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Well posedness of solutions for volterra-type integrodifferential equations in Hilbert spaces

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Abstract
The aim of this paper is to prove the well posedness of mild and strong solutions of a Volterra integrodifferential equation with nonlocal condition. Our analysis is based on semigroup theory and Banach fixed point theorem and inequalities are established by Gronwall and B. G. Pachpatte.

Keywords: volterra integro-differential, strong solutions, continuous dependence, Pachpatte's inequality, nonlocal condition

Introduction
Suppose $(H, \langle\ ,\ \rangle)$ be a Hilbert space and Y$= C([t_0, t_0+\alpha]; H)$ be a Hilbert space of all continuous functions from $[t_0, t_0+\alpha]$ into H, endowed with the inner product

$$\langle x, x\rangle_Y = sup\{\langle x(t), x(t)\rangle : x \in Y\}, \quad 0 \le t_0 \le t_0 + \alpha.$$

In this paper we study the well posedness of mild and strong solutions of a nonlocal problem of the type:

$$x(t)' + Ax(t) = f\left(t, x(t), \int_{t_0}^{t} h(t, s, x(s))ds\right), \quad t \in [t_0, t_0+\alpha], \qquad (1.1)$$

$$x(t_0) + g(x) = x_0, \qquad (1.2)$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t),\ t \ge 0$, on a Hilbert space H and the nonlinear operators $f: [t_0, t_0+\alpha] \times H \times H \to H,\ g: Y \to H$, h$: [t_0, t_0+\alpha] \times [t_0, t_0+\alpha] \times H \to H$ are continuous and x_0 is a given element of H. In this paper, $B_r(z) = \{z: \langle z, z\rangle \le r\}$ denotes the closed ball in H, Z$= C([t_0, t_0+\alpha]; B_r(z))$ denotes the complete metric space with metric $d(x, y) = \langle (x-y), (x-y)\rangle_Z^{1/2} = \sup_{t\in[t_0, t_0+\alpha]} \left\{\langle (x(t)-y(t)), (x(t)-y(t))\rangle^{1/2} : x, y \in Z\right\}$.

The equation of these type or their special forms with a classical condition or boundary conditions have been studied by many authors, using different techniques, see [5, 8, 9].
The nonlocal condition, which is a generalization of the classical initial, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski [3]. Existence of mild, strong and classical solutions for differential and integrodifferential equations in abstract spaces with nonlocal conditions has received much attention in recent years. We refer to the papers of Byszewski [3, 4], Balachandran and Chandrasekaran [1], K. Balachandran [2] and Y. Lin and J. H. Liu [6]. Here our approach the conditions on functions are different from those in [2].
The objective of the present paper, we study the existence of mild and strong solutions of the problem $(1.1) - (1.2)$. The main tool employed in our analysis is based on the Banach fixed point theorem and the results are obtained by using semigroup theory.
The paper is organized as follows. In section 2, we present the preliminaries and hypotheses, Section 3 deals with main results and in Section 4, we discuss the continuous dependence of the solution.

Preliminaries and main results
Before proceeding to the main results, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 1. A continuous solution x of the integral equation

$$x(t) = T(t-t_0)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s) f\left(s, x(s), \int_{t_0}^{s} h(s,\tau,x(\tau))d\tau\right) ds \quad (1.3)$$

$t \in [t_0, t_0+\alpha]$,

is said to be a mild solution of problem $(1.1)-(1.2)$ on $[t_0, t_0+\alpha]$.

Definition 2. A function x is said to be a strong solution of problem $(1.1)-(1.2)$ on $[t_0, t_0 + \alpha]$ if x is differentiable almost everywhere on $[t_0, t_0+\alpha]$, $x' \in L^1([t_0, t_0+\alpha], H)$, $x(t_0) + g(x) = x_0$ and $x(t)' + Ax(t) = f\left(t, x(t), \int_{t_0}^{t} h(t,s,x(s))ds\right)$, $t \in [t_0, t_0+\alpha]$.

Let us denote

$$M = \max_{t\in[0,\alpha]} \langle T(t), T(t)\rangle,$$

$$L_1 = \max_{t_0\le t\le t_0+\alpha} \langle f(t,0,0), f(t,0,0)\rangle,$$

$$K_1 = \max_{t_0\le s,t\le t_0+\alpha} \langle h(t,s,0), h(t,s,0)\rangle,$$

$$G_1 = \max_{x\in Y} \langle g(x), g(x)\rangle.$$

We list the following hypotheses for our

(H_1) $g: Z \subset Y \to H$ and there exists a constant $G > 0$ such that

$$\langle (g(x_1)-g(x_2)), (g(x_1)-g(x_2))\rangle \le G\langle (x_1(t)-x_2(t)), (x_1(t)-x_2(t))\rangle_Z$$

for $x_1, x_2 \in Z$.

(H_2) $-A$ is the infinitesimal generator of a C_0 semigroup $T(t), t \ge 0$ on H.

(H_3) The constants $\langle x_0, x_0\rangle$, r, α, L_f^1, L_f^2, K, M, N, L_1, K_1, G and G_1 satisfy the following conditions:

$$M[\langle x_0, x_0\rangle + G_1 + L_f^1 r\alpha + L_f^2 K r\alpha^2 + L_f^2 K_1 \alpha^2 + L_1\alpha] \le r,$$

and

$$[MG + ML_f^1\alpha + ML_f^2 K\alpha^2] < 1.$$

Existence of mild and strong solutions
Now we first prove the result of existence of mild solutions.

(i) hypotheses $(H_1)-(H_3)$ hold,

(ii) $f:[t_0,t_0+\alpha]\times H\times H\to H$ is continuous and there exist constants $L_f^1, L_f^2>0$ such that$\langle (f(t,x_1,y_1)-f(t,x_2,y_2)),(f(t,x_1,y_1)-f(t,x_2,y_2))\rangle \le L_f^1\langle (x_1-x_2),(x_1-x_2)\rangle + L_f^2\langle (y_1-y_2),(y_1-y_2)\rangle$,

(iii) h: $[t_0,t_0+\alpha]\times[t_0,t_0+\alpha]\times H\to H$ is continuous and there exists constant $K>0$ such that

$\langle (h(t,s,x_1)-h(t,s,x_2)),(h(t,s,x_1)-h(t,s,x_2))\rangle \le K\langle (x_1-x_2),(x_1-x_2)\rangle$.

Then problem $(1.1)-(1.2)$ has a unique mild solution on $[t_0,t_0+\alpha]$.

Proof. Define an operator B: $Z\to Z$ by

$$(Bz)(t)=T(t-t_0)[x_0-g(z)]+\int_{t_0}^{t}T(t-s)f\left(s,z(s),\int_{t_0}^{s}h(s,\tau,z(\tau))d\tau\right)ds \qquad (1.4)$$

$t\in[t_0,t_0+\alpha]$.

Further, from our assumptions, we have

$$\langle (Bz)(t),(Bz)(t)\rangle \le M[\langle x_0,x_0\rangle+G_1]+\int_{t_0}^{t}M\left\langle f\left(s,z(s),\int_{t_0}^{s}h(s,\tau,z(\tau))d\tau\right),f\left(s,z(s),\int_{t_0}^{s}h(s,\tau,z(\tau))d\tau\right)\right\rangle ds$$

$$\le M[\langle x_0,x_0\rangle+G_1]+M\int_{t_0}^{t}[\langle\{f\left(s,z(s),\int_{t_0}^{s}h(s,\tau,z(\tau))d\tau\right)-f(s,0,0)\},\{f\left(s,z(s),\int_{t_0}^{s}h(s,\tau,z(\tau))d\tau\right)-f(s,0,0)\}\rangle+\langle f(s,0,0),f(s,0,0)\rangle]ds$$

$$\le M[\langle x_0,x_0\rangle+G_1]+M\int_{t_0}^{t}[L_f^1\langle z(s),z(s)\rangle+L_f^2\int_{t_0}^{s}\langle\{h(s,\tau,z(\tau))-h(s,\tau,0)$$

$$+h(s,\tau,0)\},\{h(s,\tau,z(\tau))-h(s,\tau,0)+h(s,\tau,0)\}\rangle d\tau+L_1]ds$$

$$\le M[\langle x_0,x_0\rangle+G_1]+M\int_{t_0}^{t}[L_f^1r+L_f^2Kr(s-t_0)+L_f^2K_1(s-t_0)+L_1]ds$$

$$\le M[\langle x_0,x_0\rangle+G_1]+M[L_f^1r\alpha+L_f^2Kr\alpha^2+L_f^2K_1\alpha^2+L_1\alpha]$$

$$\le r, \qquad (1.5)$$

for $z\in Z$. The equation (1.5) shows that the operator B maps Z into itself.

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Now for every $z_1,z_2\in Z$ and $t\in[t_0,t_0+\alpha]$, we obtain

$$\langle ((Bz_1)(t)-(Bz_2)(t)),((Bz_1)(t)-(Bz_2)(t))\rangle$$

$$\leq \langle T(t-t_0), T(t-t_0)\rangle\langle (g(z_1)-g(z_2)), (g(z_1)-g(z_2))\rangle$$

$$+ \int_{t_0}^{t}\langle T(t-s)\left[f\left(s, z_1(s), \int_{t_0}^{s} h(s,\tau,z_1(\tau))d\tau\right) - f\left(s, z_2(s), \int_{t_0}^{s} h(s,\tau,z_2(\tau))d\tau\right)\right],$$

$$T(t-s)\left[f\left(s, z_1(s), \int_{t_0}^{s} h(s,\tau,z_1(\tau))d\tau\right) - f\left(s, z_2(s), \int_{t_0}^{s} h(s,\tau,z_2(\tau))d\tau\right)\right]\rangle ds$$

$$\leq MG\langle (z_1-z_2), (z_1-z_2)\rangle_Y + M\int_{t_0}^{t}[L_f^1\langle (z_1-z_2), (z_1-z_2)\rangle_Y$$

$$+L_f^2K\langle (z_1-z_2), (z_1-z_2)\rangle_Y (s-t_0)]ds$$

$$\leq MG\langle (z_1-z_2), (z_1-z_2)\rangle_Y + M[L_f^1(t-t_0) + L_f^2K\alpha^2]\langle (z_1-z_2), (z_1-z_2)\rangle_Z$$

$$\leq [MG + ML_f^1\alpha + ML_f^2K\alpha^2]\langle (z_1-z_2), (z_1-z_2)\rangle_Z. \quad (1.6)$$

If we take $q = MG + ML_f^1\alpha + ML_f^2K\alpha^2$, then

$$\langle (Bz_1 - Bz_2), (Bz_1 - Bz_2)\rangle_Z \leq q\langle (z_1-z_2), (z_1-z_2)\rangle_Z$$

with $0 < q < 1$. This shows that the operator B is a contraction on the complete metric space Z. By the Banach fixed point theorem, the function B has a unique fixed point in the space Z and this point is the mild solution of problem $(1.1)-(1.2)$ on $[t_0, t_0+\alpha]$.

Next we prove that the problem $(1.1)-(1.2)$ has a strong solution.

Theorem 2. Suppose that

(i) hypotheses $(H_1)-(H_3)$ hold,

(ii) $(H, \langle\, , \rangle)$ is a reflexive Hilbert space and $x_0 \in D(A)$, the domain of A,

(iii) $g(\cdot) \in D(A)$,

(iv) $f: [t_0, t_0+\alpha] \times H \times H \to H$ is continuous and there exist constants $L_f^i > 0,\ i = 1, 2$ and $L_t > 0$ such that

$$\langle (f(t_1,x_1,y_1) - f(t_2,x_2,y_2)), (f(t_1,x_1,y_1) - f(t_2,x_2,y_2))\rangle$$

$$\leq L_t|t_1-t_2| + L_f^1\langle (x_1-x_2), (x_1-x_2)\rangle + L_f^2\langle (y_1-y_2), (y_1-y_2)\rangle,$$

(v) h: $[t_0, t_0+\alpha] \times [t_0, t_0+\alpha] \times H \to H$ is continuous and there exist constants $K_t, K > 0$ such that

$$\langle (h(t_1,s,x_1) - h(t_2,s,x_2)), (h(t_1,s,x_1) - h(t_2,s,x_2))\rangle$$

$$\leq K_t|t_1-t_2| + K\langle (x_1-x_2), (x_1-x_2)\rangle.$$

Then x is a unique strong solution of problem $(1.1)-(1.2)$ on $[t_0, t_0+\alpha]$.

Proof. Since all the assumptions of Theorem 1 are satisfied then the problem $(1.1)-(1.2)$ has a unique mild solution belonging to Z which we denote it by $u.$ Now we will show that u is unique strong solution of problem $(1.1)-(1.2)$ on $[t_0, t_0+\alpha]$.

Take

$$L_2 = \max_{t_0 \le t \le t_0+\alpha} \langle f(t,u(t),0), f(t,u(t),0)\rangle,$$

$$K_2 = \max_{t_0 \le s,\ t \le t_0+\alpha} \langle h(t,s,u(s)), h(t,s,u(s))\rangle.$$

Then for any $t \in [t_0, t_0+\alpha]$ and $\theta \in \mathbb{R}$ with $t+\theta \in [t_0, t_0+\alpha]$, we obtain

$$u(t+\theta)-u(t) = T(t-t_0)[T(\theta)-I]x_0 - T(t-t_0)[T(\theta)-I]g(u)$$

$$-T(t+\theta-t_0)[g(u(t+\theta))-g(u(t))] + \int_{t_0}^{t_0+\theta} T(t+\theta-s)$$

$$\times\left[f\left(s,u(s),\int_{t_0}^{s} h(s,\tau,u(\tau))d\tau\right) - f(s,u(s),0) + f(s,u(s),0)\right] ds$$

$$+\int_{t_0}^{t} T(t-s) f\left(s+\theta, u(s+\theta), \int_{t_0}^{s+\theta} h(s+\theta,\tau,u(\tau))d\tau\right) ds$$

$$-\int_{t_0}^{t} T(t-s) f\left(s,u(s),\int_{t_0}^{s} h(s,\tau,u(\tau))d\tau\right) ds$$

$$= T(t-t_0)[T(\theta)-I]x_0 - T(t-t_0)[T(\theta)-I]g(u)$$

$$-T(t+\theta-t_0)[g(u(t+\theta))-g(u(t))] + \int_{t_0}^{t_0+\theta} T(t+\theta-s)$$

$$\times\left[f\left(s,u(s),\int_{t_0}^{s} h(s,\tau,u(\tau))d\tau\right) - f(s,u(s),0) + f(s,u(s),0)\right] ds$$

$$+\int_{t_0}^{t} T(t-s)[f\left(s+\theta, u(s+\theta), \int_{t_0}^{s+\theta} h(s+\theta,\tau,u(\tau))d\tau\right)$$

$$-f\left(s,u(s),\int_{t_0}^{s} h(s,\tau,u(\tau))d\tau\right)]ds, \qquad (1.7)$$

where I is the identity operator

Using our assumptions and equation $(1.7),$ we observe that

$$\langle (u(t+\theta)-u(t)), (u(t+\theta)-u(t))\rangle$$

$$\le M\theta\langle Ax_0, Ax_0\rangle + M\theta\langle Ag(u), Ag(u)\rangle + MG\langle (u(t+\theta)-u(t)), (u(t+\theta)-u(t))\rangle$$

$$+\int_{t_0}^{t_0+\theta} M[L_t|s-s| + L_f^1\langle (u(s)-u(s)), (u(s)-u(s))\rangle$$

$$+L_f^2\langle \int_{t_0}^{s}(h(s,\tau,u(\tau))d\tau - 0),\ \int_{t_0}^{s}(h(s,\tau,u(\tau))d\tau - 0)\rangle + L_2]ds$$

$$+\int_{t_0}^{t} M[L_t|s+\theta-s| + L_f^1\langle(u(s+\theta)-u(s)),(u(s+\theta)-u(s))\rangle$$

$$+L_f^2\langle\left(\int_{t_0}^{s} h(s+\theta,\tau,u(\tau))d\tau - \int_{t_0}^{s} h(s,\tau,u(\tau))d\tau\right),\left(\int_{t_0}^{s} h(s+\theta,\tau,u(\tau))d\tau - \int_{t_0}^{s} h(s,\tau,u(\tau))d\tau\right)\rangle$$

$$+L_f^2\langle\int_{s}^{s+\theta} h(s+\theta,\tau,u(\tau))d\tau, \int_{s}^{s+\theta} h(s+\theta,\tau,u(\tau))d\tau,\rangle]ds$$

$$\leq \frac{M\theta\langle Ax_0,Ax_0\rangle+M\theta\langle Ag(u),Ag(u)\rangle}{1-MG} + \frac{1}{1-MG}\int_{t_0}^{t_0+\theta} M\left[L_f^2\int_{t_0}^{s} K_2 d\tau + L_2\right]ds$$

$$+\frac{1}{1-MG}\int_{t_0}^{t} M[L_t\theta + L_f^1\langle(u(s+\theta)-u(s)),(u(s+\theta)-u(s))\rangle$$
$$+L_f^2\int_{t_0}^{s}[K_t|s+\theta-s| + K\langle(u(\tau)-u(\tau)),(u(\tau)-u(\tau))\rangle]d\tau + L_f^2\int_{s}^{s+\theta} K_2 d\tau]ds$$

$$\leq \frac{M\theta\langle Ax_0,Ax_0\rangle+M\theta\langle Ag(u),Ag(u)\rangle}{1-MG} + \frac{1}{1-MG}M\left[L_f^2K_2\alpha\theta + L_2\theta\right]$$

$$+\frac{1}{1-MG}\int_{t_0}^{t} M\left[L_t\theta + L_f^1\langle(u(s+\theta)-u(s)),(u(s+\theta)-u(s))\rangle + L_f^2K_t\theta\alpha + L_f^2K_2\theta\right]ds$$

$$\leq \frac{M\theta\langle Ax_0,Ax_0\rangle+M\theta\langle Ag(u),Ag(u)\rangle}{1-MG} + \frac{ML_f^2K_2\theta\alpha+ML_2\theta+ML_t\theta\alpha+ML_f^2K_t\theta\alpha^2+ML_f^2K_2\theta\alpha}{1-MG}$$

$$+\frac{ML_f^1}{1-MG}\int_{t_0}^{t}\langle(u(s+\theta))-u(s),(u(s+\theta))-u(s)\rangle ds$$

$$\leq P\theta + Q\int_{t_0}^{t}\langle(u(s+\theta))-u(s),(u(s+\theta))-u(s)\rangle ds, \qquad (1.8)$$

where $P = \frac{M\langle Ax_0,Ax_0\rangle+M\langle Ag(u),Ag(u)\rangle+ML_f^2K_2\alpha+ML_2+ML_t\alpha+ML_f^2K_t\alpha^2+ML_f^2K_2\alpha}{1-MG}$ and $Q = \frac{ML_f^1}{1-MG}$.

Using Gronwall' s inequality (with $c = P\theta$), we get

$$\langle(u(t+\theta)-u(t)),(u(t+\theta)-u(t))\rangle \leq P\theta e^{Q\alpha}, \quad \text{for} \quad t\in[t_0,t_0+\alpha].$$

Therefore, u is Lipschitz continuous on $[t_0,t_0+\alpha]$. The Lipschitz continuity of u on $[t_0,t_0+\alpha]$ combined with (iv) and (v) of Theorem 2, gives that

$$t \to f\left(t,u(t),\int_{t_0}^{t} h(t,s,u(s))ds\right)$$

is Lipschitz continuous on $[t_0,t_0+\alpha]$. By Theorem 4.2.11 of [9], we observe that the equation

$$y(t)' + Ay(t) = f\left(t, u(t), \int_{t_0}^{t} h(t,s,u(s))ds\right), t \in [t_0, t_0+\alpha]$$

$$y(t_0) = u_0 - g(u)$$

has a unique strong solution $y(t)$ on $[t_0, t_0+\alpha]$ satisfying the equation

$$y(t) = T(t-t_0)u_0 - T(t-t_0)g(u) + \int_{t_0}^{t} T(t-s)f\left(s, u(s), \int_{t_0}^{t} h(s,\tau,u(\tau))d\tau\right)ds$$

$$= u(t), \quad t \in [t_0, t_0+\alpha].$$

Consequently, $u(t)$ is the strong solution of problem $(1.1)-(1.2)$ on $[t_0, t_0+\alpha]$.

We require the following Lemma known as the Pachpatte's inequality in our further discussion.

Lemma 1. (see, [7], p. 758) Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}^+$, for which the inequality

$$u(t) \le u_0 + \int_0^t p(s)\left[u(s) + \int_0^s q(\tau)u(\tau)d\tau\right]ds,$$

holds for all $t \in \mathbb{R}^+$, where u_0 is a nonnegative constant, then

$$u(t) \le u_0\left[1 + \int_0^t p(s)exp\left(\int_0^s (p(\tau)+q(\tau))d\tau\right)\right]ds,$$

holds for all $t \in \mathbb{R}^+$.

Continuous dependence of a mild solution.

Theorem 3. Assume that the hypotheses $(H_1)-(H_3)$ hold and $x_0 \in H$. Assume that the functions x_1 and x_2 satisfy the equation (1.1) for $t_0 \le t \le t_0+\alpha$ with $x_1(t_0) + g(x_1) = x_0^*$ and $x_2(t_0) + g(x_2) = x_0^{**}$ respectively, and $x_1(t), x_2(t) \in Y$, then they are continuously dependent.

Proof. Suppose that the functions x_1 and x_2 satisfy the equation (1.1) for $t_0 \le t \le t_0+\alpha$ with $x_1(t_0) + g(x_1) = x_0^*$ and $x_2(t_0) + g(x_2) = x_0^{**}$ respectively. Then by 1.4; we obtain

$$x_1(t) = T(t-t_0)[x_0^* - g(x_1)] + \int_{t_0}^{t} T(t-s)f\left(s, x_1(s), \int_{t_0}^{s} h(s,\tau,x_1(\tau))d\tau\right)ds, \qquad (1.9)$$

and

$$x_2(t) = T(t-t_0)[x_0^{**} - g(x_2)] + \int_{t_0}^{t} T(t-s)f\left(s, x_2(s), \int_{t_0}^{s} h(s,\tau,x_2(\tau))d\tau\right)ds, \qquad (1.10)$$

From (1.9), (1.10) and hypotheses, we obtain for $t_0 \le t \le t_0+\alpha$

$$\langle (x_1(t)-x_2(t)), (x_1(t)-x_2(t)) \rangle$$

$$\le \langle T(t-t_0), T(t-t_0)\rangle [\langle (x_0^* - x_0^{**}), (x_0^* - x_0^{**})\rangle + \langle (g(x_1)-g(x_2)), (g(x_1)-g(x_2))\rangle]$$
$$+ \int_{t_0}^{t} \langle T(t-s), T(t-s)\rangle \langle \left[f\left(s, x_1(s), \int_{t_0}^{s} h(s,\tau,x_1(\tau))d\tau - f\left(s, x_2(s), \int_{t_0}^{s} h(s,\tau,x_2(\tau))d\tau\right)\right)\right],$$
$$\left[f\left(s, x_1(s), \int_{t_0}^{s} h(s,\tau,x_1(\tau))d\tau - f\left(s, x_2(s), \int_{t_0}^{s} h(s,\tau,x_2(\tau))d\tau\right)\right)\right]\rangle ds$$

$$\leq M\langle(x_0^* - x_0^{**}), (x_0^* - x_0^{**})\rangle + MG\langle(x_1(t) - x_2(t)), (x_1(t) - x_2(t))\rangle$$

$$+M \int_{t_0}^{t} \Big[L_f^1\langle(x_1(s) - x_2(s)), (x_1(s) - x_2(s))\rangle + L_f^2 K \int_{t_0}^{s} \langle(x_1(\tau) - x_2(\tau)), (x_1(\tau) - x_2(\tau))\rangle d\tau\Big] ds$$

$$\langle(x_1(t) - x_2(t)), (x_1(t) - x_2(t))\rangle$$

$$\leq \frac{M}{1-MG}\langle(x_0^* - x_0^{**}), (x_0^* - x_0^{**})\rangle + \int_{t_0}^{t} \frac{ML_f^1}{1-MG}\Big[\langle(x_1(s) - x_2(s)), (x_1(s) - x_2(s))\rangle + \int_{t_0}^{s} \frac{L_f^2 K}{L_f^1}\langle(x_1(\tau) - x_2(\tau)), (x_1(\tau) - x_2(\tau))\rangle d\tau\Big] ds. \qquad (1.11)$$

Now, applying Lemma 1 with $u(t) = \langle(x_1(t) - x_2(t)), (x_1(t) - x_2(t))\rangle$ to the inequality (1.11), we get

$$\langle(x_1(t) - x_2(t)), (x_1(t) - x_2(t))\rangle$$

$$\leq \frac{M}{1-MG}\langle(x_0^* - x_0^{**}), (x_0^* - x_0^{**})\rangle\Big[1 + \int_{t_0}^{t} \frac{ML_f^1}{1-MG} exp\Big(\int_{t_0}^{s}\Big[\frac{ML_f^1}{1-MG} + \frac{L_f^2 K}{L_f^1}\Big] d\tau\Big)\Big] ds$$

$$\leq \frac{M}{1-MG}\langle(x_0^* - x_0^{**}), (x_0^* - x_0^{**})\rangle\Big[1 + \frac{M(L_f^1)^2}{M(L_f^1)^2 + (1-MG)L_f^2 K} exp\Big(\Big[\frac{ML_f^1}{1-MG} + \frac{L_f^2 K}{L_f^1}\Big]\alpha\Big)\Big].$$

This proves the theorem.

Conclusion

The paper proved the well posedness of mild and strong solutions of a Volterra integrodifferential equations (1.1) – (1.2) with nonlocal condition. The paper also discussed the Banach fixed point theorem by using semigroup theory. In this way one can prove the well posedness of mild and weak solutions of these equations with nonlocal condition.

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