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# Compact weighted type composition operators from $Q_K(p,q)$ spaces to Bloch-type spaces

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**Abstract**
For a non-negative integer n; the generalized weighted composition operator $D^n_{\varphi,\psi}$ is defined by $D^n_{\varphi,\psi}f = \psi\cdot\left(f^{(n)}\circ\varphi\right), f\in H(\mathbb{D})$, where $\psi$ be holomorphic map of the open unit disk $\mathbb{D}, \varphi$ a holomorphic self-map $\mathbb{D}$ and $H(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. In this paper, we characterize compactness of $D^n_{\varphi,\psi}$ from $Q_K(p,q)$ spaces to Bloch-type spaces.

**Keywords:** Compact weighted, Bloch-type spaces, $Q_K(p,q)$

**Introduction**
Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}, \partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measure on $\mathbb{D}$( i.e. $A(\mathbb{D})=1), H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. Let $0<p<\infty, -2<q<\infty$ and $K:[0,\infty]\to[0,\infty]$ a non-decreasing continuous function. A function $f\in H(\mathbb{D})$ is in $Q_K(p,q)$ if

$$M(f)=\left\{\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\right\}^{1/p}<\infty,$$

Where

$$g(z,a)=\log\left|\frac{1-\bar{a}z}{a-z}\right|=\log\frac{1}{|\eta_a(z)|}$$

is the Green function in $\mathbb{D}$ with logarithmic singularity at $a$.
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Throughout this paper, we assume that

$$\int_0^1 (1-r^2)^qK(-\log r)rdr<\infty, \tag{1}$$

since otherwise $Q_K(p,q)$ consists only of constant functions (see [19]). For $1\le p<\infty, Q_K(p,q)$ is a Banach space with respect to the norm $\parallel f\parallel_{Q_K(p,q)}=|f(0)|+M(f)$. If $K(x)=x^s, s\ge 0$, then the spaces $Q_K(p,q)$ reduces to $F(p,q,s)$ which was introduced by $R$. Zhao in [18], is known as general family of function spaces. Let $\omega$ be a strictly positive continuous function on $\mathbb{D}$. If $\omega(z)=\omega(|z|)$ for every $z\in\mathbb{D}$, we call it a radial weight. A radial weight $\omega$ is called typical if it is non-increasing with respect to $|z|$ and $\omega(z)\to 0$ as $|z|\to 1$. For a typical weight $\omega$, the Bloch-type space $\mathcal{B}_\omega$ on $\mathbb{D}$ is a Banach space of $f\in H(\mathbb{D})$ such that

$$\parallel f\parallel_{\mathcal{B}_\omega}=|f(0)|+\sup_{z\in\mathbb{D}}\omega(z)|f'(z)|.$$

The weighted spaces of controlled growth $\mathcal{A}_\omega$ consists of $f\in H(\mathbb{D})$ such that $\parallel f\parallel_{\mathcal{A}\omega}=\sup_{z\in\mathbb{D}}\omega(z)|f(z)|<\infty$.
Let $\psi\in H(\mathbb{D})$ and $\varphi$ be a holomorphic map of $\mathbb{D}$. For a non-negative integer $n$, we define a linear operator $D^n_{\varphi,\psi}$ as follows:

$D^n_{\varphi,\psi}f = \psi \cdot (f^{(n)} \circ \varphi)$, $f \in H(\mathbb{D})$.

For example, if $n = 0$ and $\psi \equiv 1$, then we obtain the composition operator $C_\varphi$ induced by $\varphi$, defined as $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. If $\psi = 1$ and $\varphi(z) = z$, then $D^n_{p,\psi} = D^n$, the differentiation operator defined as $D^n f = f^{(n)}$. If $n = 0$, then we get the weighted composition operator $\psi C_\varphi$ defined as $\psi C_\varphi f = \psi \cdot (f \circ \varphi)$.
For more about operators of the type $D^n_{p,\psi}$ we refer [1-17]. In this paper, we characterize boundedness and compactness of $D^n_{\varphi,\psi}$ from $Q_K(p,q)$ spaces to Bloch-type spaces.
Throughout this paper constants are denoted by $C$, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that $B \lesssim A \lesssim B$, where $A \lesssim B$ means that there is a positive constant $C$ such that $A \le CB$.

**Compactness of $D^n_{\varphi,\psi}$**
In this section, we characterize compactness of $D^n_{\phi,\psi}$ from $Q_K(p,q)$ spaces to Bloch-type spaces.
For the purpose we need several lemmas. The first among these can be found in [19].

**Lemma 1.** Let $0 < p < \infty, -2 < q < \infty, \alpha = (2+q)/p, K: [0,\infty] \to [0,\infty]$ a non-decreasing continuous function such that (1) holds and $\in Q_K(p,q)$. Then (a) $Q_K(p,q) \subset B^\alpha$, (b) $Q_K(p,q) = B^\alpha$ if and only if

$$\int_0^1 (1-r^2)^{-2} K(-\log r) r dr < \infty. \tag{2}$$

Moreover, $\| f \|_{\mathcal{B}^\alpha} \lesssim \| f \|_{Q_K(p,q)}$.

**Lemma 2.** Let $0 < p < \infty$ and $-2 < q < \infty, K: [0,\infty] \to [0,\infty]$ is a non-decreasing continuous function such that (1) holds and $f \in Q_K(p,q)$.
Then

$$|f(z)| \lesssim \begin{cases} \| f \|_{Q_K(p,q)} & \text{if } 2+q < p; \\ \log_2 \dfrac{2}{1-|z|^2} \| f \|_{Q_K(p,q)} & \text{if } 2+q = p; \\ \dfrac{\| f \|_{Q_K(p,q)}}{(1-|z^2)^{\frac{2+q}{p}-1}} & \text{if } 2+q > p; \end{cases}$$

And

$$\left|f^{(n)}(z)\right| \lesssim \frac{\| f \|_{Q_K(p,q)}}{(1-|z|^2)^{\frac{2+q}{p}+n-1}} \text{ if } n \ge 1.$$

The next lemma is an extension of a well known result on compactness of composition operators of the Hardy spaces $H^p$ (see [1], proposition 3.11). The proof follows using the standard arguments of proposition 3.11 in [1]. We omit the details.

**Lemma 3.** Let $0 < p < \infty$ and $-2 < q < \infty, K: [0,\infty] \to [0,\infty]$ is a non-decreasing continuous function such that (1) holds, $\omega$ a radial weight, $\psi \in H(\mathbb{D}), n \in \mathbb{N}$ or $n = 0$ and $\varphi$ be a holomorphic self-map on $\mathbb{D}$. Then $D^n_{\varphi,\psi}: Q_K(p,q) \to B_\omega$ is compact if and only if for every bounded sequence $\{f_k\}$ in $Q_K(p,q)$ which converges to zero on compact subset of $\mathbb{D}$ as $k \to \infty$, we have $\|D^n_{\varphi,\psi} f_k\|_{\mathcal{B}_\omega} \to 0$ as $k \to \infty$.
The next two lemmas are inspired from lemmas 3.6 and 3.7 in [5].

**Lemma 4.** Let $0 < p < \infty$ and $-2 < q < \infty$ such that $2+q < p$ and $K: [0,\infty] \to [0,\infty]$ be a non-decreasing continuous function such that (1) holds. Then every bounded sequence in $Q_K(p,q)$ has a subsequence that converges uniformly on $\overline{\mathbb{D}}$.
Proof. Let $\{f_m\}$ be a sequence in $Q_K(p,q)$ and $M$ be a positive number such that $\|f_m\|_{Q_K(p,q)} \le M$ for all $m \in \mathbb{N}$. Since $Q_K(p,q) \subset \mathcal{B}^{(2+q)/p}$ and $(2+q)/p < 1$, we have $|f_m(z) - f_m(w)| \lesssim |z-w|^{(p-2-q)/p}$ for all $z, w \in \mathbb{D}$) and for every $m \in \mathbb{N}$. Thus the family $\{f_m : m \in \mathbb{N}\}$ is equicontinuous. Since $\| f \|_\infty \lesssim \| f \|_{\mathcal{B}(2+q)/p} \lesssim f \|_{Q_K(p,q)}$ for every analytic $f$ on $\mathbb{D}$, the family $\{f_m : m \in \mathbb{N}\}$ is bounded in $C(\overline{\overline{\mathbb{D}}})$. Hence by Arzela-Ascoli Theorem the result follows.

**Lemma 5.** Let $0 < p < \infty$ and $-2 < q < \infty$ such that $2+q < p$, $K: [0,\infty] \to [0,\infty]$ be a non-decreasing continuous function such that (1) holds and $T: Q_K(p,q) \to X$ is a bounded linear operator, where $X$ is a normed linear space. Then $T: Q_K(p,q) \to X$ is compact if and only if $\|Tf_m\| \to 0$ for every bounded sequence $\{f_m\}$ in $Q_K(p,q)$ that converges uniformly on $\overline{\mathbb{D}}$.
Proof. If $T: Q_K(p,q) \to X$ is compact, then it is obvious that $\|Tf_m\| \to 0$ for every bounded sequence $\{f_m\}$ in $Q_K(p,q)$ that converges uniformly on $\overline{\mathbb{D}}$. Conversely, suppose $T: Q_K(p,q) \to X$ is not compact. Then there is a bounded sequence $\{f_m\}$ in $Q_K(p,q)$ such that $\{Tf_m\}$ has no convergent subsequence. By Lemma 5, $\{f_m\}$ has a subsequence $\{g_m\}$ such that $g_m \to g$ uniformly on $\overline{\mathbb{D}}$. Clearly, $g \in Q_K(p,q)$ and sequence $\{g_m - g\}$ is bounded in $Q_K(p,q)$ and converges to zero uniformly on $\overline{\mathbb{D}}$. By

assumption $\|T(g_m - g)\| \to 0$ as $m \to \infty$. Thus the subsequence $\{T(g_m)\}$ of $\{T(f_m)\}$ converges in $X$, a contradiction. Hence we are done.

**Theorem 1.** Let $0 < p < \infty$ and $-2 < q < \infty, K: [0,\infty] \to [0,\infty]$ is a non-decreasing continuous function such that (1) holds, $\omega$ a radial weight, $\psi \in H(\mathbb{D}), n \in \mathbb{N}$ or $n = 0$ and $\varphi$ be a holomorphic self-map on $\mathbb{D}$.

(1) If $n \in \mathbb{N}$, or $n = 0$ and $2 + q > p$, then $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}_\omega$ is compact if and only if $\psi \in \mathcal{B}_\omega, \psi\varphi' \in \mathcal{A}_\omega$ and

(a) $\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} = 0,$

(b) $\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\varphi'(z)\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} = 0.$

(2) If $2 + q < p$, then $D^0_{\varphi,\psi}: Q_K(p,q) \to B_\omega$ is compact if and only if $\psi \in \mathcal{B}_\omega, \psi\varphi' \in \mathcal{A}_\omega$ and

(c) $\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{\varphi}}} = 0.$

(3) If $2 + q = p$ and condition (2) of Lemma 1 holds, then $D^0_{\varphi,\psi} : Q_K(p,q) \to \mathcal{B}_\omega$ is compact if and only if $\psi \in \mathcal{B}_\omega, \psi\varphi' \in \mathcal{A}_\omega$ and

(d) $\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} = 0,$

(e) $\lim_{|\varphi(z)|\to 1} \omega(z)|\psi'(z)|\log\left(\frac{2}{1-|\varphi(z)|^2}\right) = 0.$

Proof. (1). Suppose that $n \in \mathbb{N}$, or $n = 0$ and $2 + q > p$ and $D^n_{\varphi,\psi} : Q_K(p,q) \to B_\omega$ is compact. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{\left(1-\overline{\varphi(zk)}z\right)^{\frac{2+q}{p}}} \tag{3}$$

And

$$g_k(z) = \frac{1-|\varphi(z^k)|^2}{\left(1-\overline{\varphi(zk)}z\right)^{\frac{2+q}{p}}} - \frac{2+q}{2+q+np}\left(\frac{(1-|\varphi(z_k)|^2)^2}{(1-\varphi(z_k))^{\frac{2+q}{p}+1}}\right. \tag{4}$$

Then $\{f_k\}$ and $\{g_k\}$ are norm bounded sequences in $Q_K(p,q)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By Lemma 3, we have

$$\lim_{k\to\infty} \|D^n_{\varphi,\psi} f_k\|_{\mathcal{B}_\omega} = 0 \text{ and } \lim_{k\to\infty} \|D^n_{\varphi,\psi} g_k\|_{\mathcal{B}_\omega} = 0 \tag{5}$$

Thus we have that

$$\frac{p}{2+q+np} \prod_{j=1}^{n+1} \left(\frac{2+q}{p} + j - 1\right) \left|\overline{\varphi(zk)}\right|^{n+1} \frac{\omega(z_k)|\psi(z_k)\varphi'(z_k)|}{|1-|\varphi(z_k)|^2|^{2+q+n}} \|D^n_{\varphi,\psi} g_k\|_{\mathcal{B}_\omega} \tag{6}$$

And

$$\|D^n_{\varphi,\psi} f_k\|_{\mathcal{B}_\omega} \gtrsim \quad | \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1\right) \overline{\varphi(\lambda)})^n \frac{\omega(z_k)\psi'(\lambda)}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n-1}} + \prod_{j=1}^{n+1} \left(\frac{2+q}{p} + j - 1\right) \overline{\varphi(\lambda)})^{n+1} \frac{\omega(z_k)\psi(\lambda)\varphi'(\lambda)}{(1-|\varphi(\lambda)|^2)^{\frac{2+q}{p}+n}} | \tag{7}$$

Taking $k \to \infty$ on both sides of (6) and (7) and employing (5), we get (a) and (b). Moreover, as in Theorem 1, we can prove that $\psi \in \mathcal{B}_\omega$ and $\psi\varphi' \in \mathcal{A}_\omega$. Conversely, suppose that $\psi \in \mathcal{B}_\omega, \psi\varphi' \in \mathcal{A}_\omega$ and $(a)$ and $(b)$ hold. Let $(f_k)$ be a sequence in $Q_K(p,q)$ such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ and $\|f_k\|_{Q_K(p,q)} \lesssim 1$. Then $f_k^{(n)}$ and $f_k^{(n+1)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Since $(a)$ and $(b)$ hold, for every $\epsilon > 0$, there exist $\delta \in (0,1)$ such that

$$\frac{\omega(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} < \epsilon \text{ and } \frac{\omega(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} < \epsilon,$$

when $\delta < |\varphi(z)| < 1$. Since $\psi \in \mathcal{B}, \psi\varphi' \in \mathcal{A}^1_\infty$, we have $N_1 = \sup_{z\in\mathbb{D}} \omega(z)|\psi'(z)| < \infty$ and $N_2 = \sup_{z\in\mathbb{D}} \omega(z)|\psi(z)\varphi'(z)| < \infty$. This together with Lemma 3 yields

$$\begin{aligned}\|D^n_{\varphi,\psi} f_k\|_{\mathcal{B}_\omega} &\le \sup_{z\in\mathbb{D}} \omega(z)\left|\left(D^n_{\varphi,\psi} f_k\right)'(z)\right| \\ &\le \sup_{z\in\mathbb{D}} \omega(z)\left|\psi'(z) f_k^{(n)}(\varphi(z))\right| + \sup_{z\in\mathbb{D}} \omega(z)\left|\psi(z)\varphi'(z) f_k^{(n+1)}(\varphi(z))\right| \\ &\le \sup_{|\varphi(z)|\le\delta} \omega(z)|\psi'(z)|\left|f_k^{(n)}(\varphi(z))\right| + \sup_{|\varphi(z)|>\delta} \omega(z)|\psi'(z)|\left|f_k^{(n)}(\varphi(z))\right| \\ &+ \sup_{|\varphi(z)|\le\delta} \omega(z)|\psi(z)\varphi'(z)|\left|f_k^{(n+1)}(\varphi(z))\right| \\ &+ \sup_{|\varphi(z)|>\delta} \omega(z)|\psi(z)\varphi'(z)|\left|f_k^{(n+1)}(\varphi(z))\right| \\ &\le N_1 \sup_{|\varphi(z)|\le\delta} \left|f_k^{(n)}(\varphi(z))\right| + \sup_{|\varphi(z)|>\delta} \frac{\omega(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}}\end{aligned}$$

$$+N_2 \sup_{|\varphi(z)|\le\delta} \left|f_k^{(n+1)}(\varphi(z))\right| + \sup_{|\varphi(z)|>\delta} \frac{\omega(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}}$$

$$\le N_1 \sup_{|\varphi(z)|\le\delta} \left|f_k^{(n)}(\varphi(z))\right| + N_2 \sup_{|\varphi(z)|\le\delta} \left|f_k^{(n+1)}(\varphi(z))\right| + 2\epsilon$$

From the arbitrariness of $\epsilon$, and the fact that $f_k^{(n)}$ and $f_k^{(n+1)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\lim_{k\to\infty} \|D^n_{\varphi,\psi} f_k\|_{\mathcal{B}_\omega} = 0$ and so by Lemma 4, $D^n_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}_\omega$ is compact.

(2). The necessity in condition (2) can been proved in the same way as the proof above. We omit the details. Assume that $2+q < p, \psi \in \mathcal{B}_\omega$, $\psi\varphi' \in \mathcal{A}_\omega$ and (c) holds. Let $\{f_m\}$ be a bounded sequence in $Q_K(p,q)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. To show that $D^0_{\varphi,\psi}: Q_K(p,q) \to B_\omega$ is compact, we need to show that if $M = \sup_m \|f_m\|_{Q_K(p,q)} < \infty$ and $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|D^0_{\varphi,\psi} f_m\|_{\mathcal{B}_\omega} \to 0$ as $m \to \infty$. This amounts to show that

$$\sup_{z\in\mathbb{D}} \omega(z)|\psi(z)\varphi'(z) f_m'(\varphi(z))| \to 0 \tag{8}$$

And

$$\sup_{z\in\mathbb{D}} \omega(z)|\psi'(z) f_m(\varphi(z))| \to 0 \tag{9}$$

as $n \to \infty$. By Lemma 6, we have $\sup_{z\in\bar{D}} |f_m(z)| \to 0$ as $n \to \infty$. So

$$\sup_{z\in\mathbb{D}} \omega(z)|\psi'(z) f_m(\varphi(z))| \le \| \psi \|_{\mathcal{B}} \sup_{z\in\bar{\mathbb{D}}} |f_m(z)| \to 0$$

as $n \to \infty$. Thus (33) holds. If $|\varphi(z)| \le r < 1$, then

$$\omega(z)|\psi(z)\varphi'(z) f_m'(\varphi(z))| \le \|\psi\varphi'\|_{\mathcal{A}_\omega} \max_{|z|\le r} |f_m'(z)|$$

If $|\varphi(z)| > r$, then

$$\omega(z)|\psi(z)\varphi'(z) f_m'(\varphi(z))| \le \frac{\omega(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}}.$$

Thus

$$\begin{aligned}\sup_{z\in\mathbb{D}} \omega(z)|\psi(z)\varphi'(z) f_m'(\varphi(z))| \le \quad & \| \psi\varphi' \|_{\mathcal{A}_\omega} \max_{|z|\le r} |f_m'(z)| \\ & + \sup_{|\varphi(z)|>r} \frac{\omega(z)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}}.\end{aligned} \tag{10}$$

First letting $n \to \infty$ and subsequently $r$ increases to 1, in (10) obtains that (8) holds. (3). To prove (3), we assume that $\{z_m\}$ is a sequence in ID for which $|\varphi(z_m)| \to 1$ and consider the function

$$g_m(z) = \frac{3}{\mu_m}\left(\log\frac{2}{1-\overline{\varphi(z_m)}z}\right)^3 - \frac{2}{\mu_m^2}\left(\log\frac{2}{1-\overline{\varphi(z_m)}z}\right)^3$$

where $\mu_m = \log\frac{2}{1-|\varphi(z_m)|^2}$. Then

$$g_m'(z) = \frac{2\overline{\varphi(z_m)}}{1-\overline{\varphi(z_m)}z}\left\{\frac{1}{\mu_m}\left(\log\frac{2}{1-\overline{\varphi(z_m)}}\right) - \frac{1}{\mu_m^2}\left(\log\frac{2}{1-\overline{\varphi(z_m)}z}\right)^2\right\}.$$

By a direct calculation, we have $\sup_m\|g_m\|_{\mathcal{B}} \leq C < \infty$. Since condition (2) of Lemma 1 holds, we have $g_m \in Q_K(p,q)$ and $\sup_m\|f_m\|_{Q_K(p,q)}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Note that

$$g_m'(\varphi(z_m)) = 0 \text{ and } g_m(\varphi(z_m)) = \log\left(\frac{2}{1-|\varphi(z_m)|}\right).$$

So if $D^0_{\varphi,\psi}: Q_K(p,q) \to \mathcal{B}_\omega$ is compact, then $\|D^0_{\varphi,\psi}g_m\|_{\mathcal{B}_\omega} \to 0$, and consequently,

$$\omega(z_m)|\psi'(z_m)|\log\left(\frac{2}{1-|\varphi(z_m)|^2}\right) = \omega(z_m)\left|\left(D^n_{\varphi,\psi}g_m\right)'(z_m)\right| \to 0$$

as $n \to \infty$. Conversely, suppose that $\{f_m\}$ is a bounded sequence in $Q_K(p,q)$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. Let $M = \sup_m\|f_m(z)\|_{Q_K(p,q)} < \infty$. By Lemma 2, we have

$$\sup_m|f_m(z)| \leq CM\log\frac{2}{1-|z|^2}.$$

If $|\varphi(z)| \leq r < 1$, then

$$\omega(z)|\psi'(z)f_m(\varphi(z))| \leq \| \psi \|_{\mathcal{B}_\omega} \max_{|z|\leq r}|f_m(z)|$$

If $|\varphi(z)| > r$, then

$$\omega(z)|\psi'(z)f_m(\varphi(z))| \leq CM\omega(z)|\psi'(z)|\log\frac{2}{1-|\varphi(z)|^2}$$

Thus

$$\sup_{z\in\mathbb{D}}\omega(z)|\psi'(z)f_m(z)| \leq \| \psi \|_{\mathcal{B}_\omega} \max_{|z|\leq r}|f_m(z)| + CM\omega(z)|\psi'(z)|\log\frac{2}{1-|\varphi(z)|^2} \to 0$$

as $n \to \infty$. Similarly, we can prove that

$$\sup_{z\in\mathbb{D}}\omega(z)|\psi(z)\varphi'(z)f_m'(z)| \to 0$$

as $n \to \infty$. Hence $\|D^0_{\phi,\psi}f_m\|_{B,w} + 0$ and this completes the proof

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