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Combinatorial study of 2-iterated 2D-Appell polynomials and related polynomials

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Abstract
In this paper we are interested by 2-iterated 2D-Appel polynomials, Using Bell polynomials, we revisit the proof of the explicit formulae and provide additional recurrence relation to these known in the literature. We end this study by the application of these results to 2D-Euler-Appell and 2D-Hermite-Appell polynomials.

Keywords: Bell polynomials, 2-iterated 2D-Appel polynomials, Gould-Hopper polynomials, 2D-Euler-Appell polynomials, 2D-Hermite-Appell polynomials

Introduction

Let $n \geq 1$ be an integer, a partition of n is a representation of n as sum of integers $k_j \geq 1$ without considering the order of terms of this sum [1]. These terms are called summands of the partition. Consequently we can write

$$n = k_1 + k_2 + \cdots + k_m, k_1 \geq k_2 \geq \cdots \geq k_m \geq 1.$$

Giving a solution of (1) , is equivalent to giving a solution with $k_j \geq 0$ of

$$n = k_1 + 2k_2 + \cdots + nk_n.$$

If the partition has k summands we add the following condition

$$k = k_1 + k_2 + \cdots + k_n.$$

Let $p(n)$ be the number of partitions of n and let $P(n,k)$ be the number of partitions of n with m summands. It is clear that $p(n) = \sum_{k=1}^{n} P(n,k)$. For example we have $p(4) = 5, p(5) = 7$, $P(5,4) = 1$ and $P(5,2) = 2$. Let $s_n(k)$ be the set of all partitions of n which have k summands, then the cardinal of $s_n(k)$ is $P(n,k)$. For any sequence $(x_n)_{n\in\mathbb{N}}$ of complex numbers, where $\mathbb{N}$ is the set of positive integer. The exponential partial Bell polynomials $B_{n,k} := B_{n,k}(x_1, x_2, \ldots)$ are defined by the expression [1].

$$B_{n,k} = \frac{n!}{k!}\sum_{\pi_n(k)} \binom{k}{k_1, \ldots k_{n-k+1}} \prod_{r=1}^{n-k+1} \left(\frac{x_r}{r!}\right)^{k_r}. \quad (1)$$

Where $\binom{k}{k_1, \ldots, k_n} = \frac{k!}{k_1! \ldots k_n!}$ is the multinomial coefficient. Polynomials $B_{n,k}$ follow from the generating function

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots)\frac{t^n}{n!}.$$

Bell polynomials play an important role in number theory and combinatorial analysis, such us for computing the coefficients of the inverse or the composition of generating functions. With these combinatorial polynomials we study the 2-iterated 2D-Appell polynomials to give the combinatorial formulations and investigate extended families of polynomials. Let the exponential generating functions.

$A_1(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!}$ and $A_2(t) = \sum_{n\geq 0} b_n \frac{t^n}{n!}$,

with $a_0 b_0 \neq 0$. The 2-iterated 2D-Appell polynomials are defined by the generating function:

$$A_1(t)A_2(t)e^{xt+yt^j} = \sum_{n\geq 0} A_n^{[2]j}(x,y)\frac{t^n}{n!}. \quad (2)$$

$A_n^{[2]}(x) = A_n^{[2]}(x,0)$ is the 2-iterated-Appell polynomials given by the generating function

$$A_1(t)A_2(t)e^{xt} = \sum_{n\geq 0} A_n^{[2]j}(x)\frac{t^n}{n!}.$$

The q-analog are investigated in [8] and the big (p,q)-Appell polynomials are studied in [9]. We recall that Gould-Hopper polynomials are defined us

$$H_n^{(j)}(x,y) = n!\sum_{k=0}^{\left[\frac{n}{j}\right]} \frac{x^{n-jk}y^k}{k!(n-jk)!}, \quad (3)$$

where $j \neq 1$ a positive integer. These polynomials are pointed by the generating function

$$e^{xt+yt^j} = \sum_{n\geq 0} H_n^{(j)}(x,y)\frac{t^n}{n!}.$$

Four simplifying calculus let $A(t) = A_1(t)A_2(t) = \sum_{n\geq 0} c_n \frac{t^n}{n!}$. It follows from Cauchy product of exponential generating functions that

$$c_n = \sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k}.$$

According to Gould-Hopper polynomials the following reformulation of $A_n^{[2]j}(x,y)$ holds true.

$$A_n^{[2]j}(x,y) = \sum_{k=0}^{n}\binom{n}{k} c_{n-k} H_k^{(j)}(x,y).$$

$A_n^{[2]j}(x,y)$ is one family of numerous extensions Gould-Hopper polynomials; one can consult [3, 7] and reference therein. In this work we revisit the proof of the explicit formula (3), we derive a new reformulation of $A_n^{[2]j}(x,y)$ and we give additional recurrence relation to these developed in [2].

Combinatorial formulation of 2-iterated 2d-appell polynomials

For any generating function $f(x,t) = \sum_{n\geq 0} \gamma_n(x)t^n$, $e^{f(t)}$ is calculated as follows

$$e^{f(t)} = \sum_{n\geq 0}\sum_{k=0}^{n} B_{n,k}(1!\gamma_1, 2!\gamma_2, \ldots)\frac{t^n}{n!},$$

With the convention that $B_{0,0} = 1$ and $B_{n,0} = 0$ for $n \geq 1$. For the general case gof one consult [4,7], to see the q-analog we refer to [5]. Let the arithmetical functions q_j and δ_j to be $q_j(n) = \frac{n}{j-1}$ and

$$\delta_j(n) = \begin{cases} 1, if\ q_j(n) \in \mathbb{N} \\ 0, otherwise \end{cases},$$

It is obvious that $q_j(n-k) - 1 = q_j(n-k-j+1)$. In the following proposition we give another formulation of $H_n^{(j)}(x,y)$ slightly different from expression (3).

Proposition 2.1. We have $H_0^{(j)}(x,y) = 1$ and

$$H_n^{(j)}(x,y) = n!\sum_{k=0}^{n} \frac{\delta_j(n-k)}{k!}\binom{k}{q_j(n-k)} x^{k-q_j(n-k)} y^{q_j(n-k)}.$$

Proof. We have

$$e^{xt+yt^j} = 1 + \sum_{n\geq 1}\sum_{k=1}^{n} B_{n,k}(x, 0, \ldots, 0, j!\,y, 0, \ldots)\frac{t^n}{n!}.$$

But

$B_{n,k}(x,0,\ldots,0,j!\,y,0,\ldots) = \frac{n!}{k!}\sum_{\substack{k_1+k_2=k \\ k_1+jk_j=n}} \binom{k}{k_j} x^{k_1} y^{k_j}.$

The solution of the system

$$\begin{cases} k_1 + k_j = k \\ k_1 + jk_j = n \end{cases}$$

if exist is the pair $(k-l,l)$ with $(j-1)l = n-k$, so $\delta_j(n-k)q_j(n-j)$ is an integer and $B_{n,k}$ takes the form

$B_{n,k}(x,0\ldots,0,y,0,\ldots) = \frac{n!}{k!}\delta_j(n-k)\binom{k}{q_j(n-k)} x^{k-q_j(n-k)} y^{q_j(n-k)}.$

One remark that $B_{n,k} = 0$ for $q_j(n-k) \notin \mathbb{N}$. Furthermore the series expansion of e^{xt+yt^j} is

$e^{xt+yt^j} = 1 + \sum_{n\geq 1}\sum_{k=0}^{n} \frac{\delta_j(n-k)}{k!}\binom{k}{q_j(n-k)} x^{k-q_j(n-k)} y^{q_j(n-k)} t^n.$ □

To return back to identity (3); one just take $r = \delta_j(n-k)q_j(n-k)$ and remark that r bally all the integers between 0 and $\left[\frac{n}{j}\right]$. Let the symbol $\sum_{l,k}^{n}$ to designate the abbreviation of the sum $\sum_{l=0}^{n}\sum_{k=0}^{l}$, under those considerations the following theorem holds true.

Theorem 2.2. We have $A_0^{[2]j}(x,y) = c_0$ and

$A_n^{[2]j}(x,y) = \sum_{l,k}^{n}\binom{k}{n}\binom{n}{l}\binom{k}{q_j(l-k)}\frac{\delta_j(l,k)l!}{k!} c_{n-l} x^{k-q_j(l-k)} y^{q_j(l-k)}.$

Proof. The result follows from Cauchy product of the generating functions e^{x+yt^j} and $A(t)$.□
In this paragraph we reproduce identities concerning the partial derivatives (obtained by differentiating generating functions [2]) directly from the derivation of the polynomials $A_n^{[2]j}(x,y)$. We have

$\frac{\partial}{\partial x}A_n^{[2]j}(x,y) = n\sum_{l-1,k-1}^{n}\binom{n-1}{l-1}\binom{k-1}{k-q_j(l-k)-1}\frac{(l-1)!\delta_j(l-k)}{(k-1)!} c_{n-l} x^{k-1-q_j(l-k)} y^{q_j(l-k)}$

And then

$\frac{\partial}{\partial x}A_n^{[2]j}(x,y) = nA_{n-1}^{[2]j}(x,y).$

Since $q_j(l-k) - 1 = q_j(l-k-j+1)$, thus

$\frac{\partial}{\partial y}A_n^{[2]j}(x,y) = \sum_{l,k}^{n}\binom{n}{l}\binom{k-1}{q_j(l-j-k+1)}\frac{q_j(l-j-k+1)l!}{(k-1)!} c_{n-l} x^{k-1-q_j(l-j-k+1)} y^{q_j(l-j-k+1)}.$

On remark that

$l!\binom{n}{l} = \frac{n!(l-j)!}{(n-j)!}\binom{n-j}{l-j},$

then

$\frac{\partial}{\partial y}A_n^{[2]j}(x,y) = \sum_{l-j,k-1}^{n}\binom{n-j}{l-j}\binom{k-1}{q_j(l-j-k+1)}\frac{q_j(l-j-k+1)(l-j)!}{(k-1)!} c_{n-l} x^{k-1-q_j(l-j-k+1)} y^{q_j(l-j-k+1)}.$

and

$\frac{\partial}{\partial y}A_n^{[2]j}(x,y) = \frac{n!}{(n-j)!}A_{n-1}^{[2]j}(x,y).$

Additional Recurrence Relations
Recently Khan-Wani [2] studied the 2-iterated 2D-Appell polynomials and obtained the recurrence relation

$(x+\alpha_0+\beta_0)A_n^{[2]j}(x,y) + \sum_{k=1}^{n}\binom{n}{k}(\alpha_k+\beta_k)A_{k-1}^{[2]j}(x,y) + jy\frac{n!}{n-j+1}A_{n+1-j}^{[2]j}(x,y) = A_{n+1}^{[2]j}(x,y),$

where the coefficients α_k and β_k are given by
$\frac{A_1'(t)}{A_1(t)} = \sum_{k\geq 0} \alpha_k \frac{t^k}{k!}$ *and* $\frac{A_2'(t)}{A_2(t)} = \sum_{k\geq 0} \beta_k \frac{t^k}{k!}$.

Without lost generality one can suppose that $a_0 = b_0 = 1$, thereafter we have

$$A_1^{-1}(t) = \sum_{n\geq 0} \sum_{k=0}^{n} (-1)^k k!\, B_{n,k}(a_1, a_2, \dots) \frac{t^n}{n!},$$

$$A_2^{-1}(t) = \sum_{n\geq 0} \sum_{k=0}^{n} (-1)^k k!\, B_{n,k}(b_1, b_2, \dots) \frac{t^n}{n!}$$

and

$$A^{-1}(t) = \sum_{n\geq 0} \sum_{k=0}^{n} (-1)^k k!\, B_{n,k}(c_1, c_2, \dots) \frac{t^n}{n!}.$$

These formulae are obtained by the expression of the inverse of a generating function detailed in [6]. The functions $A_1'(t), A_2'(t)$ and $A'(t)$ generate respectively numbers a_{n+1}, b_{n+1} and c_{n+1}. Using Cauchy product generating functions, we conclude that

$$\alpha_n = \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} (-1)^k k!\, a_{n-l+1} B_{l,k}(a_1, a_2, \dots),$$

$$\beta_n = \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} (-1)^k k!\, a_{n-l+1} B_{l,k}(b_1, b_2, \dots)$$

and

$$\frac{A'(t)}{A(t)} = \sum_{n\geq 0} \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} (-1)^k k!\, c_{n-l+1} B_{l,k}(c_1, c_2, \dots) \frac{t^n}{n!}.$$

The following theorem gives additional recurrence relation

Theorem 3.1. For $n \geq 1$ we have

$$A_n^{[2]j}(x, y) = c_n - \sum_{k=0}^{n-1} \binom{n}{k} H_{n-k}^{(j)}(-x, -y) A_k^{[2]j}(x, y).$$

Proof. It follows from the generating function of Gould-Hopper polynomials that

$$e^{-xt-yt^j} = \sum_{n\geq 0} H_n^{(j)}(-x, -y) \frac{t^n}{n!}$$

and

$$\left(\sum_{n\geq 0} H_n^{(j)}(-x, -y) \frac{t^n}{n!}\right) \left(\sum_{n\geq 0} A_n^{[2]j}(x, y) \frac{t^n}{n!}\right) = \sum_{n\geq 0} c_n \frac{t^n}{n!}.$$

By Cauchy product of generating functions we have

$$\sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^{(j)}(-x, -y) A_k^{[2]j}(x, y) \frac{t^n}{n!} = \sum_{n\geq 0} c_n \frac{t^n}{n!}$$

and then

$$c_n = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^{(j)}(-x, -y) A_k^{[2]j}(x, y). \square$$

Corollary 3.2. For n even and j odd, the following identity holds true
$A_n^{[2]j}(x, y) = c_n - \sum_{k=0}^{n-1} \binom{n}{k} H_{n-k}^{(j)}(x, y) A_k^{[2]j}(x, y).$

Proof. To get the proof, just remark that $H_n^{(j)}(-x, -y) = H_n^{(j)}(x, y)$ for n even and j odd. □

Let d_n be the sequence defined by $d_0 = x + c_1$,

$$d_{j-1} = jy + \sum_{l=0}^{j-1} \sum_{k=0}^{l} \binom{j-1}{l} (-1)^k k!\, c_{j-l} B_{l,k}(c_1, c_2, \dots)$$

and for other values

$d_n = \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l} (-1)^k k!\, B_{l,k}(c_1, c_2, \ldots).$

Theorem 3.3. For $n \geq 0$ we have

$A_{n+1}^{[2]j}(x,y) = \sum_{k=0}^{n} \binom{n}{k} d_k A_{n-k}^{[2]j}(x,y).$

Proof. The differentiation operation implies that

$\frac{\partial}{\partial} A(t) e^{xt+yt^j} = DA(t) e^{x+yt^j} + (x + jyt^{j-1}) A(t) e^{xt+yt^j}$

and
$\frac{\partial}{\partial} A(t) e^{xt+yt^j} = \left(x + jyt^{j-1} + \frac{A'(t)}{A(t)}\right) A(t) e^{xt+yt^j}.$

It is obvious to remark that

$x + jyt^{j-1} + \frac{A'(t)}{A(t)} = \sum_{n \geq 0} d_n \frac{t^n}{n!}$

and the desired result follows.

Results for certain 2d-families
In this section we apply the obtained result to 2D-Euler-Appell and 2D-Hermite-Appell polynomials.

2D-Euler-Appell polynomials.
Taking $A_2(t) = \frac{2}{e^t + 1}$ in generating function (2), we obtain generating function of 2D Euler-Appell polynomials ${}_E A_n^{(j)}(x,y)$. In this case we have

$A_2(t) = \sum_{n \geq 0} E_n \frac{t^n}{n!} \; and \; c_n = \sum_{k=0}^{n} \binom{n}{k} a_k E_{n-k}.$

Theorem 4.1.

${}_E A_n^{(j)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} a_k E_{n-k} - \sum_{k=0}^{n-1} \binom{n}{k} H_{n-k}^{(j)}(-x,-y) \quad {}_E A_k^{(j)}(x,y)$

and

${}_E A_{n+1}^{(j)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} d_k \quad {}_E A_{n-k}^{(j)}(x,y).$

For $A_1(t) = 1$, we connect Euler numbers to 2D Euler- Appell polynomials as follows:
$E_n = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^{(j)}(-x,-y) \quad {}_E A_k^{(j)}(x,y).$

4.2. 2D Hermite -Appell polynomials.
2D Hermite – Appell polynomials ${}_H A_n^{(j)}(x,y)$ are obtained as:

$A_2(t) = e^{xt - \frac{t^2}{2} + yt^j} = \sum_{n \geq 0} \quad {}_H A_n^{(j)}(x,y) \frac{t^n}{n!}.$

To treat these polynomials, we proceed by the following way:

$e^{xt - \frac{t^2}{2} + yt^j} = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} B_{n,k} \frac{t^n}{n!},$

with $B_{n,k} = B_{n,k}(x, -1, 0, \ldots, 0, j!\, y, 0, \ldots)$. Since we have

$B_{n,k} = n! \sum_{\substack{k_1 + k_2 + k_j = k \\ k_1 + 2k_2 + jk_j = n}} \frac{(-1)^{k_2} x^{k_1} y^{k_j}}{2^{k_2} k_1! k_2! k_j!}.$

Then ${}_H A_0^{(j)}(x,y) = b_0$ and

$$ {}_HA_n^{(j)}(x,y) = \sum_{l=0}^{n}\sum_{k=1}^{l}\sum_{\substack{k_1+k_2+k_j=0\\ k_1+2k_2+jk_j=l}} \binom{n}{l}\frac{(-1)^{k_2}l!b_{n-l}x^{k_1}y^{k_j}}{2^k k_1!k_2!k_j!}. $$

One remark that $e^{xt-\frac{t^2}{2}+yt^j} = e^{-\frac{t^2}{2}}e^{xt+yt^j}$. Consequently we obtain

$$ e^{xt-\frac{t^2}{2}+yt^j} = \left(\sum_{n\geq 0}\left(-\frac{1}{2}\right)^{\frac{n}{2}}\frac{n!I_{\frac{n}{2}}t^n}{\left(\frac{n}{2}\right)!n!}\right)\left(\sum_{n\geq 0}H_n^{(j)}(x,y)\frac{t^n}{n!}\right), $$

where $I_{\frac{n}{2}} = 1$ if n is even and 0 if n is odd. Thereafter we obtain

$$ e^{xt-\frac{t^2}{2}+yt^j} = \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}\right)^{\frac{k}{2}}\frac{k!I_{\frac{k}{2}}}{\left(\frac{k}{2}\right)!}H_{n-k}^{(j)}(x,y)\frac{t^n}{n!}. $$

The comparison between the two series expansions of $e^{xt-\frac{t^2}{2}+yt^j}$ allow to get

$$ \sum_{k=1}^{n}\sum_{\substack{k_1+k_1+k_j=k\\ k_1+2k_2+jk_j=n}}\frac{(-1)^{k_2}x^{k_1}y^{k_j}}{2^{k_2}k_1!k_2!k_j!} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}\right)^{\frac{k}{2}}\frac{k!I_{\frac{k}{2}}}{\left(\frac{k}{2}\right)!}H_{n-k}^{(j)}(x,y). $$

In this case

$$ c_n = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}\right)^{\frac{k}{2}}\frac{k!I_{\frac{k}{2}}}{\left(\frac{k}{2}\right)!}H_{n-k}^{(j)}(x,y). $$

Then

$$ {}_HA_n^{(j)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}\right)^{\frac{k}{2}}\frac{k!I_{\frac{k}{2}}}{\left(\frac{k}{2}\right)!}H_{n-k}^{(j)}(x,y) - \sum_{k=0}^{n-1}\binom{n}{k}H_{n-k}^{(j)}(-x,-y)\quad {}_HA_k^{(j)}(x,y) $$

and

$$ {}_HA_n^{(j)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}d_k \quad {}_HA_{n-k}^{(j)}(x,y). $$

Conclusion

With the Bell polynomials, we can find in particular the series expansions of e^{xt+yt^j} and $e^{xt-\frac{t^2}{2}+yt^j}$ to deduce another proof of the identity (3) and the explicit formula 2-iterated 2D-Appell polynomials. These results cover the expression of 2D-Euler-Appell and 2D-Hermite-Appell polynomials.

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