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On some solutions of the Schröder equation in Hilbert spaces

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Abstract
In this paper we prove the results on solutions of the Schröder equation (1.1) defined on cones in Hilbert spaces and having some properties related with monotonicity and boundedness.

Keywords: Schröder functional equation, monotonic and bounded solutions, Krein-Rutman theorem

Introduction

Consider the Schröder equation

$$g(f(x)) = \lambda g(x), \tag{1.1}$$

where g is an arbitrary function and the function f is given. In [4; Ch. VI, §4] equation (1.1) is considered in the real case, among others in the class of functions such that the function "$x \to g(x)/x$" is monotonic (see also [5; section 2.3F]). This class of functions is related with classes of convex (concave) functions. In the paper we propose to study equation (1.1) for functions defined on cones in Hilbert spaces under an assumption which in the real case means that the function "$x \to g(x) - x$" is either monotonic or bounded.

Suppose $(H, \langle\, , \rangle)$ be a Hilbert space and $C \neq \{\theta\}$ be a closed cone in H with non empty interior, i.e. (cf. [3; Definition 2.1]), C is a closed subset of H such that $C + C \subset C$, $tC \subset C$ for each $t \geq 0$, $C \cap (-C) = \{\theta\}$ and $Int\ C \neq \emptyset$. We define a (partial) order $\leq$ on H by

$$\mathrm{x} \leq \mathrm{y} \Leftrightarrow \mathrm{y} - \mathrm{x} \in \mathrm{C}$$

and note the following simple lemma (cf. [3; p. 208]).

Lemma 1.
Let $x_n \in H$ for $n \in \mathbb{N}$. If $\lim_{n\to\infty} x_n = \theta$, then for each $a \in Int\ C$ there exists an $N \in \mathbb{N}$ such that $x_n \leq a$ for $n \geq N$.

Suppose $T: H \to H$ be a completely continuous linear operator, i.e. T is linear and maps bounded subsets of H into relatively compact ones. We assume additionally that

$$TC \subset C$$

and for each $x \in C \setminus \{\theta\}$ there exists a positive integer n such that $T^n x \in Int\ C$. By the Krein-Rutman theorem [3; Theorem 6.3] there exists exactly one vector $v \in Int\ C$ and exactly one continuous linear functional $h : H \to \mathbb{R}$ such that

$$Tv = \lambda v$$

$$h(Tx) = \lambda h(x), x \in H,$$

$$h(x) > 0, x \in C \setminus \{\theta\},$$

$$\langle v, v \rangle = 1, h(v) = 1,$$

here λ denotes the spectral radius of T:

$$\lambda = \lim_{n\to\infty} \langle T^n, T^n \rangle^{\frac{1}{2n}}.$$

Of course $\lambda > 0$.

Suppose

$\lambda \neq 1$

and let $f: C \to C$ be a function such that $f(\theta) = \theta$ and

$$\lim_{n\to\infty} f^n(x) = \theta, x \in C \ (1.2)$$

Lemma 2.
Assume $g: C \to \overline{\mathbb{R}}$ be a monotonic solution of (1.1) such that $g(a) = 0$ for some $a \in Int\ C$. If either g is increasing and $g(\theta) > -\infty$, or g is decreasing and $g(\theta) < \infty$, then $g = 0$.

Proof. We may (and we do) suppose, that g is increasing. Then $g(\theta) \leq g(a) = 0$, In particular $g(\theta)$ is finite. Therefore, since g is a solution of (1.1), $g(\theta) = 0$. Now, if $x \in C$ is arbitrarily fixed then according to (1.2) and Lemma 1 there exists a positive integer n such that

$$f^n(x) \leq a$$

Hence

$$0 = \lambda^n g(\theta) \leq \lambda^n g(x) = g\big(f^n(x)\big) \leq g(a) = 0$$

and $g(x) = 0$.

Arguing similarly, also we can prove the following lemma.

Lemma 3.
Suppose $g: C \to \overline{\mathbb{R}}$ be a monotonic solution of (1.1) such that $|g(a)| < \infty$ for some $a \in Int\ C$. If either g is increasing and $g(\theta) > -\infty$, or g is decreasing and $g(\theta) < \infty$, then g is finite-valued.

Denoting

$$T_0 \coloneqq T|_C, h_0 \coloneqq h|_C,$$

we have the following result.

Theorem 1. Let the function f is increasing and $f - T_0$ is monotonic. Then:
i). For each $x \in C$ the sequence

$$\big(h(f^n(x))/\lambda^n\big)_{n\in\mathbb{N}} \ (1.3)$$

is monotonic and the function $g_0: C \to [0, \infty]$ given by the formula

$$g_0(x) \coloneqq \lim_{n\to\infty} \frac{h(f^n(x))}{\lambda^n} \ (1.4)$$

is an increasing solution of (1.1).
ii). Assume $f - T_0$ is increasing. Then the function $g_0 - h_0$ is increasing, and if $g: C \to \overline{\mathbb{R}}$ is a solution of (1.1) such that $g - h_0$ is increasing and $g(\theta) > -\infty$ [resp. $g - h_0$ is decreasing and $g(\theta) < \infty$] then $g_0 \leq g$ [resp. $g \leq g_0$] and $g(a) = g_0(a) < \infty$ for some $a \in Int\ C$ implies $g = g_0$.
iii). Assume $f - T_0$ is decreasing. Then g_0 is finite-valued, the function $g_0 - h_0$ is decreasing and if $g: C \to \overline{\mathbb{R}}$ is a solution of (1.1) such that $g - h_0$ is increasing and $g(\theta) > -\infty$ [resp. $g - h_0$ is decreasing and $g(\theta) < \infty$] then $g_0 \leq g$ [resp. $g \leq g_0$] and $g(a) = g_0(a) < \infty$ for some $a \in Int\ C$ implies $g = g_0$.

Proof. Denote

$F := f - T_0.$

Since $F(\theta) = \theta$ and F is monotonic, we have

$\theta \leq F$ or $F \leq \theta,$

i.e.

$T_0 \leq f$ or $f \leq T_0.$

In the first case

$h(f(x)) \geq h(Tx) = \lambda h(x), x \in C,$

which shows that for each $x \in C$ the sequence (1.3) is increasing. In the second case it is a decreasing sequence. Moreover,

$$g_0(f(x)) = \lim_{n\to\infty} \frac{h(f^{n+1}(x))}{\lambda^n} = \lambda \lim_{n\to\infty} \frac{h(f^{n+1}(x))}{\lambda^{n+1}} = \lambda g_0(x)$$

for each $x \in C$, i.e. g_0 is a solution of (1.1).
Hence the function g_0 is increasing. Using induction it is easy to check that

$\frac{h(f^n(x))}{\lambda^n} = h(x) + \sum_{k=o}^{n-1} \frac{h\left(F\left(f^k(x)\right)\right)}{\lambda^{k+1}}, x \in C, n \in \mathbb{N}.$ (1.5)

Hence

$\sum_{k=o}^{\infty} \frac{h\left(F\left(f^k(x)\right)\right)}{\lambda^{k+1}} = g_0(x) - h(x), x \in C.$

Consequently, if F is increasing [resp. decreasing] then so is $g_0 - h_0$.
Assume now that F is increasing and let $g: C \to \overline{\mathbb{R}}$ be a solution of (1.1) such that $g - h_0$ is increasing and $g(\theta) > -\infty$. Then $h_0 \leq g$ and, consequently,

$\frac{h(f^n(x))}{\lambda^n} \leq \frac{g(f^n(x))}{\lambda^n} = g(x), x \in C, n \in \mathbb{N},$ (1.6)

hence $g_0 \leq g$. Suppose now that $g(a) = g_0(a) < \infty$ for some $a \in Int\ C$. According to Lemma 3, g_0 is finite-valued. Denoting

$G := g - h_0$

We have

$\frac{G(f^n(x))}{\lambda^n} = g(x) - \frac{h(f^n(x))}{\lambda^n}, x \in C, n \in \mathbb{N},$

Hence

$$\lim_{n\to\infty} \frac{G(f^n(x))}{\lambda^n} = g(x) - g_0(x), x \in C. \tag{1.7}$$

In particular, $g - g_0$ is an increasing function. Since it is also a non-negative solution of (1.1) and vanishes at a, according to Lemma 2 it vanishes everywhere , i.e. $g = g_0$. In the case where $g - h_0$ is decreasing we argue similarly.
Finally assume that F is decreasing. As we noted, for each $x \in C$ the sequence (1.3) is then decreasing and, consequently, g_0 is now finite-valued. Let $g: C \to \overline{\mathbb{R}}$ be a solution of (1.1) such that $g - h_0$ is increasing. Then (1.6) holds and this gives $g_0 \leq g$. Suppose now that $g(a) = g_0(a)$ for some $a \in Int\ C$. As previously (cf. in particular (1.7)) we see that $g - g_0$ is a solution of (1.1) which is increasing, non-negative and vanishes at a. An application of Lemma 2 gives $g = g_0$ and ends the proof.

Remark 1. If $\lambda \in (0, \infty)$ and $T: \mathbb{R} \to \mathbb{R}$ is given by the formula $Tx = \lambda x$ then the vectors obtained from the Krein-Rutman theorem for this T (and $C = [0, \infty)$) are: $v = 1$, $h = id_{\mathbb{R}}$. Consequently, we have the following corollary.

Corollary 1. Suppose $\lambda \in (0,1)$ and assume that $f:[0,\infty) \to [0,\infty)$ is an increasing function such that $\lim_{n\to\infty} f^n(x) = 0, x \in [0,\infty)$, and the function

"$x \to f(x) - \lambda x, x \in [0,\infty)$" (1.8)

is monotonic. Then:

(i) For each $x \in [0,\infty)$ the sequence $(f^n(x)/\lambda^n)_{n\in\mathbb{N}}$ is monotonic and the function $g_0:[0,\infty) \to [0,\infty]$ defined by

$g_0(x) := \lim_{n\to\infty} \frac{f^n(x)}{\lambda^n}$ (1.9)

(1.9)

is an increasing solution of (1.1).
(ii) Assume (1.8) is increasing. Then the function "$x \to g_0(x) - x, x \in [0,\infty)$" is increasing, and if $g:[0,\infty) \to \overline{\mathbb{R}}$ is a solution of (1.1) such that

"$x \to g(x) - x, x \in [0,\infty)$" (1.10)

is increasing and $g(0) > -\infty$ [resp. (1.10) is decreasing and $g(0) < \infty$] then $g_0 \le g$ [resp. $g \le g_0$] and

$g(a) = g_0(a) < \infty$ for some $a \in (0,\infty)$ implies $g = g_0$.

(iii) Assume (1.8) is decreasing. Then g_0 is finite-valued, the function "$x \to g_0(x) - x, x \in [0,\infty)$" is decreasing, and if $g:[0,\infty) \to \overline{\mathbb{R}}$ is a solution of (1.1) such that (1.10) is increasing and $g(0) > -\infty$ [resp. (1.10) is decreasing and $(0) < \infty$] then $g_0 \le g$ [resp. $g \le g_0$] and

$g(a) = g_0(a) < \infty$ for some $a \in (0,\infty)$ implies $g = g_0$.

Remark 2. Suppose $\lambda \in (0,\infty)$. If a function $f:[0,\infty) \to [0,\infty)$ is convex [resp. concave], $f(0) = 0$ and $\lambda x \le f(x)$ [resp. $f(x) \le \lambda x$] for $x \in [0,\infty)$, then the function "$x \to f(x) - \lambda x, x \in [0,\infty)$" is increasing [resp. decreasing].

Proof. Consider the case where f is convex, fix $x, y \in [0,\infty)$ such that $x < y$ and let $x = ty$. Then

$f(x) - \lambda x = f(ty) - \lambda ty \le tf(y) - \lambda ty = t(f(y) - \lambda y) \le f(y) - \lambda y.$

Remark 3. In the real case we have a condition of Seneta (see [5; Theorem 1.3.2]) which guarantees that g_0 is finite-valued. The following example shows that the solution g_0 need be neither convex nor concave.

Example 1. The functions

$$-F(x) := \begin{cases} 0 \; for \; x \in [0,2], \\ \frac{1}{2}x - 1 \; for \; x \in (2,4], \\ 1 \; for \; x \in (4,\infty) \end{cases}$$

And

$f(x) := \frac{1}{2}x + F(x), x \in [0,\infty)$,

are increasing, and

$0 \le f(x) \le \frac{1}{2}x, x \ge 0.$

Since

$$f^n(x) = \begin{cases} 2^{-n}x \; for \; x \in [0,2], \\ 2^{-n+1} \; for \; x \in (2,4], \\ 2^{-n}x - 2^{-n+1} \; for \; x \in (4,6], \end{cases}$$

and $\lambda = \frac{1}{2}$ the function $g_0|_{[0,6]}$ given by (1.9) is of the form

$$g_0(x) = \begin{cases} x \, for \, x \in [0,2], \\ 2 \, for \, x \in (2,4], \\ x-2 \, for \, x \in (4,6]. \end{cases}$$

Therefore $g_0|_{[0,6]}$ is neither convex nor concave. (Let us observe even more: the function "$x \to g(x)/x, x \in (0,6]$" is not monotonic). Consequently g_0 is also neither convex nor concave.
The next example shows that equation (1.1) may have a lot of solutions g such that the function $g - h_0$ is increasing.

Example 2. Suppose $\lambda \in (0,1)$ and $f(x) = \lambda x, x \geq 0$. Then $g_0(x) = x, x \geq 0$. Using a standard argument (see , e.g., [5; the proof of Theorem 2.2.3]) it is easy to prove that if $a > 0$ and $\tilde{g}: [\lambda a, a] \to \mathbb{R}$ is a function such that the function "$x \to \tilde{g}(x) - x, x \in [\lambda a, a]$" is increasing,

$$\tilde{g}(\lambda a) = \lambda \tilde{g}(a)$$

And
$$x \leq \tilde{g}(x), x \in [\lambda a, a],$$

then there exists exactly one solution $g: [0,\infty) \to \mathbb{R}$ of (1.1) such that $g|_{[\lambda a, a]} = \tilde{g}$; moreover the function "$x \to g(x) - x, x \in [0,\infty)$" is increasing. In particular, there are solutions $g_1, g_2: [0,\infty) \to \mathbb{R}$ of (1.1) such that $g_1(a) = g_2(a)$, functions "$x \to g_i(x) - x, x \in [0,\infty)$", $i \in \{1,2\}$, are increasing, but $g_1 \neq g_2$.

2. Now we pass to solutions g of (1.1) such that $g - h_0$ is bounded. Suppose $f: C \to C$ be an arbitrary function.

Theorem 2. Let $\lambda > 1$ and assume $f - T_0$ be bounded. Then:
i). For each $x \in C$ the sequence (1.3) converges.
ii). The function $g_0: C \to [0,\infty)$ given by the formula (1.4) is a non-zero solution of (1.1) such that $g_0 - h_0$ is bounded.
iii). If $g: C \to \mathbb{R}$ is a solution of (1.1) such that for some $\xi \in \mathbb{R}$ the function $g - \xi h_0$ is bounded then $g = \xi g_0$.

Proof. Replacing $F := f - T_0$ and taking into account boundedness of this function we infer that the series
$$\sum_{k=0}^{\infty} \frac{h \circ F \circ f^k}{\lambda^{k+1}}$$

uniformly and absolutely converges and its sum is a bounded function. Therefore and from (1.5) it follows that for each $x \in C$ the sequence (1.3) converges and $g_0 - h_0$ is bounded function. In particular, $g_0 \neq 0$.
Suppose now that $g: C \to \mathbb{R}$ is a solution of (1.1) such that the function $\chi := g - \xi h_0$ is bounded (by a constant L). Then

$$\langle \left\{ g(x) - \xi \frac{h(f^n(x))}{\lambda^n} \right\}, \left\{ g(x) - \xi \frac{h(f^n(x))}{\lambda^n} \right\} \rangle = \langle \frac{\chi(f^n(x))}{\lambda^n}, \frac{\chi(f^n(x))}{\lambda^n} \rangle \leq L \frac{1}{\lambda^n} \; x \in C, n \in \mathbb{N},$$

Hence
$$g = \xi g_0.$$

We should mention here that the idea of examining the Schröder equation (1.1) with the aid of the Krein-Rutman theorem has come up while the author was thinking on generalization to the infinite-dimensional case of some results from the papers [1] by F. M. Hoppe and [2] by A. Joffe and F. Spitzer where the finite-dimensional case is considered with the aid of the Frobenius theory.

Conclusion
The paper proved the results on solutions of the Schröder equation (1.1) defined on cones in Hilbert spaces. The paper discussed some monotonic solutions. In this way one can prove many results on solutions of this equation.

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