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## Weighted zygmond type estimates for the volterra convolution type

**Mamatov Tulkin and Sharipova Nargiza**

### Abstract

Weighted Zygmund-type estimates are obtained for mixed fractional integrals of the Volterra convolution type for a function of two variables defined by a mixed modulus of continuity.

**Keywords:** weight, mixed modulus of continuity, Zygmund type estimate, two variable function, Volterra convolution type

### 1. Introduction

An important stage in the study of fractional integro-differentiation of functions from generalized Hölder spaces is the obtaining of Zygmund-type estimates, i.e. estimate of the modulus of continuity of a fractional integral through the modulus of continuity of the original function. A similar problem can be considered completely solved for the Hölder space of functions of one variable and power weights (see <sup>[1, 2]</sup>), as well as for the Hölder space of functions of two variables and power weights (see <sup>[5, 18]</sup>). Obtaining an estimate of the Zygmund type for mixed fractional integrals with arbitrary kernels and has not been studied.

The main focus of the work is to obtain an estimate of the type of Zygmund majorizing a mixed modulus of continuity  $\omega(\rho\tilde{K}\varphi; h, \eta)$  mixed fractional integral with a Volterra convolution type of weight integral constructions from a mixed modulus of continuity  $\omega(\rho\varphi; h, \eta)$  its density  $\varphi(x, y)$  with weight  $\rho(x, y)$ . These Zygmund type bounds and action theorems directly affect the nature of the improvement of the modulus of continuity by mixed fractional integration of the Volterra convolution type:

$$(\tilde{K}\varphi)(x, y) = \int_0^x \int_0^y k(x-t)k(y-s)\varphi(t, s)dt ds \quad (1)$$

here we consider the degenerate kernels, as well as each  $k(x)$  and  $k(y)$  assumed to be close in some sense to a power function. In this paper, we deal with arbitrary kernels, i.e. not necessarily power. We will consider the operator (1) in a rectangle  $Q = \{(x, y): 0 < x < b, 0 < y < d\}$ .

### Preliminary

In this section, we present some well-known results and notation that we will need to present the issues under consideration (see <sup>[2]</sup>). We follow the papers <sup>[3]</sup> in the definitions below. Positive constants which can be different at different places will be denoted by  $C$ .

**Definition 1.** We say that  $\psi(x) \in W_\mu = W_\mu([0, l])$  if  $\psi(x) \in C([0, l])$ ,  $\psi(0) = 0, \psi(x) > 0$  for  $x > 0, \psi(x)$  is almost increasing while  $\psi(x)/x^\mu$  is almost decreasing and there exists  $C$  such that

$$\left| \frac{\psi(x) - \psi(y)}{x - y} \right| \leq C \frac{\psi(x^*)}{x^*}, \quad x^* = \max(x, y) \quad (2)$$

We remind that non-negative function  $\psi(x), 0 \leq x \leq l, 0 < l \leq \infty$  is called almost

Increasing (decreasing) if  $\psi(x) \leq C\psi(y)$  for all  $x \leq y$  ( $x \geq y$  resp.) this notion being due to S. Bernstein.

**Definition 2.** We say that  $\psi(x) \in W_\mu^*$  if  $\psi(x) \in W_\mu$  and  $\psi(x)/x^{\mu-\varepsilon}$  is almost increasing for all  $\varepsilon > 0$ . We shall also need the following modification of the Definition 2.

**Definition 3.** We say that a non-negative function  $k(x)$  on  $[0, l]$  belongs to the class  $V_\lambda$ ,  $\lambda > 0$ , if

- 1)  $k(x) \neq 0$ ,  $x^\lambda k(x)$  - is almost increases and  $x^\lambda k(x)|_{x=0} = 0$ ;
- 2) there exists  $\varepsilon$ ,  $0 < \varepsilon < \lambda$ , such that  $x^{\lambda-\varepsilon} k(x)$  is almost decreases;
- 3) there exists  $C$  such that
 
$$\left| \frac{k(x_1) - k(x_2)}{x_1 - x_2} \right| \leq C \frac{k(x^*)}{x^*}, \quad x^* = \max(x_1, x_2). \tag{3}$$

**Remark 1.**  $x^\lambda k(x) \in W_\lambda^* \Rightarrow k(x) \in V_\lambda$  and  $k(x) \in V_\lambda \Rightarrow k(x) \in W_\lambda$ .

**Definition 4.** Let given bounded on  $[a, b]$  function  $\varphi(x)$ . Under modulus of continuity  $\omega(\varphi)$  understood expression

$$\sup_{h \in [0, \delta]} |\varphi(x+h) - \varphi(x)| = \omega(\varphi; \delta), \quad 0 < \delta \leq b - a.$$

**Definition 3.** We denote by  $\Phi^1$  function class  $\omega(\delta) \in (0, b - a]$  and satisfying the conditions

- 1)  $\omega(\delta) > 0$  in  $(0, b - a]$ ,  $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$ ;
- 2)  $\omega(\delta) \uparrow$  in  $(0, b - a]$ ;
- 3)  $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ .

Below in the estimates we need inequalities:

1) if  $\omega(\varphi; h)$  is modulus continuity, then we have

$$x_2 \omega(\varphi; x_1) \leq C x_1 \omega(\varphi; x_2), \quad x_2 \leq x_1; \tag{4}$$

2) if  $k(x) \in V_\lambda$ , then

$$x_1^\lambda k(x_1) \leq C x_2^\lambda k(x_2) \text{ and } \exists 0 < \varepsilon < \lambda, \quad x_2^{\lambda-\varepsilon} k(x_2) \leq C x_1^{\lambda-\varepsilon} k(x_1), \quad x_1 \leq x_2; \tag{5}$$

3) if  $\psi(x) \in W_\mu$  then

$$y^\mu \psi(x) \leq C x^\mu \psi(y), \quad x \geq y; \tag{6}$$

4) if  $0 < \alpha < 1$  then  $\frac{\omega(\varphi, h)}{h^\alpha} \leq C \int_0^h \frac{\omega(\varphi, t)}{t^{1+\alpha}} dt$ ;

5) if  $\psi(x) \in W_\mu$  with  $0 < \mu < 1$  then (1) holds with  $x^*$  replaced both by  $x$  and  $y$ :

$$\left| \frac{\psi(x) - \psi(y)}{x - y} \right| \leq C \frac{\psi(x)}{x}, \quad \left| \frac{\psi(x) - \psi(y)}{x - y} \right| \leq C \frac{\psi(y)}{y}; \tag{7}$$

6) if  $\lambda \leq 1$ , then

$$|x^\lambda - y^\lambda| \leq C(x - y)y^{\lambda-1}, \quad x \geq y > 0, \tag{8}$$

if  $\lambda \geq 0$ , then

$$|x^\lambda - y^\lambda| \leq C(x - y)x^{\lambda-1}, \quad x \geq y > 0. \tag{9}$$

**Lemma 1.** Let  $k(x) \in V_\lambda$ ,  $\lambda > 0$  and  $\omega(x) \geq 0$  be an almost increases function. Then for any  $0 < x < l/2$ , rightly

$$\omega(x)k(x) \leq C \int_x^l t^{-1} \omega(t)k(t)dt. \tag{10}$$

Below we follow some technical estimations suggested in [5] for the case of one-dimensional Riemann-Liouville fractional integrals. We denote

$$B(x, y; t, s) = \frac{\rho(x, y) - \rho(t, s)}{\rho(t, s)(x-t)^{1+\alpha}(y-s)^{1+\beta}}, \tag{11}$$

where  $0 < \alpha, \beta < 1, 0 < t < x < b, 0 < s < y < d$  and

$$B_1(x, t) = \frac{\rho(x) - \rho(t)}{\rho(t)(x-t)^{1+\alpha}} \text{ and } B_2(y, s) = \frac{\rho(y) - \rho(s)}{\rho(s)(y-s)^{1+\beta}}. \tag{12}$$

In the case  $\rho(x, y) = \rho(x)\rho(y)$  we have

$$B(x, y; t, s) = B_1(x, t)B_2(y, s) + \frac{B_1(x, t)}{(y-s)^{1+\beta}} + \frac{B_2(y, s)}{(x-t)^{1+\alpha}}.$$

Let also

$$D_1(x, h, t) = B_1(x+h, t) - B_1(x, t), \quad t, x, x+h \in [0, b], \quad h > 0, \\ D_2(y, \eta, s) = B_2(y+\eta, s) - B_2(y, s), \quad s, y, y+\eta \in [0, d], \quad \eta > 0.$$

**Lemma 2.** Let  $\rho(x) = \psi(x-a), \psi(x) \in W_\mu, \mu \in \mathbb{R}^1$  and  $k(x) \in V_\lambda, \lambda > 0$ . Then

$$|B_1(x, t)| \leq C \left( \frac{x-a}{t-a} \right)^{\max(\mu-1, 0)} \frac{x-t}{t-a} k(x-t), \tag{13}$$

$$|D_1(x, h, t)| \leq C \left( \frac{x+h-a}{t-a} \right)^{\max(\mu-1, 0)} \frac{h}{t-a} k(x+h-t). \tag{14}$$

**Proof.** Let  $\mu < 1$ , then from (2) and (8) имеем

$$|B_1(x, t)| = \left| \frac{\psi(x-a) - \psi(t-a)}{\psi(t-a)} k(x-t) \right| \leq \frac{x-t}{t-a} k(x-t).$$

If  $1 \leq \mu$ , then from inequalities (2) and (6), we have

$$|B_1(x, t)| \leq \left| \frac{\psi(x-a)}{\psi(t-a)} - 1 \right| k(x-t) \leq C \left| \frac{(x-a)^\mu - (t-a)^\mu}{(t-a)^\mu} \right| k(x-t) \leq C \left( \frac{x-a}{t-a} \right)^{\mu-1} \frac{x-t}{t-a} k(x-t).$$

Now let's estimate  $D_1(x, h, t)$ . Let  $\mu < 1$  then from inequalities (2), (6), (3) and (5), we obtain

$$|D_1(x, h, t)| \leq \left| \frac{\psi(x+h-a) - \psi(x-a)}{\psi(t-a)} \right| k(x+h-t) + \\ + \left| \frac{\psi(x-a) - \psi(t-a)}{\psi(t-a)} \right| |k(x+h-t) - k(x-t)| \leq C \left[ \frac{h}{x-a} \frac{\psi(x-a)}{\psi(t-a)} k(x+h-t) + \right. \\ \left. + \frac{(x-a)\psi(t-a)k(x+h-t)}{(t-a)\psi(t-a)(x+h-t)} h \right] \leq C \left[ \frac{h}{x-a} \left( \frac{x-a}{t-a} \right)^\mu k(x+h-t) + \right. \\ \left. + \frac{h}{t-a} k(x+h-t) \right] = C \frac{h}{t-a} k(x+h-t) \left[ \left( \frac{x-a}{t-a} \right)^{\mu-1} + 1 \right].$$

Since  $\mu < 1$  then

$$|D_1(x, h, t)| \leq C \frac{h}{t-a} k(x+h-t) \left[ \left( \frac{t-a}{x-a} \right)^{1-\mu} + 1 \right] = C_1 \frac{h}{t-a} k(x+h-t).$$

If  $1 \leq \mu$  then

$$|D_1(x, h, t)| \leq C \left[ h \frac{\psi(x+h-a)}{\psi(t-a)} \frac{k(x+h-t)}{x+h-a} + h \frac{\psi(x-a)}{\psi(t-a)} \frac{k(x+h-t)}{x-a} \right] \leq \\ \leq C \left[ \left( \frac{x+h-a}{t-a} \right)^{\mu-1} \frac{h}{t-a} k(x+h-t) + h \left( \frac{x-a}{t-a} \right)^{\mu-1} \frac{k(x+h-t)}{t-a} \right] \leq Ch \left( \frac{x+h-a}{t-a} \right)^{\mu-1} \frac{k(x+h-t)}{t-a}.$$

Similar estimates hold for  $B_2(y, s)$  and  $D_2(y, \eta, s)$  with  $\rho(y) = y^\nu$ .

Consider a one-dimensional integral operator of the Volterra type convolutions

$$(K\varphi)(x) = \int_0^x k(x-t)\varphi(t)dt, \quad 0 < x < b. \tag{15}$$

The following statements are known (see [2]). We use the schemes of the proofs to make the presentation easier for two-dimensional case. The following theorem provides a Zygmund type estimate for the integral (15).

**Theorem 1.** Let  $k(x) \in V_\lambda$ ,  $0 < \lambda < 1$  and  $\varphi(x) \in C([0, b])$ ,  $\varphi(0) = 0$ . Then for integral (15), the following estimate is valid

$$\omega(K\varphi, h) \leq C \left( hk(h)\omega(\varphi, h) + h \int_h^b \frac{k(t)\omega(\varphi, t)}{t} dt \right) \tag{16}$$

**Proof.** The integral (15) will be presented in the form

$$(K\varphi)(x) = \varphi(0) \int_0^x k(x-t)dt + \int_0^x k(x-t)[\varphi(t) - \varphi(0)]dt.$$

Since  $\varphi(0) = 0$ , then we have  $f(x) = \int_0^x g(t)k(x-t)dt$ , where  $\varphi(t) - \varphi(0) = g(t)$ .

Let  $h > 0$ ,  $x, x+h \in [0, b]$ . Consider the difference

$$|f(x+h) - f(x)| \leq C \left| \int_{-h}^0 k(h+t)[g(x-t) - g(x)]dt + \int_0^x (k(h+t) - k(t))[g(x-t) - g(x)]dt + g(x) \int_x^{x+h} k(t)dt \right| = A_1 + A_2 + A_3.$$

Taking (5) and increasing of  $\omega(\varphi, t)$  into account, we have for  $A_1$

$$A_1 \leq \int_0^h \omega(\varphi, t)k(h-t)dt \leq C\omega(\varphi, h)k(h) \int_0^h \left(\frac{h}{h-t}\right)^\lambda dt \leq Chk(h)\omega(\varphi, h)$$

For  $A_2$  applying (3) and (5), we obtain in the case  $h \geq x$

$$\begin{aligned} A_2 &\leq C \int_0^x \frac{k(h+t)}{h+t} \omega(\varphi; t)dt \leq Ch^{1+\lambda-\varepsilon}k(h) \int_0^x \frac{\omega(\varphi; t)}{(h+t)^{1+\lambda-\varepsilon}} dt = \\ &= Chk(h) \int_0^{\frac{x}{h}} \frac{\omega(\varphi, ht)dt}{(1+t)^{1+\lambda-\varepsilon}} \leq Chk(h) \int_0^1 \frac{\omega(\varphi, ht)}{(1+t)^{1+\lambda-\varepsilon}} dt \leq Chk(h)\omega(\varphi, h). \end{aligned} \tag{17}$$

In the  $h < x$  we write  $A_2 \leq \int_0^h + \int_h^{x-h} = A'_2 + A''_2$ . For  $A'_2$  the estimate (17) is valid, while for  $A''_2$ , we have

$$A''_2 \leq Ch \int_h^x \omega(\varphi, t) \frac{k(t)}{t} dt \leq Ch \int_h^b \frac{\omega(\varphi, t)}{t} k(t)dt.$$

As regards  $A_3$ , we have in the case  $x \leq h$ :

$$A_3 \leq C\omega(\varphi, x)k(x+h)(x+h)^\lambda \int_x^{x+h} \frac{dt}{t^\lambda} \leq C\omega(\varphi, h)k(h)h.$$

If  $x > h$ , we use Lemma 1 to obtain

$$A_3 \leq C\omega(\varphi, x)hk(x) \leq Ch \int_x^b \frac{k(t)}{t} \omega(\varphi; t)dt \leq Ch \int_x^b \frac{k(t)}{t} \omega(\varphi; t)dt.$$

Gathering all the estimates for  $A_1, A_2, A_3$ , we arrive at (16).

**Theorem 2.** Let  $k(x) \in V_\lambda$ ,  $0 < \lambda < 1$ ,  $\rho(x) = \psi(x-a)$ ,  $\psi(x) \in W_\mu$ ,  $0 < \mu < 1 + \lambda$ . Assume that

1)  $\rho(x)\varphi(x) \in C([a, b])$  and  $\rho(x)\varphi(x)|_{x=a} = 0$ ;

2)  $\int_0^{b-a} t^{-\gamma} \omega(\rho\varphi, t) dt < \infty$ ,  $\gamma = \max(1, \mu)$ . Then the following Zygmund type estimate holds:

$$\omega(\rho K\varphi, h) \leq Ch^\gamma k(h) \int_0^h \frac{\omega(\rho\varphi, t)}{t^\gamma} dt + Ch \int_h^{b-a} \frac{\omega(\rho\varphi, t)}{t} k(t) dt, \tag{18}$$

if  $0 < \mu < 1 + \lambda$ .

**Proof.** Let  $\varphi_0(x) = \rho(x)\varphi(x)$  and  $a = 0$  for simplicity. We have

$$\rho(x)(K\varphi)(x) = \int_0^x k(x-t)\varphi_0(t) dt + \int_0^x B(x,t)\varphi_0(t) dt = F_1(x) + F_2(x),$$

where  $B(x,t) = \frac{\Psi(x) - \Psi(t)}{\Psi(t)} k(x-t)$ .

Since  $\varphi_0 \in C([0, b])$  and  $\varphi_0(0) = 0$  the first term  $F_1(x)$  is covered by Theorem 1. To estimate  $\omega(F_2, h)$  we represent the difference  $F_2(x+h) - F_2(x)$  as

$$F_2(x+h) - F_2(x) = \int_x^{x+h} B(x+h,t)\varphi_0(t) dt + \int_0^x D(x,h,t)\varphi_0(t) dt,$$

where  $D(x,h,t) = B(x+h,t) - B(x,t)$ .

Further

$$F_2(x+h) - F_2(x) = \int_x^{x+h} |B(x+h,t)| \omega(\varphi_0, t) dt + \int_0^x |D(x,h,t)| \omega(\varphi_0, t) dt = I_1 + I_2.$$

Estimate of  $I_1$ . Let  $0 < \mu < 1$  at first. Using (13) we have

$$I_1 \leq C \int_x^{x+h} \frac{x+h-t}{t} k(x+h-t) \omega(\varphi_0, t) dt \leq Chk(h) \int_x^{x+h} \frac{\omega(\varphi_0, t)}{t} dt.$$

Given  $\frac{\omega(\varphi_0, t)}{t}$  decreasing, then we obtain

$$I_1 \leq Chk(h) \int_x^{x+h} \frac{\omega(\varphi_0, t-x)}{t-x} dt = Chk(h) \int_0^h \frac{\omega(\varphi_0, t)}{t} dt.$$

If  $1 \leq \mu < 2$ , then taking (13) into account, we have

$$I_1 \leq C \int_x^{x+h} \left(\frac{x+h}{x}\right)^{\mu-1} \frac{x+h-t}{t} k(x+h-t) \omega(\varphi_0, t) dt.$$

In the case  $x \leq h$

$$I_1 \leq Ch^\mu k(h) \int_x^{x+h} \frac{\omega(\varphi_0, t)}{t^\mu} dt \leq Ch^\mu k(h) \int_x^{x+h} \frac{\omega(\varphi_0, t-x)}{(t-x)^\mu} dt = Ch^\mu k(h) \int_0^h \frac{\omega(\varphi_0, t)}{t^\mu} dt.$$

In the case  $x \geq h$

$$\begin{aligned} I_1 &\leq Ch(x+h)^{\mu-1} k(h) \int_x^{x+h} \frac{\omega(\varphi_0, t)}{t^\mu} dt = C \frac{hk(h)}{(x+h)^{1-\mu}} \int_{-h}^0 \frac{\omega(\varphi_0, x-t)}{(x-t)^\mu} dt \leq C_1 x^{\mu-1} hk(h) \int_0^h \frac{\omega(\varphi_0, x+t)}{(x+t)^{\mu-1}} \frac{dt}{x+t} \leq \\ &\leq Chk(h) \int_0^h \frac{\omega(\varphi_0, t+x)}{t+x} dt \leq Chk(h) \int_0^h \frac{\omega(\varphi_0, t)}{t} dt. \end{aligned}$$

Gathering estimates of  $I_1$  then we have by  $0 < \mu < 2$

$$I_1 \leq Ch^\gamma k(h) \int_0^h \frac{\omega(\varphi_0, t)}{t^\gamma} dt, \text{ where } \gamma = \max(1, \mu).$$

Estimate of  $I_2$ . By  $0 < \mu < 1$  used (14) and we have

$$I_2 \leq Ch \int_0^x \frac{\omega(\varphi_0, t)}{t} k(x+h-t) dt. \tag{19}$$

If  $x \leq h$ , then

$$I_2 \leq Chk(h) \int_0^x \frac{\omega(\varphi_0, t)}{t} dt \leq Chk(h) \int_0^h \frac{\omega(\varphi_0, t)}{t} dt.$$

In the case  $x > h$ , we represent (19) as

$$I_2 \leq Ch \left( \int_0^h + \int_h^{\frac{x+h}{2}} + \int_{\frac{x+h}{2}}^x \right) \frac{\omega(\varphi_0, t)}{t} k(x+h-t) dt = \Delta_1 + \Delta_2 + \Delta_3.$$

Since  $t \leq x$  then immediately follow  $\Delta_1 \leq Chk(h) \int_0^h \frac{\omega(\varphi_0, t)}{t} dt$ . For  $\Delta_2$  taking into account that  $x+h-t \geq t$  we obtain

$$\Delta_2 \leq Ch \int_h^{\frac{x+h}{2}} \frac{\omega(\varphi_0, t)}{t} k(t) \left( \frac{t}{x+h-t} \right)^{\lambda-\varepsilon} dt \leq Ch \int_h^{\frac{x+h}{2}} \frac{\omega(\varphi_0, t)}{t} k(t) dt.$$

For  $\Delta_3$  taking into account that  $x+h-t \leq t$ , so

$$\Delta_3 \leq Ch \int_{\frac{x+h}{2}}^x \frac{\omega(\varphi_0, x+h-t)}{x+h-t} k(x+h-t) dt.$$

After replacing  $x+h-t = t$  we obtain

$$\Delta_3 \leq Ch \int_h^{\frac{x+h}{2}} \frac{\omega(\varphi_0, t)}{t} k(t) dt \leq Ch \int_h^b \frac{\omega(\varphi_0, t)}{t} k(t) dt.$$

Thus for  $I_2$  by  $0 < \mu < 1$  we have

$$I_2 \leq C \left( hk(h) \int_0^h \frac{\omega(\varphi_0, t)}{t} dt + h \int_h^b \frac{\omega(\varphi_0, t)}{t} k(t) dt \right).$$

Let  $1 \leq \mu < 2$ . Taking into account (14) we have

$$I_2 \leq Ch \int_0^x \left( \frac{x+h}{t} \right)^{\mu-1} \frac{\omega(\varphi_0, t)}{t} k(x+h-t) dt. \tag{20}$$

In the case  $x \leq h$  we have

$$\begin{aligned} I_2 &\leq Ch^\mu \int_0^x \frac{\omega(\varphi_0, t)}{t^\mu} k(x+h-t) dt \leq Ch^{\mu+1} k(h) \int_0^x \frac{\omega(\varphi_0, t)}{t^\mu (x+h-t)} dt \leq Ch^\mu k(h) \int_0^x \frac{\omega(\varphi_0, t)}{t^\mu} dt \leq \\ &\leq Ch^\mu k(h) \int_0^h \frac{\omega(\varphi_0, t)}{t^\mu} dt. \end{aligned}$$

If  $x > h$ , then we represent the right-hand side of inequality (20) as the sum of three terms

$$f_2'' \leq Ch \left( \int_0^h + \int_h^{\frac{x+h}{2}} + \int_{\frac{x+h}{2}}^x \right) \left( \frac{x+h}{t} \right)^{\mu-1} \frac{\omega(\varphi_0, t)}{t} k(x+h-t) dt = T_1 + T_2 + T_3.$$

Taking into account that  $x+h \leq 2(x+h-t)$  in  $T_1$  we obtain

$$T_1 \leq Ch \int_0^h \left( \frac{x+h-t}{t} \right)^{\mu-1} \frac{\omega(\varphi_0, t)}{t} k(x+h-t) dt \leq Ch \int_0^h k(h) \frac{\omega(\varphi_0, t)}{t} \left( \frac{h}{t} \right)^{\mu-1} dt \leq Ch^\mu k(h) \int_0^h \frac{\omega(\varphi_0, t)}{t^\mu} dt.$$

Since  $x + h \geq 2t$  than for  $T_2$  we have

$$T_2 \leq Ch \int_h^{\frac{x+h}{2}} \left(\frac{x+h}{t}\right)^{\mu-1} \frac{\omega(\varphi_0, t)}{t} k(x+h-t) dt \leq Ch \int_h^{\frac{x+h}{2}} t^{\mu-1} k(t) \frac{\omega(\varphi_0, t)}{t^\mu} dt \leq Ch \int_h^b \frac{\omega(\varphi_0, t)}{t} k(t) dt.$$

To estimate  $T_3$ , we notice that  $t \geq x + h - t$  in  $T_3$ , so  $\frac{\omega(\varphi_0, t)}{t} \leq C \frac{\omega(\varphi_0, x+h-t)}{(x+h-t)}$ . Then follows that

$$T_3 \leq Ch \int_{\frac{x+h}{2}}^x \left(\frac{x+h}{t}\right)^{\mu-1} \frac{\omega(\varphi_0, x+h-t)}{x+h-t} k(x+h-t) dt \leq Ch \int_{\frac{x+h}{2}}^x \frac{\omega(\varphi_0, x+h-t)}{x+h-t} k(x+h-t) dt,$$

because,  $x + h \leq 2t$ . Having replaced  $\xi = x + h - t$  and passing again to the variable  $t$ , we obtain

$$T_3 \leq Ch \int_h^b \frac{\omega(\varphi_0, t)}{t} k(t) dt.$$

Collecting all the estimates of  $I_2$  by  $1 \leq \mu < 1 + \lambda$ , we have

$$I_2 \leq C \left( h^\gamma k(h) \int_0^h \frac{\omega(\varphi_0, t)}{t^\gamma} dt + h \int_h^b \frac{k(t)}{t} \omega(\varphi_0, t) dt \right), \quad \gamma = \max(1, \mu).$$

Gathering estimates  $I_1$  and  $I_2$  for  $0 \leq \mu < 1 + \lambda$  we have

$$|F_2(x+h) - F_2(x)| \leq C \left( h^\gamma k(h) \int_0^h \frac{\omega(\varphi_0, t)}{t^\gamma} dt + h \int_h^b \frac{k(t)}{t} \omega(\varphi_0, t) dt \right), \quad \text{where } \gamma = \max(1, \mu).$$

Let a continuous function  $\varphi(x, y)$  be defined in  $\mathbb{R}^2$ . We introduce the necessary notation

$$\left(\Delta_h \varphi\right)(x, y) = \varphi(x+h, y) - \varphi(x, y), \quad \left(\Delta_\eta \varphi\right)(x, y) = \varphi(x, y+\eta) - \varphi(x, y),$$

$$\left(\Delta_{h,\eta} \varphi\right)(x, y) = \varphi(x+h, y+\eta) - \varphi(x, y+\eta) - \varphi(x+h, y) + \varphi(x, y),$$

and

$$\varphi(x+h, y+\eta) = \left(\Delta_{h,\eta} \varphi\right)(x, y) + \left(\Delta_\eta \varphi\right)(x, y) + \left(\Delta_h \varphi\right)(x, y) + \varphi(x, y). \tag{21}$$

Now we introduce the following characteristics:

1) Private modules of continuity

$$\omega(\varphi; \delta, 0) = \sup_y \sup_{0 \leq h \leq \delta} \left| \left(\Delta_h \varphi\right)(x, y) \right| \quad \text{and} \quad \omega(\varphi; 0, \sigma) = \sup_x \sup_{0 \leq \eta \leq \sigma} \left| \left(\Delta_\eta \varphi\right)(x, y) \right|;$$

2) Mixed modulus continuity of order 1.1

$$\omega(\varphi; \delta, \sigma) = \sup_{x,y} \sup_{\substack{0 \leq h \leq \delta \\ 0 \leq \eta \leq \sigma}} \left| \left(\Delta_{h,\eta} \varphi\right)(x, y) \right|, \quad \text{where } 0 < \delta \leq b, \quad 0 < \sigma \leq d.$$

It follows from the definition  $\omega^{1,1}(\varphi; \delta, \sigma)$  that this function belongs in each variable  $\Phi^1$ . In addition, we note that there is an inequality

$$\omega^{1,1}(\varphi; \delta, \sigma) \leq 2 \min \left\{ \omega(\varphi; \delta, 0), \omega(\varphi; 0, \sigma) \right\}. \tag{22}$$

### 3. Main result

Here we generalize Theorem 1 to the case  $k(x_1, x_2) = k(x_1)k(x_2)$ .

**Theorem 3.** Let  $k(x_i) \in V_{\lambda_i}, 0 < \lambda_i < 1, i = 1, 2, \varphi(x_1, x_2) \in C(Q)$  and  $\varphi(x_1, x_2)|_{x_1=a, x_2=c} = 0$ . Then Zygmund type estimates are valid

$$\begin{aligned} \omega(\tilde{K}\varphi; h, 0) &\leq C_1 \left[ hk(h) \omega(\varphi; h, d) + h \int_h^{b-a} \frac{k(t)^{1,1}}{t} \omega(\varphi; t, d) dt \right], \\ \omega(\tilde{K}\varphi; 0, \eta) &\leq C_2 \left[ \eta k(\eta) \omega(\varphi; b, \eta) + \eta \int_\eta^{d-c} \frac{k(s)^{1,1}}{s} \omega(\varphi; b, s) ds \right], \\ \omega(\tilde{K}\varphi; h, \eta) &\leq C_3 \left[ h\eta k(h)k(\eta) \omega(\varphi; h, \eta) + h\eta k(\eta) \int_h^{b-a} \frac{k(t)^{1,1}}{t} \omega(\varphi; t, \eta) dt + hk(h)\eta \int_\eta^{d-c} \frac{k(s)^{1,1}}{s} \omega(\varphi; h, s) ds + \right. \\ &\quad \left. + h\eta \int_h^{b-a} \int_\eta^{d-c} \frac{k(t)k(s)^{1,1}}{ts} \omega(\varphi; t, s) dt ds \right]. \end{aligned}$$

**Proof.** We will not prove this theorem. For a proof of this theorem, see [13].

Now we generalize Theorem 2 to the cases  $k(x_1, x_2) = k(x_1)k(x_2)$  and  $\rho(x_1, x_2) = \psi(x_1 - a)\psi(x_2 - c)$ .

**Theorem 4.** Let  $k(x_i) \in V_{\lambda_i}$ ,  $0 < \lambda_i < 1$ , and  $\psi(x_i) \in W_{\mu_i}$ ,  $0 < \mu_i < 2$ , where  $i = 1, 2$ . Assume that  $\varphi_0(x_1, x_2) = \rho(x_1, x_2)\varphi(x_1, x_2) \in C(Q)$  and  $\varphi_0(x_1, x_2)|_{x_1=a, x_2=c} = 0$ ;

$$\int_0^{b-a} \int_0^{d-c} \frac{\omega(\varphi_0; t, s)^{1,1}}{t^\delta s^\sigma} dt ds < \infty, \text{ where } \delta = \max(1, \mu_1), \sigma = \max(1, \mu_2).$$

Then the following Zygmund type estimates holds

$$\omega(\rho K\varphi; h, 0) \leq C_1 \left[ h^\delta k(h) \int_0^h \frac{\omega(\varphi_0; t, d-c)^{1,1}}{t^\delta} dt + h \int_h^{b-a} \frac{\omega(\varphi_0; t, d-c)^{1,1}}{t} k(t) dt \right], \tag{23}$$

$$\omega(\rho K\varphi; 0, \eta) \leq C_2 \left[ \eta^\sigma k(\eta) \int_0^\eta \frac{\omega(\varphi_0; b-a, s)^{1,1}}{s^\sigma} ds + \eta \int_\eta^{d-c} \frac{\omega(\varphi_0; b-a, s)^{1,1}}{s} k(s) ds \right], \tag{24}$$

$$\begin{aligned} \omega(\rho K\varphi; h, \eta) &\leq C_3 \left[ h^\delta \eta^\sigma k(h)k(\eta) \int_0^h \int_0^\eta \frac{\omega(\varphi_0; t, s)^{1,1}}{t^\delta s^\sigma} dt ds + h\eta^\sigma k(\eta) \int_h^{b-a} \int_0^\eta \frac{\omega(\varphi_0; t, s)^{1,1}}{ts^\sigma} k(t) dt + \right. \\ &\quad \left. + h^\delta \eta k(\eta) \int_0^h \int_\eta^{d-c} \frac{\omega(\varphi_0; t, s)^{1,1}}{t^\delta s} k(s) dt ds + h\eta \int_h^{b-a} \int_\eta^{d-c} \frac{\omega(\varphi_0; t, s)^{1,1}}{ts} k(t)k(s) dt ds \right], \end{aligned} \tag{25}$$

если  $0 < \mu_i < 1 + \lambda_i, i = 1, 2$ .

**Proof.** Let  $\varphi_0(x_1, x_2) = \rho(x_1, x_2)\varphi(x_1, x_2)$ . We denote

$$G(x_1, x_2) = (\rho K\varphi)(x_1, x_2) = \int_a^{x_1} \int_c^{x_2} \frac{\rho(x_1, x_2)}{\rho(t, s)} \varphi_0(t, s) k(x_1 - t) k(x_2 - s) dt ds. \tag{26}$$

Since we consider the decaying weight function, i.e., the weight function in the form  $\rho(x_1, x_2) = \psi(x_1 - a)\psi(x_2 - c)$ , then the equality

$$\begin{aligned} \psi(x_1)\psi(x_2) &= [\psi(x_1) - \psi(t)][\psi(x_2) - \psi(s)] + \psi(s)[\psi(x_1) - \psi(t)] + \\ &+ \psi(t)[\psi(x_2) - \psi(s)] + \psi(t)\psi(s). \end{aligned} \tag{27}$$

Equality (26) can be represented as

$$\begin{aligned} G(x_1, x_2) &= \int_a^{x_1} \int_c^{x_2} \varphi_0(t, s) k(x_1 - t) k(x_2 - s) dt ds + \int_a^{x_1} \int_c^{x_2} B(x_1, x_2; t, s) \varphi_0(t, s) dt ds = \\ &= G_1(x_1, x_2) + G_2(x_1, x_2). \end{aligned} \tag{28}$$

Here, the question of estimating the modulus of continuity for the first term is solved by us in Theorem 3.



Taking into account (27), we represent  $G_2(x_1, x_2)$  in the following form

$$G_2(x_1, x_2) = \int_a^{x_1} \int_c^{x_2} B(x_1, t)B(x_2, \tau)\varphi_0(t, s)dtds + \int_a^{x_1} \int_c^{x_2} B(x_1, t)k(x_2 - s)\varphi_0(t, s)dtds + \int_a^{x_1} \int_c^{x_2} B(x_2, s)k(x_1 - t)\varphi_0(t, s)dtds = f_1(x_1, x_2) + f_2(x_1, x_2) + f_3(x_1, x_2), \tag{29}$$

where

$$B(x_1, t) = \frac{\Psi(x_1 - a) - \Psi(t - a)}{\Psi(t - a)}k(x_1 - t), \quad B(x_2, s) = \frac{\Psi(x_2 - c) - \Psi(s - c)}{\Psi(s - c)}k(x_2 - s).$$

Let us estimate each term  $f_i(x_1, x_2), i = 1, 2, 3$  separately.

Let  $x_1, x_2 + h \in [a, b]$ . Consider the difference

$$f_1(x_1 + h, x_2) - f_1(x_1, x_2) = \int_{x_1}^{x_1+h} \int_c^{x_2} B(x_1 + h, t)B(x_2, s)\varphi_0(t, s)dtds + \int_a^{x_1} \int_c^{x_2} D(x_1, h, t)B(x_2, s)\varphi_0(t, s)dtds.$$

So  $\varphi_0(x_1, x_2)|_{x_1=a, x_2=c} = 0$ , then inequality holds

$$|f_1(x_1 + h, x_2) - f_1(x_1, x_2)| \leq \int_{x_1}^{x_1+h} \int_c^{x_2} |B(x_1 + h, t)| |B(x_2, s)| \omega(\varphi_0; t - a, s - c)dtds + \int_a^{x_1} \int_c^{x_2} |D(x_1, h, t)| |B(x_2, s)| \omega(\varphi_0; t - a, s - c)dtds.$$

We estimate integral

$$\int_c^{x_2} |B(x_2, s)| \omega(\varphi_0; t - a, s - c)ds. \tag{30}$$

Let  $0 < \mu_2 < 1$ . From Lemma 2, we have

$$\begin{aligned} \int_c^{x_2} |B(x_2, s)| \omega(\varphi_0; t - a, s - c)ds &\leq C \int_c^{x_2} \frac{(x_2 - s)k(x_2 - s)^{1,1}}{s - c} \omega(\varphi_0; t - a, s - c)ds \leq \\ &\leq C(x_2 - c)k(x_2 - c) \int_0^{x_2 - c} \frac{\omega(\varphi_0; t - a, s)^{1,1}}{s} ds \leq C(x_2 - c)k(x_2 - c) \omega(\varphi_0; t - a, x_2 - c) \leq C \omega(\varphi_0; t - a, d - c). \end{aligned}$$

For  $1 \leq \mu_2 < 2$  we taking into account (13), we have

$$\begin{aligned} \int_c^{x_2} |B(x_2, s)| \omega(\varphi_0; t - a, s - c)ds &\leq C \int_c^{x_2} \left(\frac{x_2 - c}{s - c}\right)^{\mu_2 - 1} \frac{x_2 - s}{s - c} k(x_2 - s) \omega(\varphi_0; t - a, s - c)ds = \\ &= C \int_0^{x_2 - c} \left(\frac{x_2 - c}{s}\right)^{\mu_2 - 1} \frac{\omega(\varphi_0; t - a, s)^{1,1}}{s} (x_2 - s + c)k(x_2 - s + c)ds \leq \\ &\leq C(x_2 - c)k(x_2 - c) \int_0^{x_2 - c} \frac{\omega(\varphi_0; t - a, s)^{1,1}}{s} ds \leq \\ &\leq C(x_2 - c)k(x_2 - c) \omega(\varphi_0; t - a, x_2 - c) \leq C_1 \omega(\varphi_0; t - a, d - c). \end{aligned}$$

Using estimates of (30) and estimates for  $I_1, I_2$  from Theorem 2, we obtain estimate by  $0 \leq \mu_1 < 1 + \lambda_1$

$$|f_1(x_1 + h, x_2) - f_1(x_1, x_2)| \leq C \left( h^\delta k(h) \int_0^h \frac{\omega(\varphi_0; t, d - c)^{1,1}}{t^\delta} dt + h \int_h^{b-a} \frac{k(t)^{1,1}}{t} \omega(\varphi_0; t, d - c) dt \right),$$

where  $\delta = \max(1, \mu_1)$ .

Now, we estimate  $f_2(x_1, x_2)$ . Let  $h > 0; x_1, x_1 + h \in [a, b]$ . Consider the difference

$$|f_2(x_1 + h, x_2) - f_2(x_1, x_2)| \leq \int_{x_1}^{x_1+h} \int_c^{x_2} |B(x_1 + h, t)| k(x_2 - s) \omega(\varphi_0; t - a, s - c) dt ds + \int_a^{x_1} \int_c^{x_2} |D(x_1, h, t)| k(x_2 - s) \omega(\varphi_0; t - a, s - c) dt ds.$$

So

$$\int_c^{x_2} \omega(\varphi_0; t - a, s - c) k(x_2 - s) ds \leq C \omega(\varphi_0; t - a, x_2 - c) \int_c^{x_2} k(x_2 - s) ds \leq C \omega(\varphi_0; t - a, x_2 - c) k(x_2 - c) (x_2 - c)^{\lambda_2} \int_c^{x_2} \frac{ds}{(x_2 - s)^{\lambda_2}} \leq C \omega(\varphi_0; t - a, d - c).$$

Then we have

$$|f_2(x_1 + h, x_2) - f_2(x_1, x_2)| \leq C \left( \int_{x_1}^{x_1+h} |B(x_1 + h, t)| \omega(\varphi_0; t - a, d - c) dt + \int_a^{x_1} |D(x_1, h, t)| \omega(\varphi_0; t - a, d - c) dt \right)$$

It is easy to verify the validity of the assessment by taking into account the estimates  $I_1$  and  $I_2$  from Theorem 2, if  $0 < \mu_1 < 1 + \lambda_1$

$$|f_2(x_1 + h, x_2) - f_2(x_1, x_2)| \leq C \left( h^\delta k(h) \int_0^h \frac{\omega(\varphi_0; t, d - c)}{t^\delta} dt + h \int_h^{b-a} \frac{\omega(\varphi_0; t, d - c)}{t} k(t) dt \right),$$

где  $\delta = \max(\mu_1, 1)$ .

Now  $f_3$ , let's evaluate. Imagine the difference

$$f_3(x_1 + h, x_2) - f_3(x_1, x_2) = \int_{x_1}^{x_1+h} \int_c^{x_2} \varphi_0(t, s) k(x_1 + h - t) B(x_2, s) dt ds + \int_a^{x_1} \int_c^{x_2} [k(x_1 + h - t) - k(x_1, t)] \varphi_0(t, s) B(x_2, s) dt ds.$$

Since  $\varphi_0(x_1, c) = 0$  we have

$$|f_3(x_1 + h, y) - f_3(x_1, y)| \leq \int_{-h}^0 \int_c^{x_2} |\varphi_0(x_1 - t, s) - \varphi_0(x_2, c)| |B(x_2, s)| k(h + t) dt ds + \int_0^{x_1-a} \int_c^y |k(h + t) - k(t)| |\varphi_0(x - t, \tau) - \varphi_0(x, c)| |B(y, \tau)| dt d\tau \leq C \left( \int_{-h}^0 \int_c^{x_2} \omega(\varphi_0; t, s - c) |B(x_2, s)| k(h + t) dt ds + \int_0^{x_1-a} \int_c^{x_2} |k(h + t) - k(t)| \omega(\varphi_0; t, s - c) |B(x_2, s)| dt ds \right).$$

We obtain

$$|f_3(x_1 + h, x_2) - f_3(x_1, x_2)| \leq C \left( \int_{-h}^0 \omega(\varphi_0; t, d - c) k(h + t) dt + \int_0^{x_1-a} \omega(\varphi_0; t, d - c) |k(h + t) - k(t)| dt \right).$$

Using the estimates  $A_1, A_2$  from Theorem 1, we obtain the estimate

$$|f_3(x_1 + h, x_2) - f_3(x_1, x_2)| \leq C \left( hk(h) \omega(\varphi_0; h, d - c) + h \int_h^{b-a} \frac{\omega(\varphi_0; t, d - c)}{t} k(t) dt \right).$$

Gathering estimates for  $f_1(x_1, x_2)$ ,  $f_2(x_1, x_2)$  and  $f_3(x_1, x_2)$  we obtain

$$|G_2(x_1 + h, x_2) - G_2(x_1, x_2)| \leq C \left[ h^\delta k(h) \int_0^h \frac{\omega(\varphi_0; t, d - c)}{t^\delta} dt + h \int_h^{b-a} \frac{\omega(\varphi_0; t, d - c)}{t} k(t) dt \right],$$

where  $\delta = \max(\mu_1, 1)$ .

The estimate

$$|G_2(x_1, x_2 + \eta) - G_2(x_1, x_2)| \leq C \left[ \eta^\sigma k(\eta) \int_0^\eta \frac{\omega(\varphi_0; b - a, s)}{s^\sigma} ds + \eta \int_\eta^{d-c} \frac{\omega(\varphi_0; b - a, s)}{s} k(s) ds \right],$$

$\sigma = \max(1, \mu_2)$  is symmetrical obtained.

Now, we prove the validity of the estimate (25). Let be  $h, \eta > 0$ ,  $x_1, x_1 + h \in [a, b]$ ,  $x_2, x_2 + \eta \in [c, d]$ . Consider the difference

$$\begin{aligned} \left( \Delta_{h, \eta}^{1,1} G_2 \right) (x_1, x_2) \leq C & \left\{ \int_{x_1}^{x_1+h} \int_{x_2}^{x_2+\eta} B(x_2 + h, t) B(x_2 + \eta, s) \varphi_0(t, s) dt ds + \right. \\ & + \int_a^{x_1} \int_c^{x_2} D(x_1, h, t) D(x_2, \eta, s) \varphi_0(t, s) dt ds + \int_{x_1}^{x_1+h} \int_c^{x_2} B(x_1 + h, t) D(x_2, \eta, s) \varphi_0(t, s) dt ds + \\ & + \int_a^{x_1} \int_{x_1}^{x_1+\eta} D(x_1, h, t) B(x_2 + \eta, s) \varphi_0(t, s) dt ds + \int_{x_1}^{x_1+h} \int_{-\eta}^0 k(\eta + s) B(x_1 + h, t) \varphi_0(t, s) dt ds + \\ & + \int_{x_1}^{x_1+h} \int_0^{x_2-c} B(x_1 + h, t) [k(\eta + s) - k(s)] \varphi_0(t, s) dt ds + \int_a^{x_1} \int_{-\eta}^0 D(x_1, h, t) k(\eta + s) \varphi_0(t, s) dt ds + \\ & + \int_a^{x_1} \int_0^{x_2-c} D(x_1, h, t) [k(\eta + s) - k(s)] \varphi_0(t, s) dt ds + \int_{-h}^0 \int_{x_2}^{x_2+\eta} k(h + t) B(x_2 + \eta, s) \varphi_0(t, s) dt ds + \\ & + \int_0^{x_1-a} \int_{x_2}^{x_2+\eta} [k(h + t) - k(t)] B(x_2 + \eta, s) \varphi_0(t, s) dt ds + \int_{-h}^0 \int_c^{x_2} k(h + t) D(x_2, \eta, s) \varphi_0(t, s) dt ds + \\ & \left. \int_0^{x_1-a} \int_c^{x_2} [k(h + t) - k(t)] D(x_2, \eta, s) \varphi_0(t, s) dt ds \right\}. \end{aligned}$$

Since  $\varphi_0(x_1, x_2) |_{x_1=a, x_2=c} = 0$  then the inequalities

$$\begin{aligned} \left| \left( \Delta_{h, \eta}^{1,1} G_2 \right) (x_1, x_2) \right| \leq C & \left[ \int_{x_1}^{x_1+h} \int_{x_2}^{x_2+\eta} B(x_2 + h, t) B(x_2 + \eta, s) \omega(\varphi_0; t - a, s - c) dt ds + \right. \\ & + \int_a^{x_1} \int_c^{x_2} D(x_1, h, t) D(x_2, \eta, s) \omega(\varphi_0; t - a, s - c) dt ds + \int_{x_1}^{x_1+h} \int_c^{x_2} B(x_1 + h, t) D(x_2, \eta, s) \omega(\varphi_0; t - a, s - c) dt ds + \\ & + \int_a^{x_1} \int_{x_1}^{x_1+\eta} D(x_1, h, t) B(x_2 + \eta, s) \omega(\varphi_0; t - a, s - c) dt ds + \int_{x_1}^{x_1+h} \int_{-\eta}^0 k(\eta + s) B(x_1 + h, t) \omega(\varphi_0; t - a, s) dt ds + \\ & + \int_{x_1}^{x_1+h} \int_0^{x_2-c} B(x_1 + h, t) [k(\eta + s) - k(s)] \omega(\varphi_0; t - a, s) dt ds + \int_a^{x_1} \int_{-\eta}^0 D(x_1, h, t) k(\eta + s) \omega(\varphi_0; t - a, s) dt ds + \end{aligned}$$

$$\begin{aligned}
 & + \int_a^{x_1} \int_0^{x_2-c} D(x_1, h, t) [k(\eta + s) - k(s)] \omega(\varphi_0; t - a, s) dt ds + \int_{-h}^0 \int_{x_2}^{x_2+\eta} k(h+t) B(x_2 + \eta, s) \omega(\varphi_0; t, s - c) dt ds + \\
 & + \int_0^{x_1-a} \int_{x_2}^{x_2+\eta} [k(h+t) - k(t)] B(x_2 + \eta, s) \omega(\varphi_0; t, s - c) dt ds + \int_{-h}^0 \int_c^{x_2} k(h+t) D(x_2, \eta, s) \omega(\varphi_0; t, s - c) dt ds + \\
 & \left. \int_0^{x_1-a} \int_c^{x_2} [k(h+t) - k(t)] D(x_2, \eta, s) \omega(\varphi_0; t, s - c) dt ds \right|. \tag{31}
 \end{aligned}$$

We skip the details of calculating each term in inequality (31). The calculation is carried out in the standard way using Lemma 2, inequalities (4) - (9) and using the estimates  $A_1, A_2, A_3$  from Theorem 1, it is easy to verify the inequality

$$\begin{aligned}
 \left| \left( \Delta_{h,\eta}^{1,1} G_2 \right) (x_1, x_2) \right| \leq C_3 & \left[ h^\delta \eta^\sigma k(h) k(\eta) \int_0^h \int_0^\eta \frac{\omega(\varphi_0; t, s)}{t^\delta s^\sigma} dt ds + h \eta^\sigma k(\eta) \int_h^{b-a} \int_0^\eta \frac{\omega(\varphi_0; t, s)}{ts^\sigma} k(t) dt ds + \right. \\
 & \left. + h^\delta \eta k(\eta) \int_0^h \int_\eta^{d-c} \frac{\omega(\varphi_0; t, s)}{t^\delta s} k(s) dt ds + h \eta \int_h^{b-a} \int_\eta^{d-c} \frac{\omega(\varphi_0; t, s)}{ts} k(t) k(s) dt ds \right],
 \end{aligned}$$

if  $0 < \mu_i < 1 + \lambda_i, i = 1, 2$ .

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