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Weighted zygmund type estimates for the volterra convolution type

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Abstract
Weighted Zygmund-type estimates are obtained for mixed fractional integrals of the Volterra convolution type for a function of two variables defined by a mixed modulus of continuity.

Keywords: weight, mixed modulus of continuity, Zygmund type estimate, two variable function, Volterra convolution type

1. Introduction

An important stage in the study of fractional integro-differentiation of functions from generalized Hölder spaces is the obtaining of Zygmund-type estimates, i.e. estimate of the modulus of continuity of a fractional integral through the modulus of continuity of the original function. A similar problem can be considered completely solved for the Hölder space of functions of one variable and power weights (see) [1, 2], as well as for the Hölder space of functions of two variables and power weights (see) [5, 18]. Obtaining an estimate of the Zygmund type for mixed fractional integrals with arbitrary kernels and has not been studied.

The main focus of the work is to obtain an estimate of the type of Zygmund majorizing a mixed modulus of continuity $\omega(\rho\tilde{K}\varphi;\, h,\eta)$ mixed fractional integral with a Volterra convolution type of weight integral constructions from a mixed modulus of continuity $\omega(\rho\varphi; h,\eta)$ its density $\varphi(x,y)$ with weight $\rho(x,y)$. These Zygmund type bounds and action theorems directly affect the nature of the improvement of the modulus of continuity by mixed fractional integration of the Volterra convolution type:

$$\left(\tilde{K}\varphi\right)(x,y)=\int_0^x\int_0^y k(x-t)k(y-s)\varphi(t,s)dtds\,, \tag{1}$$

here we consider the degenerate kernels, as well as each $k(x)$ and $k(y)$ assumed to be close in some sense to a power function. In this paper, we deal with arbitrary kernels, i.e. not necessarily power. We will consider the operator (1) in a rectangle $Q=\{(x,y): 0<x<b,\ 0<y<d\}$.

Preliminary

In this section, we present some well-known results and notation that we will need to present the issues under consideration (see [2]). We follow the papers [3] in the definitions below. Positive constants which can be different at different places will be denoted by C.

Definition 1. We say that $\psi(x)\in W_\mu=W_\mu([0,l])$ if $\psi(x)\in C([0,l])$, $\psi(0)=0, \psi(x)>0$ for $x>0, \psi(x)$ is almost increasing while $\psi(x)/x^\mu$ is almost decreasing and there exists C such that

$$\left|\frac{\psi(x)-\psi(y)}{x-y}\right|\le C\frac{\psi(x^*)}{x^*},\quad x^*=\max(x,y). \tag{2}$$

We remind that non-negative function $\psi(x), 0\le x\le l$, $0<l\le\infty$ is called almost

Increasing (decreasing) if $\psi(x) \le C\psi(y)$ for all $x \le y$ ($x \ge y$ resp.) this notion being due to S. Bernstein.

Definition 2. We say that $\psi(x) \in W_\mu^*$ if $\psi(x) \in W_\mu$ and $\psi(x)/x^{\mu-\varepsilon}$ is almost increasing for all $\varepsilon > 0$.
We shall also need the following modification of the Definition 2.

Definition 3. We say that a non-negative function $k(x)$ on $[0,l]$ belongs to the class V_λ, $\lambda > 0$, if
1) $k(x) \ne 0$, $x^\lambda k(x)$ - is almost increases and $x^\lambda k(x)\big|_{x=0} = 0$;
2) there exists ε, $0 < \varepsilon < \lambda$, such that $x^{\lambda-\varepsilon} k(x)$ is almost decreases;
3) there exists C such that

$$\left|\frac{k(x_1) - k(x_2)}{x_1 - x_2}\right| \le C \frac{k(x^*)}{x^*}, \; x^* = \max(x_1, x_2). \tag{3}$$

Remark 1. $x^\lambda k(x) \in W_\lambda^* \Rightarrow k(x) \in V_\lambda$ and $k(x) \in V_\lambda \Rightarrow k(x) \in W_\lambda$.

Definition 4. Let given bounded on [*a*, *b*] function $\varphi(x)$. Under modulus of continuity $\varphi(x)$ understood expression $\sup\limits_{h \in [0,\delta]} |\varphi(x+h) - \varphi(x)| = \omega(\varphi;\, \delta)$, $0 < \delta \le b - a$.

Definition 3. We denote by Φ^1 function class $\omega(\delta) \in (0, b-a]$ and satisfying the conditions
1) $\omega(\delta) > 0$ in (0, *b*–*a*], $\lim\limits_{\delta \to 0} \omega(\delta) = 0$;
2) $\omega(\delta) \uparrow$ in(0, *b*–*a*];
3) $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$.

Below in the estimates we need inequalities:
1) if $\omega(\varphi; h)$ is modulus continuity, then we have

$$x_2 \omega(\varphi;\, x_1) \le C x_1 \omega(\varphi;\, x_2), \; x_2 \le x_1; \tag{4}$$

2) if $k(x) \in V_\lambda$, then

$$x_1^\lambda k(x_1) \le C x_2^\lambda k(x_2) \text{ and } \exists\, 0 < \varepsilon < \lambda,\; x_2^{\lambda-\varepsilon} k(x_2) \le C x_1^{\lambda-\varepsilon} k(x_1),\; x_1 \le x_2; \tag{5}$$

3) if $\psi(x) \in W_\mu$ then

$$y^\mu \psi(x) \le C x^\mu \psi(y), \quad x \ge y; \tag{6}$$

4) if $0 < \alpha < 1$ then $\dfrac{\omega(\varphi, h)}{h^\alpha} \le C \displaystyle\int_0^h \frac{\omega(\varphi, t)}{t^{1+\alpha}} dt$;

5) if $\psi(x) \in W_\mu$ with $0 < \mu < 1$ then (1) holds with x^* replaced both by x and y:

$$\left|\frac{\psi(x) - \psi(y)}{x - y}\right| \le C \frac{\psi(x)}{x}, \qquad \left|\frac{\psi(x) - \psi(y)}{x - y}\right| \le C \frac{\psi(y)}{y}; \tag{7}$$

6) if $\lambda \le 1$, then

$$|x^\lambda - y^\lambda| \le C(x-y) y^{\lambda-1}, \quad x \ge y > 0, \tag{8}$$

if $\lambda \ge 0$, then

$$|x^\lambda - y^\lambda| \le C(x-y) x^{\lambda-1}, \quad x \ge y > 0. \tag{9}$$

Lemma 1. Let $k(x) \in V_\lambda$, $\lambda > 0$ and $\omega(x) \ge 0$ be an almost increases function. Then for any $0 < x < l/2$, rightly

$$\omega(x) k(x) \le C \int_x^l t^{-1} \omega(t) k(t) dt. \tag{10}$$

Below we follow some technical estimations suggested in [5] for the case of one-dimensional Riemann-Liouville fractional integrals. We denote

$$B(x,y;t,s)=\frac{\rho(x,y)-\rho(t,s)}{\rho(t,s)(x-t)^{1+\alpha}(y-s)^{1+\beta}}, \tag{11}$$

where $0<\alpha,\beta<1, 0<t<x<b,\ \ 0<s<y<d$ and

$$B_1(x,t)=\frac{\rho(x)-\rho(t)}{\rho(t)(x-t)^{1+\alpha}} \text{ and } B_2(y,s)=\frac{\rho(y)-\rho(s)}{\rho(s)(y-s)^{1+\beta}}. \tag{12}$$

In the case $\rho(x,y)=\rho(x)\rho(y)$ we have

$$B(x,y;t,s)=B_1(x,t)B_2(y,s)+\frac{B_1(x,t)}{(y-s)^{1+\beta}}+\frac{B_2(y,s)}{(x-t)^{1+\alpha}}.$$

Let also

$$D_1(x,h,t)=B_1(x+h,t)-B_1(x,t),\ t,x,x+h\in[0,b],\ h>0,$$
$$D_2(y,\eta,s)=B_2(y+\eta,s)-B_2(y,s),\ s,y,y+\eta\in[0,d],\eta>0.$$

Lemma 2. Let $\rho(x)=\psi(x-a), \psi(x)\in W_\mu, \mu\in \mathrm{R}^1$ and $k(x)\in V_\lambda, \lambda>0$. Then

$$|B_1(x,t)|\le C\left(\frac{x-a}{t-a}\right)^{\max(\mu-1,0)}\frac{x-t}{t-a}k(x-t), \tag{13}$$

$$|D_1(x,h,t)|\le C\left(\frac{x+h-a}{t-a}\right)^{\max(\mu-1,0)}\frac{h}{t-a}k(x+h-t). \tag{14}$$

Proof. Let $\mu<1$, then from (2) and (8) имеем

$$|B_1(x,t)|=\left|\frac{\psi(x-a)-\psi(t-a)}{\psi(t-a)}k(x-t)\right|\le\frac{x-t}{t-a}k(x-t).$$

If $1\le\mu$, then from inequalities (2) and (6), we have

$$|B_1(x,t)|\le\left|\frac{\psi(x-a)}{\psi(t-a)}-1\right|k(x-t)\le C\left|\frac{(x-a)^\mu-(t-a)^\mu}{(t-a)^\mu}\right|k(x-t)\le C\left(\frac{x-a}{t-a}\right)^{\mu-1}\frac{x-t}{t-a}k(x-t).$$

Now let's estimate $D_1(x,h,t)$. Let $\mu<1$ then from inequalities (2), (6), (3) and (5), we obtain

$$|D_1(x,h,t)|\le\left|\frac{\psi(x+h-a)-\psi(x-a)}{\psi(t-a)}\right|k(x+h-t)+$$
$$+\left|\frac{\psi(x-a)-\psi(t-a)}{\psi(t-a)}\right||k(x+h-t)-k(x-t)|\le C\left[\frac{h}{x-a}\frac{\psi(x-a)}{\psi(t-a)}k(x+h-t)+\right.$$
$$\left.+\frac{(x-a)\psi(t-a)k(x+h-t)}{(t-a)\psi(t-a)(x+h-t)}h\right]\le C\left[\frac{h}{x-a}\left(\frac{x-a}{t-a}\right)^{\mu}k(x+h-t)+\right.$$
$$\left.+\frac{h}{t-a}k(x+h-t)\right]=C\frac{h}{t-a}k(x+h-t)\left[\left(\frac{x-a}{t-a}\right)^{\mu-1}+1\right].$$

Since $\mu<1$ then

$$|D_1(x,h,t)|\le C\frac{h}{t-a}k(x+h-t)\left[\left(\frac{t-a}{x-a}\right)^{1-\mu}+1\right]=C_1\frac{h}{t-a}k(x+h-t).$$

If $1\le\mu$ then

$$|D_1(x,h,t)|\le C\left[h\frac{\psi(x+h-a)}{\psi(t-a)}\frac{k(x+h-t)}{x+h-a}+h\frac{\psi(x-a)}{\psi(t-a)}\frac{k(x+h-t)}{x-a}\right]\le$$
$$\le C\left[\left(\frac{x+h-a}{t-a}\right)^{\mu-1}\frac{h}{t-a}k(x+h-t)+h\left(\frac{x-a}{t-a}\right)^{\mu-1}\frac{k(x+h-t)}{t-a}\right]\le Ch\left(\frac{x+h-a}{t-a}\right)^{\mu-1}\frac{k(x+h-t)}{t-a}.$$

Similar estimates hold for $B_2(y,s)$ and $D_2(y,\eta,s)$ with $\rho(y)=y^{\nu}$.
Consider a one-dimensional integral operator of the Volterra type convolutions

$$(K\varphi)(x)=\int_0^x k(x-t)\varphi(t)dt,\quad 0<x<b. \tag{15}$$

The following statements are known (see [2]). We use the schemes of the proofs to make the presentation easier for two-dimensional case. The following theorem provides a Zygmund type estimate for the integral (15).

Theorem 1. Let $k(x)\in V_\lambda$, $0<\lambda<1$ and $\varphi(x)\in C([0,\,b])$, $\varphi(0)=0$. Then for integral (15), the following estimate is valid

$$\omega(K\varphi,\,h)\le C\left(hk(h)\omega(\varphi,h)+h\int_h^b\frac{k(t)\omega(\varphi,\,t)}{t}dt\right) \tag{16}$$

Proof. The integral (15) will be presented in the form

$$(K\varphi)(x)=\varphi(0)\int_0^x k(x-t)dt+\int_0^x k(x-t)[\varphi(t)-\varphi(0)]dt\ .$$

Since $\varphi(0)=0$, then we have $f(x)=\int_0^x g(t)k(x-t)dt$, where $\varphi(t)-\varphi(0)=g(t)$.

Let $h>0$, $x,x+h\in[0,b]$. Consider the difference

$$|f(x+h)-f(x)|\le C\left|\int_{-h}^0 k(h+t)[g(x-t)-g(x)]dt+\int_0^x(k(h+t)-k(t))[g(x-t)-g(x)]dt+g(x)\int_x^{x+h}k(t)dt\right|=$$
$$A_1+A_2+A_3.$$

Taking (5) and increasing of $\omega(\varphi,t)$ into account, we have for A_1

$$A_1\le\int_0^h\omega(\varphi,\,t)k(h-t)dt\le C\omega(\varphi,\,h)k(h)\int_0^h\left(\frac{h}{h-t}\right)^{\lambda}dt\le Chk(h)\omega(\varphi,\,h)$$

For A_2 applying (3) and (5), we obtain in the case $h\ge x$

$$\begin{aligned}A_2&\le C\int_0^x\frac{k(h+t)}{h+t}\omega(\varphi;\,t)dt\le Ch^{1+\lambda-\varepsilon}k(h)\int_0^x\frac{\omega(\varphi;\,t)}{(h+t)^{1+\lambda-\varepsilon}}dt=\\&=Chk(h)\int_0^{\frac{x}{h}}\frac{\omega(\varphi,\,ht)dt}{(1+t)^{1+\lambda-\varepsilon}}\le Chk(h)\int_0^1\frac{\omega(\varphi,\,ht)}{(1+t)^{1+\lambda-\varepsilon}}dt\le Chk(h)\omega(\varphi,\,h)\ .\end{aligned} \tag{17}$$

In the $h<x$ we write $A_2\le\int_0^h+\int_h^{x-a}=A_2'+A_2''$. For A_2' the estimate (17) is valid, while for A''_2, we have

$$A_2''\le Ch\int_h^x\omega(\varphi,\,t)\frac{k(t)}{t}dt\le Ch\int_h^b\frac{\omega(\varphi,\,t)}{t}k(t)dt.$$

As regards A_3, we have in the case $x\le h$:

$$A_3\le C\omega(\varphi,\,x)k(x+h)(x+h)^{\lambda}\int_x^{x+h}\frac{dt}{t^{\lambda}}\le C\omega(\varphi,\,h)k(h)h.$$

If $x>h$, we use Lemma 1 to obtain

$$A_3\le C\omega(\varphi,\,x)hk(x)\le Ch\int_x^b\frac{k(t)}{t}\omega(\varphi;t)dt\le Ch\int_x^b\frac{k(t)}{t}\omega(\varphi;t)dt\ .$$

Gathering all the estimates for A_1, A_2, A_3, we arrive at (16).

Theorem 2. Let $k(x)\in V_\lambda$, $0<\lambda<1$, $\rho(x)=\psi(x-a)$, $\psi(x)\in W_\mu$, $0<\mu<1+\lambda$. Assume that

1) $\rho(x)\varphi(x)\in C([a,b])$ and $\rho(x)\varphi(x)\big|_{x=a}=0$;

2) $\int_0^{b-a} t^{-\gamma}\omega(\rho\varphi,t)dt<\infty,\ \gamma=\max(1,\mu)$. Then the following Zygmund type estimate holds:

$$\omega(\rho K\varphi,h)\le Ch^{\gamma}k(h)\int_0^h\frac{\omega(\rho\varphi,t)}{t^{\gamma}}dt+Ch\int_h^{b-a}\frac{\omega(\rho\varphi,t)}{t}k(t)dt\,, \tag{18}$$

if $0<\mu<1+\lambda$.

Proof. Let $\varphi_0(x)=\rho(x)\varphi(x)$ and $a=0$ for simplicity. We have

$$\rho(x)(K\varphi)(x)=\int_0^x k(x-t)\varphi_0(t)dt+\int_0^x B(x,t)\varphi_0(t)dt=F_1(x)+F_2(x),$$

where $B(x,t)=\dfrac{\psi(x)-\psi(t)}{\psi(t)}k(x-t)$.

Since $\varphi_0\in C([0,b])$ and $\varphi_0(0)=0$ the first term $F_1(x)$ is covered by Theorem 1. To estimate $\omega(F_2,h)$ we represent the difference $F_2(x+h)-F_2(x)$ as

$$F_2(x+h)-F_2(x)=\int_x^{x+h}B(x+h,t)\varphi_0(t)dt+\int_0^x D(x,h,t)\varphi_0(t)dt\,,$$

where $D(x,h,t)=B(x+h,t)-B(x,t)$.

Further

$$F_2(x+h)-F_2(x)=\int_x^{x+h}|B(x+h,t)|\omega(\varphi_0,t)dt+\int_0^x|D(x,h,t)|\omega(\varphi_0,t)dt=I_1+I_2\,.$$

Estimate of I_1. Let $0<\mu<1$ at first. Using (13) we have

$$I_1\le C\int_x^{x+h}\frac{x+h-t}{t}k(x+h-t)\omega(\varphi_0,t)dt\le Chk(h)\int_x^{x+h}\frac{\omega(\varphi_0,t)}{t}dt\,.$$

Given $\dfrac{\omega(\varphi_0,t)}{t}$ decreasing, then we obtain

$$I_1\le Chk(h)\int_x^{x+h}\frac{\omega(\varphi_0,t-x)}{t-x}dt=Chk(h)\int_0^h\frac{\omega(\varphi_0,t)}{t}dt\,.$$

If $1\le\mu<2$, then taking (13)into account, we have

$$I_1\le C\int_x^{x+h}\left(\frac{x+h}{x}\right)^{\mu-1}\frac{x+h-t}{t}k(x+h-t)\omega(\varphi_0,t)dt\,.$$

In the case $x\le h$

$$I_1\le Ch^{\mu}k(h)\int_x^{x+h}\frac{\omega(\varphi_0,t)}{t^{\mu}}dt\le Ch^{\mu}k(h)\int_x^{x+h}\frac{\omega(\varphi_0,t-x)}{(t-x)^{\mu}}dt=Ch^{\mu}k(h)\int_0^h\frac{\omega(\varphi_0,t)}{t^{\mu}}dt\,.$$

In the case $x\ge h$

$$I_1\le Ch(x+h)^{\mu-1}k(h)\int_x^{x+h}\frac{\omega(\varphi_0,t)}{t^{\mu}}dt=C\frac{hk(h)}{(x+h)^{1-\mu}}\int_{-h}^{0}\frac{\omega(\varphi_0,x-t)}{(x-t)^{\mu}}dt\le C_1x^{\mu-1}hk(h)\int_0^h\frac{\omega(\varphi_0,x+t)}{(x+t)^{\mu-1}}\frac{dt}{x+t}\le$$

$$\le Chk(h)\int_0^h\frac{\omega(\varphi_0,t+x)}{t+x}dt\le Chk(h)\int_0^h\frac{\omega(\varphi_0,t)}{t}dt\,.$$

Gathering estimates of I_1 then we have by $0<\mu<2$

$$I_1\le Ch^{\gamma}k(h)\int_0^h\frac{\omega(\varphi_0,t)}{t^{\gamma}}dt\,,\ \text{where}\ \gamma=\max(1,\mu).$$

Estimate of I_2. By $0<\mu<1$ used (14) and we have

$$I_2 \le Ch\int_0^x \frac{\omega(\varphi_0,t)}{t}k(x+h-t)dt. \quad (19)$$

If $x \le h$, then

$$I_2 \le Chk(h)\int_0^x \frac{\omega(\varphi_0,t)}{t}dt \le Chk(h)\int_0^h \frac{\omega(\varphi_0,t)}{t}dt.$$

In the case $x>h$, we represent (19) as

$$I_2 \le Ch\left(\int_0^h + \int_h^{\frac{x+h}{2}} + \int_{\frac{x+h}{2}}^x\right)\frac{\omega(\varphi_0,t)}{t}k(x+h-t)dt = \Delta_1+\Delta_2+\Delta_3.$$

Since $t \le x$ then immediately follow $\Delta_1 \le Chk(h)\int_0^h \frac{\omega(\varphi_0,t)}{t}dt$. For Δ_2 taking into account that $x+h-t \ge t$ we obtain

$$\Delta_2 \le Ch\int_h^{\frac{x+h}{2}} \frac{\omega(\varphi_0,t)}{t}k(t)\left(\frac{t}{x+h-t}\right)^{\lambda-\varepsilon}dt \le Ch\int_h^b \frac{\omega(\varphi_0,t)}{t}k(t)dt.$$

For Δ_3 taking into account that $x+h-t \le t$, so

$$\Delta_3 \le Ch\int_{\frac{x+h}{2}}^x \frac{\omega(\varphi_0,x+h-t)}{x+h-t}k(x+h-t)dt.$$

After replacing $x+h-t=t$ we obtain

$$\Delta_3 \le Ch\int_h^{\frac{x+h}{2}} \frac{\omega(\varphi_0,t)}{t}k(t)dt \le Ch\int_h^b \frac{\omega(\varphi_0,t)}{t}k(t)dt.$$

Thus for I_2 by $0<\mu<1$we have

$$I_2 \le C\left(hk(h)\int_0^h \frac{\omega(\varphi_0,t)}{t}dt + h\int_h^b \frac{\omega(\varphi_0,t)}{t}k(t)dt\right).$$

Let $1 \le \mu < 2$. Taking into account (14) we have

$$I_2 \le Ch\int_0^x \left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(\varphi_0,t)}{t}k(x+h-t)dt. \quad (20)$$

In the case $x \le h$ we have

$$I_2 \le Ch^{\mu}\int_0^x \frac{\omega(\varphi_0,t)}{t^{\mu}}k(x+h-t)dt \le Ch^{\mu+1}k(h)\int_0^x \frac{\omega(\varphi_0,t)dt}{t^{\mu}(x+h-t)} \le Ch^{\mu}k(h)\int_0^x \frac{\omega(\varphi_0,t)}{t^{\mu}}dt \le$$

$$\le Ch^{\mu}k(h)\int_0^h \frac{\omega(\varphi_0,t)}{t^{\mu}}dt.$$

If $x>h$, then we represent the right-hand side of inequality (20) as the sum of three terms

$$f_2'' \le Ch\left(\int_0^h + \int_h^{\frac{x+h}{2}} + \int_{\frac{x+h}{2}}^x\right)\left(\frac{x+h}{t}\right)^{\mu-1}\frac{\omega(\varphi_0,t)}{t}k(x+h-t)dt = T_1+T_2+T_3.$$

Taking into account that $x+h \le 2(x+h-t)$ in T_1 we obtain

$$T_1 \le Ch\int_0^h \left(\frac{x+h-t}{t}\right)^{\mu-1}\frac{\omega(\varphi_0,t)}{t}k(x+h-t)dt \le \le Ch\int_0^h k(h)\frac{\omega(\varphi_0,t}{t}\left(\frac{h}{t}\right)^{\mu-1}dt \le Ch^{\mu}k(h)\int_0^h \frac{\omega(\varphi_0,t)}{t^{\mu}}dt.$$

Since $x+h \geq 2t$ than for T_2 we have

$$T_2 \leq Ch \int_h^{\frac{x+h}{2}} \left(\frac{x+h}{t}\right)^{\mu-1} \frac{\omega(\varphi_0, t)}{t} k(x+h-t)dt \leq Ch \int_h^{\frac{x+h}{2}} t^{\mu-1} k(t) \frac{\omega(\varphi_0, t)}{t^{\mu}} dt \leq Ch \int_h^b \frac{\omega(\varphi_0, t)}{t} k(t)dt.$$

To estimate T_3. we notice that $t \geq x+h-t$ in T_3, so $\frac{\omega(\varphi_0,t)}{t} \leq C\frac{\omega(\varphi_0, x+h-t)}{(x+h-t)}$. Then follows that

$$T_3 \leq Ch \int_{\frac{x+h}{2}}^{x} \left(\frac{x+h}{t}\right)^{\mu-1} \frac{\omega(\varphi_0, x+h-t)}{x+h-t} k(x+h-t)dt \leq Ch \int_{\frac{x+h}{2}}^{x} \frac{\omega(\varphi_0, x+h-t)}{x+h-t} k(x+h-t)dt,$$

because, $x+h \leq 2t$. Having replaced $\xi = x+h-t$ and passing again to the variable t , we obtain

$$T_3 \leq Ch\int_h^b \frac{\omega(\varphi_0, t)}{t} k(t)dt.$$

Collecting all the estimates of I_2 by $1 \leq \mu < 1+\lambda$, we have

$$I_2 \leq C\left(h^{\gamma} k(h)\int_0^h \frac{\omega(\varphi_0, t)}{t^{\gamma}} dt + h\int_h^b \frac{k(t)}{t} \omega(\varphi_0, t)dt \right), \quad \gamma = \max(1, \mu).$$

Gathering estimates I_1 and I_2 for $0 \leq \mu < 1+\lambda$ we have

$$|F_2(x+h) - F_2(x)| \leq C\left(h^{\gamma} k(h)\int_0^h \frac{\omega(\varphi_0, t)}{t^{\gamma}} dt + h\int_h^b \frac{k(t)}{t} \omega(\varphi_0, t)dt \right), \text{ where } \gamma = \max(1,\mu).$$

Let a continuous function $\varphi(x,y)$ be defined in R^2. We introduce the necessary notation

$$\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y) = \varphi(x+h, y) - \varphi(x,y), \left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y) = \varphi(x, y+\eta) - \varphi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y) = \varphi(x+h, y+\eta) - \varphi(x, y+\eta) - \varphi(x+h, y) + \varphi(x,y),$$

and

$$\varphi(x+h, y+\eta) = \left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y) + \left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y) + \left(\overset{1,0}{\Delta}_h \varphi\right)(x,y) + \varphi(x,y). \tag{21}$$

Now we introduce the following characteristics:

1) Private modules of continuity

$$\overset{1,0}{\omega}(\varphi;\ \delta, 0) = \sup_y \sup_{0 \leq h \leq \delta} \left| \left(\overset{1,0}{\Delta}_h \varphi\right)(x,y) \right| \text{ and } \overset{0,1}{\omega}(\varphi;\ 0, \sigma) = \sup_x \sup_{0 \leq \eta \leq \sigma} \left| \left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y) \right|;$$

2) Mixed modulus continuity of order 1.1

$$\overset{1,1}{\omega}(\varphi;\ \delta, \sigma) = \sup_{x,y} \sup_{\substack{0 \leq h \leq \delta \\ 0 \leq \eta \leq \sigma}} \left| \left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y) \right|, \text{ where } 0 < \delta \leq b,\ 0 < \sigma \leq d .$$

It follows from the definition $\overset{1,1}{\omega}(\varphi; \delta, \sigma)$ that this function belongs in each variable Φ^1. In addition, we note that there is an inequality

$$\overset{1,1}{\omega}(\varphi; \delta, \sigma) \leq 2\min\left\{ \overset{1,0}{\omega}(\varphi; \delta, 0),\ \overset{0,1}{\omega}(\varphi; 0, \sigma) \right\}. \tag{22}$$

3. Main result

Here we generalize Theorem 1 to the case $k(x_1, x_2) = k(x_1)k(x_2)$.

Theorem 3. Let $k(x_i) \in V_{\lambda_i}, 0 < \lambda_i < 1,\ i = 1,2,\ \ \varphi(x_1, x_2) \in C(Q)$ and $\varphi(x_1, x_2)|_{x_1 = a, x_2 = c} = 0$. Then Zygmund type estimates are valid

$$\overset{1,0}{\omega}(\tilde{K}\varphi;h,0)\le C_1\left[hk(h)\overset{1,1}{\omega}(\varphi;h,d)+h\int_h^{b-a}\frac{k(t)}{t}\overset{1,1}{\omega}(\varphi;t,d)dt\right],$$

$$\overset{0,1}{\omega}(\tilde{K}\varphi;0,\eta)\le C_2\left[\eta k(\eta)\overset{1,1}{\omega}(\varphi;b,\eta)+\eta\int_\eta^{d-c}\frac{k(s)}{s}\overset{1,1}{\omega}(\varphi;b,s)ds\right],$$

$$\overset{1,1}{\omega}(\tilde{K}\varphi;h,\eta)\le C_3\left[h\eta k(h)k(\eta)\overset{1,1}{\omega}(\varphi;h,\eta)+h\eta k(\eta)\int_h^{b-a}\frac{k(t)}{t}\overset{1,1}{\omega}(\varphi;t,\eta)dt+hk(h)\eta\int_\eta^{d-c}\frac{k(s)}{s}\overset{1,1}{\omega}(\varphi;h,s)ds+\right.$$
$$\left.+h\eta\int_h^{b-a}\int_\eta^{d-c}\frac{k(t)k(s)}{ts}\overset{1,1}{\omega}(\varphi;t,s)dtds\right].$$

Proof. We will not prove this theorem. For a proof of this theorem, see [13].
Now we generalize Theorem 2 to the cases $k(x_1,x_2)=k(x_1)k(x_2)$ and $\rho(x_1,x_2)=\psi(x_1-a)\psi(x_2-c)$.

Theorem 4. Let $k(x_i)\in V_{\lambda_i}$, $0<\lambda_i<1$, and $\psi(x_i)\in W_{\mu_i}$, $0<\mu_i<2$, where $i=1,2$. Assume that $\varphi_0(x_1,x_2)=\rho(x_1,x_2)\varphi(x_1,x_2)\in C(Q)$ and $\varphi_0(x_1,x_2)|_{x_1=a,x_2=c}=0$;
$\int_0^{b-a}\int_0^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{t^\delta s^\sigma}dtds<\infty$, where $\delta=\max(1,\mu_1)$, $\sigma=\max(1,\mu_2)$.
Then the following Zygmund type estimates holds

$$\overset{1,0}{\omega}(\rho K\varphi;h,0)\le C_1\left[h^\delta k(h)\int_0^h\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t^\delta}dt+h\int_h^{b-a}\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t}k(t)dt\right],\tag{23}$$

$$\overset{0,1}{\omega}(\rho K\varphi;0,\eta)\le C_2\left[\eta^\sigma k(\eta)\int_0^\eta\frac{\overset{1,1}{\omega}(\varphi_0;b-a,s)}{s^\sigma}ds+\eta\int_\eta^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;b-a,s)}{s}k(s)ds\right],\tag{24}$$

$$\overset{1,1}{\omega}(\rho K\varphi;h,\eta)\le C_3\left[h^\delta\eta^\sigma k(h)k(\eta)\int_0^h\int_0^\eta\frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{t^\delta s^\sigma}dtds+h\eta^\sigma k(\eta)\int_h^{b-a}\int_0^\eta\frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{ts^\sigma}k(t)dt+\right.$$
$$\left.+h^\delta\eta k(\eta)\int_0^h\int_\eta^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{t^\delta s}k(s)dtds+h\eta\int_h^{b-a}\int_\eta^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{ts}k(t)k(s)dtds\right],\tag{25}$$

если $0<\mu_i<1+\lambda_i$, $i=1,2$.

Proof. Let $\varphi_0(x_1,x_2)=\rho(x_1,x_2)\varphi(x_1,x_2)$. We denote

$$G(x_1,x_2)=(\rho K\varphi)(x_1,x_2)=\int_a^{x_1}\int_c^{x_2}\frac{\rho(x_1,x_2)}{\rho(t,s)}\varphi_0(t,s)k(x_1-t)k(x_2-s)dtds.\tag{26}$$

Since we consider the decaying weight function, i.e., the weight function in the form $\rho(x_1,x_2)=\psi(x_1-a)\psi(x_2-c)$, then the equality

$$\psi(x_1)\psi(x_2)=[\psi(x_1)-\psi(t)][\psi(x_2)-\psi(s)]+\psi(s)[\psi(x_1)-\psi(t)]+$$
$$+\psi(t)[\psi(x_2)-\psi(s)]+\psi(t)\psi(s).\tag{27}$$

Equality (26) can be represented as

$$G(x_1,x_2)=\int_a^{x_1}\int_c^{x_2}\varphi_0(t,s)k(x_1-t)k(x_2-s)dtds+\int_a^{x_1}\int_c^{x_2}B(x_1,x_2;t,s)\varphi_0(t,s)dtds=$$
$$=G_1(x_1,x_2)+G_2(x_1,x_2).\tag{28}$$

Here, the question of estimating the modulus of continuity for the first term is solved by us in Theorem 3.

Taking into account (27), we represent $G_2(x_1,x_2)$ in the following form

$$G_2(x_1,x_2)=\int_a^{x_1}\int_c^{x_2}B(x_1,t)B(x_2,\tau)\varphi_o(t,s)dtds+\int_a^{x_1}\int_c^{x_2}B(x_1,t)k(x_2-s)\varphi_0(t,s)dtds+$$
$$+\int_a^{x_1}\int_c^{x_2}B(x_2,s)k(x_1-t)\varphi_0(t,s)dtds=f_1(x_1,x_2)+f_2(x_1,x_2)+f_3(x_1,x_2), \quad (29)$$

where

$$B(x_1,t)=\frac{\psi(x_1-a)-\psi(t-a)}{\psi(t-a)}k(x_1-t), \quad B(x_2,s)=\frac{\psi(x_2-c)-\psi(s-c)}{\psi(s-c)}k(x_2-s).$$

Let us estimate each term $f_i(x_1,x_2),\ i=1,2,3$ separately.

Let $x_1,\ x_2+h\in[a,b]$. Consider the difference

$$f_1(x_1+h,x_2)-f_1(x_1,x_2)=\int_{x_1}^{x_1+h}\int_c^{x_2}B(x_1+h,t)B(x_2,s)\varphi_0(t,s)dtds++\int_a^{x_1}\int_c^{x_2}D(x_1,h,t)B(x_2,s)\varphi_0(t,s)dtds.$$

So $\varphi_0(x_1,x_2)|_{x_1=a,x_2=c}=0$, then inequality holds

$$\left|f_1(x_1+h,x_2)-f_1(x_1,x_2)\right|\le\int_{x_1}^{x_1+h}\int_c^{x_2}|B(x_1+h,t)||B(x_2,s)|\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds+$$
$$+\int_a^{x_1}\int_c^{x_2}|D(x_1,h,t)||B(x_2,s)|\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds.$$

We estimate integral

$$\int_c^{x_2}|B(x_2,s)|\overset{1,1}{\omega}(\varphi_0;t-a,s-c)ds. \quad (30)$$

Let $0<\mu_2<1$. From Lemma 2, we have

$$\int_c^{x_2}|B(x_2,s)|\overset{1,1}{\omega}(\varphi_0;t-a,s-c)ds\le C\int_c^{x_2}\frac{(x_2-s)k(x_2-s)}{s-c}\overset{1,1}{\omega}(\varphi_0;t-a,s-c)ds\le$$
$$\le C(x_2-c)k(x_2-c)\int_0^{x_2-c}\frac{\overset{1,1}{\omega}(\varphi_0;t-a,s)}{s}ds\le C(x_2-c)k(x_2-c)\overset{1,1}{\omega}(\varphi_0;t-a,x_2-c)\le\le C\overset{1,1}{\omega}(\varphi_0;t-a,d-c).$$

For $1\le\mu_2<2$ we taking into account (13), we have

$$\int_c^{x_2}|B(x_2,s)|\overset{1,1}{\omega}(\varphi_0;t-a,s-c)ds\le C\int_c^{x_2}\left(\frac{x_2-c}{s-c}\right)^{\mu_2-1}\frac{x_2-s}{s-c}k(x_2-s)\overset{1,1}{\omega}(\varphi_0;t-a,s-c)ds=$$
$$=C\int_0^{x_2-c}\left(\frac{x_2-c}{s}\right)^{\mu-1}\frac{\overset{1,1}{\omega}(\varphi_0;t-a,s)}{s}(x_2-s+c)k(x_2-s+c)ds\le$$
$$\le C(x_2-c)k(x_2-c)\int_0^{x_2-c}\frac{\overset{1,1}{\omega}(\varphi_0;t-a,s)}{s}ds\le$$
$$\le C(x_2-c)k(x_2-c)\overset{1,1}{\omega}(\varphi_0;t-a,x_2-c)\le C_1\overset{1,1}{\omega}(\varphi_0;t-a,d-c).$$

Using estimates of (30) and estimates for $I_1,\ I_2$ from Theorem 2, we obtain estimate by $0\le\mu_1<1+\lambda_1$

$$\left|f_1(x_1+h,\ x_2)-f_1(x_1,\ x_2)\right|\le C\left(h^\delta k(h)\int_0^h\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t^\delta}dt+h\int_h^{b-a}\frac{k(t)}{t}\overset{1,1}{\omega}(\varphi_0;t,d-c)dt\right),$$

where $\delta=\max(1,\mu_1)$.

Now, we estimate $f_2(x_1,x_2)$. Let $h>0;\ x_1,x_1+h\in[a,b]$. Consider the difference

$$\left|f_2(x_1+h,\,x_2)-f_2(x_1,x_2)\right|\le\int\limits_{x_1}^{x_1+h}\int\limits_{c}^{x_2}|B(x_1+h,t)|\,k(x_2-s)\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds+$$
$$+\int\limits_{a}^{x_1}\int\limits_{c}^{x_2}|D(x_1,h,t)|\,k(x_2-s)\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds.$$

So

$$\int\limits_{c}^{x_2}\overset{1,1}{\omega}(\varphi_0;t-a,s-c)k(x_2-s)ds\le C\overset{1,1}{\omega}(\varphi_0;t-a,x_2-c)\int\limits_{c}^{x_2}k(x_2-s)ds\le$$
$$\le C\overset{1,1}{\omega}(\varphi_0;t-a,x_2-c)k(x_2-c)(x_2-c)^{\lambda_2}\int\limits_{c}^{x_2}\frac{ds}{(x_2-s)^{\lambda_2}}\le C\overset{1,1}{\omega}(\varphi_0;t-a,d-c).$$

Then we have

$$\left|f_2(x_1+h,\,x_2)-f_2(x_1,\,x_2)\right|\le C\left(\int\limits_{x_1}^{x_1+h}|B(x_1+h,t)|\overset{1,1}{\omega}(\varphi_0;t-a,d-c)dt+\right.$$
$$\left.+\int\limits_{a}^{x_1}\left|D(x_1,h,t)\right|\overset{1,1}{\omega}(\varphi_0;t-a,d-c)dt\right)$$

It is easy to verify the validity of the assessment by taking into account the estimates I_1 and I_2 from Theorem 2, if $0<\mu_1<1+\lambda_1$

$$\left|f_2(x_1+h,\,x_2)-f_2(x_1,x_2)\right|\le C\left(h^{\delta}k(h)\int\limits_{0}^{h}\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t^{\delta}}dt+h\int\limits_{h}^{b-a}\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t}k(t)dt\right),$$

где $\delta=\max(\mu_1,1)$.

Now f_3, let's evaluate. Imagine the difference

$$f_3(x_1+h,\,x_2)-f_3(x_1,\,x_2)=\int\limits_{x_1}^{x_1+h}\int\limits_{c}^{x_2}\varphi_0(t,s)k(x_1+h-t)B(x_2,s)dtds+$$
$$+\int\limits_{a}^{x_1}\int\limits_{c}^{x_2}[k(x_1+h-t)-k(x_1,t)]\varphi_0(t,s)B(x_2,s)dtds.$$

Since $\varphi_0(x_1,c)=0$ we have

$$\left|f_3(x+h,\,y)-f_3(x,\,y)\right|\le\int\limits_{-h}^{0}\int\limits_{c}^{x_2}|\varphi_0(x_1-t,s)-\varphi(x_2,c)||B(x_2,s)|\,k(h+t)dtds+$$
$$+\int\limits_{0}^{x-a}\int\limits_{c}^{y}|k(h+t)-k(t)||\varphi_0(x-t,\tau)-\varphi_0(x,c)||B(y,\tau)|\,dtd\tau\le$$
$$\le C\left(\int\limits_{-h}^{0}\int\limits_{c}^{x_2}\overset{1,1}{\omega}(\varphi_0;t,s-c)|B(x_2,s)|\,k(h+t)dtds+\right.$$
$$\left.+\int\limits_{0}^{x_1-a}\int\limits_{c}^{x_2}|k(h+t)-k(t)|\overset{1,1}{\omega}(\varphi_0;t,s-c)|B(x_2,s)|\,dtds\right).$$

We obtain

$$\left|f_3(x_1+h,\,x_2)-f_3(x_1,\,x_2)\right|\le C\left(\int\limits_{-h}^{0}\overset{1,1}{\omega}(\varphi_0;t,d-c)k(h+t)dt+\int\limits_{0}^{x_1-a}\overset{1,1}{\omega}(\varphi_0;t,d-c)|k(h+t)-k(t)|\,dt\right).$$

Using the estimates $A_1,\ A_2$ from Theorem 1, we obtain the estimate

$$\left|f_3(x_1+h,x_2)-f_3(x_1,x_2)\right|\le C\left(hk(h)\overset{1,1}{\omega}(\varphi_0;h,d-c)+h\int_h^{b-a}\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t}k(t)dt\right).$$

Gathering estimates for $f_1(x_1,x_2)$, $f_2(x_1,x_2)$ and $f_3(x_1,x_2)$ we obtain

$$\left|G_2(x_1+h,x_2)-G_2(x_1,x_2)\right|\le C\left[h^{\delta}k(h)\int_0^h\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t^{\delta}}dt+h\int_h^{b-a}\frac{\overset{1,1}{\omega}(\varphi_0;t,d-c)}{t}k(t)dt\right],$$

where $\delta=\max(\mu_1,1)$.

The estimate

$$\left|G_2(x_1,x_2+\eta)-G_2(x_1,x_2)\right|\le C\left[\eta^{\sigma}k(\eta)\int_0^{\eta}\frac{\overset{1,1}{\omega}(\varphi_0;b-a,s)}{s^{\sigma}}ds+\eta\int_{\eta}^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;b-a,s)}{s}k(s)ds\right],$$

$\sigma=\max(1,\mu_2)$ is symmetrical obtained.

Now, we prove the validity of the estimate (25). Let be $h,\eta>0,\ x_1,x_1+h\in[a,b],\ x_2,x_2+\eta\in[c,d]$. Consider the difference

$$\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x_1,x_2)\le C\Bigg\{\int_{x_1}^{x_1+h}\int_{x_2}^{x_2+\eta}B(x_2+h,t)B(x_2+\eta,s)\varphi_0(t,s)dtds+$$

$$+\int_a^{x_1}\int_c^{x_2}D(x_1,h,t)D(x_2,\eta,s)\varphi_0(t,s)dtds+\int_{x_1}^{x_1+h}\int_c^{x_2}B(x_1+h,t)D(x_2,\eta,s)\varphi_0(t,s)dtds+$$

$$+\int_a^{x_1}\int_{x_1}^{x_1+\eta}D(x_1,h,t)B(x_2+\eta,s)\varphi_0(t,s)dtds+\int_{x_1}^{x_1+h}\int_{-\eta}^{0}k(\eta+s)B(x_1+h,t)\varphi_0(t,s)dtds+$$

$$+\int_{x_1}^{x_1+h}\int_0^{x_2-c}B(x_1+h,t)[k(\eta+s)-k(s)]\varphi_0(t,s)dtds+\int_a^{x_1}\int_{-\eta}^{0}D(x_1,h,t)k(\eta+s)\varphi_0(t,s)dtds+$$

$$+\int_a^{x_1}\int_0^{x_2-c}D(x_1,h,t)[k(\eta+s)-k(s)]\varphi_0(t,s)dtds+\int_{-h}^{0}\int_{x_2}^{x_2+\eta}k(h+t)B(x_2+\eta,s)\varphi_0(t,s)dtds+$$

$$+\int_0^{x_1-a}\int_{x_2}^{x_2+\eta}[k(h+t)-k(t)]B(x_2+\eta,s)\varphi_0(t,s)dtds+\int_{-h}^{0}\int_c^{x_2}k(h+t)D(x_2,\eta,s)\varphi_0(t,s)dtds+$$

$$\int_0^{x_1-a}\int_c^{x_2}[k(h+t)-k(t)]D(x_2,\eta,s)\varphi_0(t,s)dtds\Bigg\}.$$

Since $\varphi_0(x_1,x_2)|_{x_1=a,x_2=c}=0$ then the inequalities

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}G_2\right)(x_1,x_2)\right|\le C\left|\int_{x_1}^{x_1+h}\int_{x_2}^{x_2+\eta}B(x_2+h,t)B(x_2+\eta,s)\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds+\right.$$

$$+\int_a^{x_1}\int_c^{x_2}D(x_1,h,t)D(x_2,\eta,s)\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds+\int_{x_1}^{x_1+h}\int_c^{x_2}B(x_1+h,t)D(x_2,\eta,s)\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds+$$

$$+\int_a^{x_1}\int_{x_1}^{x_1+\eta}D(x_1,h,t)B(x_2+\eta,s)\overset{1,1}{\omega}(\varphi_0;t-a,s-c)dtds+\int_{x_1}^{x_1+h}\int_{-\eta}^{0}k(\eta+s)B(x_1+h,t)\overset{1,1}{\omega}(\varphi_0;t-a,s)dtds+$$

$$+\int_{x_1}^{x_1+h}\int_0^{x_2-c}B(x_1+h,t)[k(\eta+s)-k(s)]\overset{1,1}{\omega}(\varphi_0;t-a,s)dtds+\int_a^{x_1}\int_{-\eta}^{0}D(x_1,h,t)k(\eta+s)\overset{1,1}{\omega}(\varphi_0;t-a,s)dtds+$$

$$+\int_a^{x_1}\int_0^{x_2-c} D(x_1,h,t)[k(\eta+s)-k(s)]\overset{1,1}{\omega}(\varphi_0;t-a,s)dtds+\int_{-h}^{0}\int_{x_2}^{x_2+\eta} k(h+t)B(x_2+\eta,s)\overset{1,1}{\omega}(\varphi_0;t,s-c)dtds+$$

$$+\int_0^{x_1-a}\int_{x_2}^{x_2+\eta}[k(h+t)-k(t)]B(x_2+\eta,s)\overset{1,1}{\omega}(\varphi_0;t,s-c)dtds+\int_{-h}^{0}\int_c^{x_2} k(h+t)D(x_2,\eta,s)\overset{1,1}{\omega}(\varphi_0;t,s-c)dtds+$$

$$\left.\int_0^{x_1-a}\int_c^{x_2}[k(h+t)-k(t)]D(x_2,\eta,s)\overset{1,1}{\omega}(\varphi_0;t,s-c)dtds\right|. \quad (31)$$

We skip the details of calculating each term in inequality (31). The calculation is carried out in the standard way using Lemma 2, inequalities (4) - (9) and using the estimates $A_1,\ A_2,\ A_3$ from Theorem 1, it is easy to verify the inequality

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} G_2\right)(x_1,x_2)\right| \le C_3\left[h^\delta\eta^\sigma k(h)k(\eta)\int_0^h\int_0^\eta \frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{t^\delta s^\sigma}dtds + h\eta^\sigma k(\eta)\int_h^{b-a}\int_0^\eta \frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{ts^\sigma}k(t)dtds+\right.$$

$$\left.+h^\delta\eta k(\eta)\int_0^h\int_\eta^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{t^\delta s}k(s)dtds + h\eta\int_h^{b-a}\int_\eta^{d-c}\frac{\overset{1,1}{\omega}(\varphi_0;t,s)}{ts}k(t)k(s)dtds\right],$$

if $0<\mu_i<1+\lambda_i,\ i=1,2$.

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