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Finite groups with weakly-supplemented subgroups of prime orders

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Abstract

A subgroup H of a finite group G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G=HK$. In the paper, one interesting result with weakly-supplemented minimal subgroups of G is obtained. Some known results are generalized.

Keywords: Finite group, weakly-supplemented subgroups, supersolvable groups

1. Introduction

It is well known that a subgroup H of a finite group G is complemented in G if there exists a subgroup K of G such that $G=HK$ and $H \cap K = 1$. Such a subgroup K of G is called a complement to H in G . It is clear that the existence of complements for certain subgroups of a finite group gives a lot of useful information about its structure, such as in [1, 2, 4-7] etc. For instance, P. Hall in [4] proved that a finite group is solvable if and only if every Sylow subgroup of G is complemented. Arad and Ward in [1] proved that a finite group is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented. In particular, Hall in [5] proved that a finite G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G . Ballester-Bolínches and Guo in [2] shown that: Let P be a Sylow p -subgroups of G , where G is a finite group and p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is complemented in G , then G is p -nilpotent. Qiao, Isaacs and Li in [7] extended the above result and proved that: Let P be a Sylow p -subgroups of G , where G is a finite group and p is the smallest prime divisor of $|G|$. Let d be a power of p with $1 < d < |P|$, and assume that every subgroup of P having order d is complemented in G . Suppose also that $N_G(P)$ is p -nilpotent. Then G is p -nilpotent. In a recent paper, Kong and Liu in [6] studied finite groups for which every minimal subgroup is weakly-supplemented. A subgroup H of G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G=HK$. They proved that every minimal subgroup of G is weakly-supplemented in G if and only if G is a supersolvable group and all Sylow subgroups of G are elementary abelian.

In this note, we further investigate the influence of weakly-supplemented subgroups on the structure of finite groups along the above direction. Our main result is the following:

Theorem A. Let G be a finite group, $p \in \pi(G)$. Every subgroup of order p is weakly-supplemented in a p -solvable group G if and only if G is a p -supersolvable group with elementary abelian Sylow p -subgroups.

All groups considered in this paper are finite groups. Our notation and terminology are standard. The reader may refer to ref. [3].

2. Preliminary results

In this section, we give one lemma which are useful for our main results.

Lemma 2.1 ([6]) Let G be a group. $p \in \pi(G)$, Suppose that every subgroup of G of order p is weakly-supplemented in G . Then:

- i. If H is a subgroup of G , then every subgroup of H of order p is weakly-supplemented in H ; in particular, a Sylow p -subgroup of G is elementary abelian and weakly-supplemented in G .

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- ii. If N is a normal subgroup of G and G is p -solvable, then every subgroup of G/N of order p is weakly-supplemented in G/N .

3. Proof of the Main Theorem

Proof of Theorem A. By Lemma 2.1(i), every Sylow p -subgroup of G is elementary abelian and weakly-supplemented in G . Let us verify that G is p -supersolvable. We use induction on the order of G . By Lemma 2.1(ii), the conditions of the theorem hold for all quotient groups, so we may assume $O_{p'}(G) = \Phi(G) = 1$, $F(G) = O_p(G) = C_G(F(G))$, and N is a minimal normal subgroup of G . It follows that G has a maximal subgroup M with $G = [N]M$. Let X be a subgroup of N of prime order p . By the hypotheses of the theorem, there is a proper subgroup Y of G such that $G = XY$. The subgroup $N_1 = N \cap Y$ is normal in Y and centralized by X . Hence N_1 is normal in G . But N is a minimal normal subgroup of G . Thus $N_1 = 1$, N has prime order p , so G is p -supersolvable. Thus necessity is proved.

Now we prove sufficiency. Let G be a p -supersolvable group with elementary abelian Sylow p -subgroups. Fix an arbitrary subgroup A of prime order p . Suppose that there exists a normal subgroup $N \neq 1$ such that A is not contained in N . Then $A \cap N = 1$ and by induction, the subgroup AN/N is weakly-supplemented in G/N . Suppose that B/N is weakly-supplement for AN/N in G/N . Then $G/N = (AN/N)B/N$, $AN/N \cap B/N = N/N$, $G = (AN)B = AB$, and $A \cap B = 1$ since $|G : B| = p$. So we may assume that every normal subgroup of the group G contains A . Since every minimal normal subgroup of a p -supersolvable group either has order p or is a p' -subgroup, it follows that A is the only minimal normal subgroup of G . But A is weakly-supplemented in the Sylow p -subgroup since the latter is elementary abelian. The theorem of Gaschutz of [3, Theorem I.17.4] implies that the subgroup A is weakly-supplemented in G . Thus our proof is complete now.

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