what follows we purpose a method for computing the coefficients α_k.

n	$a_n(m)$
1	m
2	$(1/2)m^2 + (1/2)m$
3	$(1/6)m^3 + (1/2)m^2 + 1/3$
4	$-(1/3)m^4 + (5/3)m^3 + (1/8)m^2 + 7/12$

Combinatorial formulations

Let $s_n(k)$ be the set of all partitions of n which have k summands, for any sequence $(x_n)_{n\in\mathbb{N}}$ of complex numbers; the exponential partial Bell polynomials $B_{n,k} := B_{n,k}(x_1, x_2, \ldots)$ are defined by the expression [5].

$$B_{n,k} = \frac{n!}{k!}\sum_{s_n(k)}\binom{k}{k_1, \ldots k_{n-k+1}}\prod_{r=1}^{n-k+1}\left(\frac{x_r}{r!}\right)^{k_r} \quad (6)$$

Where $\binom{k}{k_1, \ldots, k_n} = \frac{k!}{k_1!\ldots k_n!}$ is the multinomial coefficient, with the convention that $B_{0,0} = 1$ and $B_{n,0} = 0$ for $n \geq 1$. Polynomials $B_{n,k}$ follow from the generating function

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}\,(x_1, x_2, \ldots)\frac{t^n}{n!}.$$

Among other values of $B_{n,k}$ we have $B_{n,k}(1,1,..,1) = S(n,k)$ is a Stirling number of second kinds defined by the generating function:

$$e^{[u(e^t-1)]} = 1 + \sum_{1\leq k\leq n} S(n,k)u^k\frac{t^n}{n!}.$$

$B_{n,k}(0!, 1!, 2!, \ldots) = |s(n,k)| = \mathfrak{s}(n,k)$ is signless Stirling numbers of first kind and $s(n,k)$ is a Stirling number of first kind defined by the generating function

$$(1+t)^u = 1 + \sum_{1\leq k\leq n} s(n,k)u^k\frac{t^n}{n!}.$$

$B_{n,k}(1,2,3,\ldots) = \binom{n}{k}k^{n-k}$ is idempotent number.
Bell polynomials play an important role in number theory and combinatorial analysis, such us for computing the coefficients of the inverse or the composition of generating functions. With these combinatorial polynomials we give the explicit formula of the polynomials $a_n(m)$ in two different ways.
For any generating function

$f(x,t) = \sum_{n\geq 1}\gamma_n(x)t^n$, $e^{f(t)}$ is calculated as follows [9].

$$e^{f(t)} = \sum_{n\geq 0}\sum_{k=0}^{n} B_{n,k}(1!\,\gamma_1, 2!\,\gamma_2, \ldots)\frac{t^n}{n!} \quad (7)$$

For the general case gof one consult [6, 8], to see the q-analog we refer to [7]. According to Bell polynomials the expression of $a_n(m)$ as a polynomial on the integer variable m is given in the following theorem.

Theorem 3.1: We have

$$a_n(m) = \frac{1}{n!}\sum_{k=0}^{n}|s(n,k)|m^k. \quad (8)$$

Proof: We have $f_m(t) = e^{m\sum_{n\geq 1}\frac{t^n}{n}}$, then $f_m(t) = \sum_{n\geq 0}\sum_{k=0}^{n} B_{n,k}(0!,1!,2!,\dots)\, m^k \frac{t^n}{n!}$, and the desired result follows. Consequently the coefficients α_k are signless Stirling numbers of first king.

Corollary 3.1: The connection of $a_n(m)$ to Bernoulli polynomials and numbers is given by the following relation.

$$\sum_{j=1}^{m-1} a_n(j) = \frac{1}{n!}\sum_{k=0}^{n}(B_{k+1}(m) - B_{k+1})\frac{|s(n,k)|}{k+1}. \quad (9)$$

Proof: We have

$\sum_{j=1}^{m-1} a_n(j) = \frac{1}{n!}\sum_{k=0}^{n}|s(n,k)|\left(\sum_{j=1}^{m-1} j^k\right)$, and then $\sum_{j=1}^{m-1} a_n(j) = \frac{1}{n!}\sum_{k=0}^{n}|s(n,k)|\frac{B_{k+1}(m)-B_{k+1}}{k+1}$.

Otherwise we have

$$\sum_{j=0}^{n} a_j(m) = \sum_{j=0}^{n}\left(\sum_{k=0}^{j}\frac{1}{j!}|s(j,k)|m^k\right).$$

Then

$$a_n(m+1) = \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\frac{1}{j!}|s(j,k)|\right)m^k.$$

But from Theorem 3.1 we get

$$a_n(m+1) = \frac{1}{n!}\sum_{k=0}^{n}|s(n,k)|(m+1)^k.$$

Since $(m+1)^k = \sum_{i=0}^{k}\binom{k}{i}m^i$, then

$$a_n(m+1) = \frac{1}{n!}\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i}|s(n,k)|\right)m^i,$$

which is the same with

$$a_n(m+1) = \frac{1}{n!}\sum_{k=0}^{n}\left(\sum_{j=k}^{n}\binom{j}{k}|s(n,j)|\right)m^k.$$

The comparison between the coefficients let us to conclude the following corollary.

Corollary 3.2 For every $0 \leq k \leq n$, the Stirling numbers of first kind satisfies the following identity

$$\sum_{j=k}^{n}\binom{j}{k}|s(n,j)| = n!\sum_{j=k}^{n}\frac{1}{j!}|s(j,k)|. \quad (10)$$

For any generating function $g(t) = \sum_{n\geq 0} b_n t^n$ and complex number α; the series expansion of the function g^α is given by

$$g^\alpha(t) = \sum_{n\geq 0}\sum_{k=0}^{n}(\alpha)_k b_0^{\alpha-k} B_{n,k}(1!\,b_1, 2!\,b_2,\dots)\frac{t^n}{n!}.$$

Theorem 3.2:

$$a_n(m) = (-1)^n\frac{(-m)_n}{n!}. \quad (11)$$

Proof: In the case $g(t) = 1 - t$ and $\alpha = -m$, we obtain $f_m(t) = g^{-m}(t)$. Then we deduce that

$$f_m(t) = \sum_{n\geq 0}\sum_{k=0}^{n}(-m)_k B_{n,k}(-1,0,0,0,\dots)\frac{t^n}{n!}.$$

Since $B_{n,n}(-1,0,0,0,\dots) = (-1)^n$ and $B_{n,k}(-1,0,0,0,\dots) = 0$ for $k \neq 0$. Then

$a_n(m) = (-1)^n \frac{(-m)_n}{n!}$.

Corollary 3.1: The connection between generalized binomial coefficient and Stirling numbers of first kind is given by the following relation.

$$(-m)_n = (-1)^n \sum_{k=0}^{n} |s(n,k)| m^k. \quad (12)$$

Proof: This result is immediate from the series expansion

$$(1-t)^{-m} = \sum_{n\geq 0} \binom{-m}{n} (-1)^n t^n$$

and the expression (11) of Theorem 3.1.

Let $m^{\bar{n}} = m(m+1)\dots(z+m-1)$ the rising factorial. Then $m^{\bar{n}} = (-1)^n(-m)_n$ and the well-known result

$$m^{\bar{n}} = \sum_{k=0}^{n} |s(n,k)| m^k$$

is obtained. Knowing that in the literature; sometimes the numbers $|s(n,k)|$ are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}$.

Perspective

We computed explicitly the coefficients of $a_n(m)$ as a polynomial of the integer variable m. Regarding the recurrence relation (1); $a_n(m)$ is a polynomial on the variable n. But what is the explicit formulae of the coefficients? To answer this question; we must investigate the generating function of the falling number $(-m)_n$. For a any fixed integer n, one can write $f_n(t) = \sum_{m\geq 0}(-m)_n t^m$. The values of $f_1(t)$ and $f_2(t)$ are illustrated in the table 3. But how about the general form of $f_n(t)$?

Table 3: We must investigate the generating function of the falling number $(-m)_n$. For a any fixed integer n, one can write $f_n(t) = \sum_{m\geq 0}(-m)_n t^m$. The values of $f_1(t)$ and $f_2(t)$ are illustrated

n	$f_n(t)$
1	$-t(t-1)^{-2}$
2	$2t^2(1-t)^{-3}$

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