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## Ops transformation

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### Abstract

Ordinary power series have important role in discrete mathematics, recently involving complicated calculations and manipulations. The more variables included, the more complicated it will be. In this article, we will construct new perspective of ordinary power series through generating new transformation which will ignore sigma notation and minimize involving too many variables when we do some complicated calculations. The result of this research is new form of ordinary power series called Ops transformation and its basic properties.

**Keywords:** Ops transformation, ordinary power series, ordinary generating function

### 1. Introduction

Generating functions are one of the most surprising and useful inventions in discrete mathematics (Meyer, 2005). Generating functions play important roles in mathematics such as solving the recurrence relations. Solving some kinds of differential equations, counting on convergence value of series, simplifying a function form and so on.

Actually counting and manipulating generating functions is not that easy. The more complicated a problem it will be the harder to manipulate. There still a lack theory discovering the method of counting the ordinary power series. There are many variables and indexes involved in general form of a power series. The simplest form of power series is the form

$$P(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \quad (1.1)$$

The above equation contains several variables involving such as  $x$ ,  $n$ ,  $a_n$ , and  $c$ . For more complicated form will take the form

$$P(x) = \sum_{n=0}^{\infty} a_n (\gamma(x) - c)^n \quad (1.2)$$

Which is replacing free variable  $x$  by some smooth function  $\gamma$ .

There are several ways to simplify the methods to solve such equations. One of them is by transforming such equations to the other isomorphics spaces which is simpler to analyze it. Wilf (1994) stated the notion of power series as following.

**Definition 1.1.** Let be a sequence of real numbers. A formal power series is an expression of form

$$a_0 + a_1x + a_2x^2 + \dots \quad (1.3)$$

Furthermore, the sequence is called coefficient sequence of the corresponding expression (1.3). It's trivial to check that the collections of power series is a real linear space which is isomorphic to the real sequence space by mapping

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$$\{a_n\} \mapsto \sum_{n=0}^{\infty} a_n x^n \tag{1.4}$$

One of the benefit of power series is to solve the recurrence relation. As an example lets take a look at Fibonacci equation

$$F_{n+2} = F_n + F_{n+1} \quad (n \geq 1, F_0 = 0, F_1 = 1) \tag{1.5}$$

Wilf (1994) solved it by modelling the relation to a generating function

$$F(x) = \sum_{n=0}^{\infty} F_n x^n$$

Multiplying (1.5) by  $x$  one can get the left side

$$F_2 x + F_3 x^2 + F_4 x^3 + \dots = \frac{F(x) - x}{x} \tag{1.6}$$

and the right side will be

$$\{F_1 x + F_2 x^2 + \dots\} + \{F_0 x + F_1 x^2 + \dots\} = F(x) + xF(x) \tag{1.7}$$

Hence the equation become

$$\frac{F(x) - x}{x} = F(x) + xF(x) \tag{1.8}$$

Therefore we conclude

$$F(x) = \frac{x}{1 - x - x^2}$$

Now we have to transform back  $F(x)$  to its original form to find out the value of coefficient  $F_n$  corresponding to McLaurin

series of  $\frac{x}{1 - x - x^2}$ .

By some algebra treatment one can get

$$\begin{aligned} \frac{x}{1 - x - x^2} &= \frac{x}{(1 - xr_+)(1 - xr_-)} \\ &= \frac{1}{(r_+ - r_-)} \left( \frac{1}{(1 - xr_+)} - \frac{1}{(1 - xr_-)} \right) \\ &= \frac{1}{\sqrt{5}} \left( \sum_{n=0}^{\infty} r_+^n x^n - \sum_{n=0}^{\infty} r_-^n x^n \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (r_+^n - r_-^n) x^n \\ &= \sum_{n=0}^{\infty} F_n x^n \end{aligned}$$

by  $r_{\pm} = (1 \pm \sqrt{5}) / 2$ .

Therefore we get the solution  $F_n = \frac{1}{\sqrt{5}}(r_+^n - r_-^n)$ ,  $n = 0, 1, 2, \dots$  (1.9)

That's one of the most popular examples of how we apply ordinary power series a.k.a. generating functions to solve recurrence relation problems.

Note that manipulating some generating functions is not always trivial especially when we have to solve the series which is not in standard form.

Here is an example of how ordinary power series takes form which is not in standard form

$$F(x) = \sum_{n=2}^{\infty} \frac{(n+1)^2}{(n+1)!} x^{4n}$$
 (1.10)

The above expression is not in standard form of generating function due to  $n$  is begun from 2 and  $x$  is powered by  $4n$ .

So this is another gap of generating function theory that will become state of art of this article topics.

Wilf (1994) has an idea to assign the power series to some analytic function by the following definition.

**Definition 1.2.** The symbol  $f \xleftarrow{Ops} \{a_n\}$  means the series  $f$  is the ordinary power series (Ops) generating function for sequence  $\{a_n\}$ .

The above definition results several properties as following.

Suppose  $f \xleftarrow{Ops} \{a_n\}$ , then what generates  $\{a_{n+1}\}$ ?

By little work we will get

$$\sum_{n=1}^{\infty} a_{n+1} x^n = \frac{1}{x} \sum_{m=1}^{\infty} a_{m+1} x^m = \frac{f(x) - f(0)}{x}$$
 (1.11)

Therefore we have

$$\left( \frac{f - a_0}{x} \right) \xleftarrow{Ops} \{a_{n+1}\}$$
 (1.12)

Inductively we get for some positive integer  $h$ .

$$\left( \frac{f - a_0 - \dots - a_{n-1} x^{n-1}}{x^n} \right) \xleftarrow{Ops} \{a_{n+h}\}$$
 (1.13)

The last expression is not simple enough and thus not so efficient to solve many generating function with complicated forms. We will find more efficient idea to simplify the expression through this topic.

**2. Material and Methods**

In this article, we will construct the definition of Ops transformation and make sure that the definition should be well defined. After that we will analyze some of its basic properties. Next we will try to apply Ops transformation and its properties to solve some ordinary power series calculations.

**3. Result and Discussion**

If we observe at (1.10) the sigma notation and variable  $x$  don't play so important role which we can ignore that for a while. So in order to do some fast, effective calculations sometime we might to eliminate  $x$  and sigma notation for a while.

Observe the following definition. Recall that

$$\mathbb{R}^{\mathbb{N}} = \{\{a_n\}: a_n \in \mathbb{R}\}$$
 (3.11)

i.e. the space of all real sequences. It's trivially one can show that  $\mathbb{R}^{\mathbb{N}}$  is a linear space over  $\mathbb{R}$ . Now we define a transformation as following

**Definition 3.1.** Ops transformation is mapping

$$Ops: \mathbb{R}^{\mathbb{N}} \times \mathbb{R} \rightarrow \mathbb{R}^*$$

by

$$Ops(\{a_n\})(x) = \sum_{n=0}^{\infty} a_n x^n \tag{3.12}$$

Its well defined because arbitrary real power series has radius of convergence. Observe sometime we can write just  $Ops(\{a_n\})$  instead of  $Ops(\{a_n\})(x)$ . We can assume  $Ops(\{a_n\})$  as a  $C^\omega(D)$  function (i.e real analytics function) from some disk  $D \subset \mathbb{R}$  determined by its radius convergence.

Another expression of ops transformation is to write it implicitly by unspecified notation as McBride(1971). Its written as

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n \tag{3.13}$$

Its fine, but we will always involve variable  $x$  in every calculation, which means the expression (3.12) still more effecient in some situations.

Now we will explore some of Ops Transformations properties

**Theorem 3.2.** For each  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ ,  $Ops(\{0, a_0, a_1, \dots\}) = xOps(\{a_n\})$

Proof: For each  $x$  we have

$$\begin{aligned} xOps(\{a_n\})(x) &= x \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 x + a_1 x^2 + \dots \\ &= 0 + a_0 x + a_1 x^2 + \dots \\ &= Ops(\{0, a_0, a_1, \dots\}) \end{aligned}$$

Inductively theorem 3.2 gives concequency as follow

**Corollary 3.3.** For each  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ ,  $Ops\left(\left\{\frac{0, 0, \dots, 0}{k\text{-terms}}, a_0, a_1, \dots\right\}\right) = x^k Ops(\{a_n\})$

As well as Benderier(2002) explains generating function, we have following properties

**Theorem 3.4.** For each  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ ,  $Ops(\{a_n\}) - a_0 = xOps(\{a_{n+1}\})$ .

Proof: For each  $x$  lies on the domain we have

$$\begin{aligned} Ops(\{a_n\})(x) - a_0 &= \sum_{n=1}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty} a_{n+1} x^{n+1} \\ &= x \sum_{n=1}^{\infty} a_{n+1} x^n \end{aligned}$$

$$= xOps(\{a_{n+1}\})(x).$$

Thus inductively we get

**Corollary 3.5.** For each  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ ,  $Ops(\{a_n\}) - \sum_{n=0}^{k-1} a_n x^n = x^k Ops(\{a_{n+k}\})$

As another consequence we also have

**Corollary 3.6.** For each  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ ,  $Ops(\{a_{n+k}\}) = \frac{Ops(\{a_n\}) - \sum_{n=0}^{k-1} a_n x^n}{x^k}$ ,  $x \neq 0$

Ops transformation is failed to be linear transformation from  $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}$  to  $\mathbb{R}^*$  because  $Ops(\{a_n\})(x+y)$  is not necessary equals to  $Ops(\{a_n\})(x) + Ops(\{a_n\})(y)$ . But if we set  $x$  to be fixed then we have

$$Ops(\{a_n\} + \{b_n\})(x) = Ops(\{a_n\})(x) + Ops(\{b_n\})(x) \tag{3.14}$$

By ignoring  $x$  for a while we can write

$$Ops(\{a_n\} + \{b_n\}) = Ops(\{a_n\})(x) + Ops(\{b_n\}) \tag{3.15}$$

Its easy to check that for each  $\alpha \in \mathbb{R}$

$$Ops(\alpha\{a_n\}) = \alpha Ops(\{a_n\}) \tag{3.16}$$

Now we shall observe what value of Ops transformation for the product of two power series. Suppose we have

$$Ops(\{a_n\})(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } Ops(\{b_n\})(x) = \sum_{n=0}^{\infty} b_n x^n \text{ then}$$

$$\begin{aligned} Ops(\{a_n\})Ops(\{b_n\})(x) &= \left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right) \\ &= \left( \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) x^n \right) \\ &= Ops \left( \sum_{k=0}^n a_k b_{n-k} \right) (x) \end{aligned}$$

This will result the following theorem

**Theorem 3.7.** For each  $\{a_n\}, \{b_n\} \in \mathbb{R}^{\mathbb{N}}$ ,  $Ops(\{a_n\})Ops(\{b_n\}) = Ops \left( \sum_{k=0}^n a_k b_{n-k} \right)$ .

Next we will observe for the derivative of Ops transformation. As we know that any power series are smooth functions, so we are always able to define the derivative of Ops transformation with respect to  $x$ .

Now lets observe the following definition

**Definition 3.8.** Let  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ , and  $c$  be an interior point of domain of  $Ops(\{a_n\}): \mathbb{R} \rightarrow \mathbb{R}$ . Define

$$D_x Ops(\{a_n\})(c) = \frac{Ops(\{a_n\})(x) - Ops(\{a_n\})(c)}{x - c}$$

The above definition is no other than definition of derivative of function in usual sense.

Now we shall explore more properties of Ops transformation related to its derivative.

**Theorem 3.9.** *If  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ , then  $D_x Ops(\{a_n\}) = Ops(\{(n + 1)a_{n+1}\})$*

For example we have  $D_x Ops(\{1\}) = (Ops(\{1\}))^2$  (K.Lando (2002)) As for consequence of the above theorem. There should be the following corollary

**Corollary 3.10.** *If  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ , then*

$$D_x^k Ops(\{a_n\}) = D_x(D_x(\dots(D_x Ops(\{a_n\}))) = Ops\left(\left\{\frac{(n+k)!}{n!} a_{n+k}\right\}\right).$$

Another useful property of Ops transformation is

**Theorem 3.11.** *If  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ , then  $x D_x Ops(\{a_n\}) = Ops(\{na_n\})$*

**Proof:** Lets observe

$$\begin{aligned} Ops(\{na_n\}) &= \sum_{n=0}^{\infty} na_n x^n \\ &= x \sum_{n=1}^{\infty} na_n x^{n-1} \\ &= x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n \\ &= x Ops(\{(n+1)a_{n+1}\}) \\ &= x D_x Ops(\{a_n\}) \end{aligned}$$

Inductively we can get the following property

**Corollary 3.12.** *If  $\{a_n\} \in \mathbb{R}^{\mathbb{N}}$ , then  $Ops(\{n^k a_n\}) = \underbrace{x D_x (x D_x (\dots x D_x Ops(\{a_n\}))}_{k \text{ times}}$ .*

Here we will give some example how to apply Ops transformation to recurrence relation. We will solve the equation

$$a_n + 3a_{n-1} - 1 = 0, a_0 = 1$$

We will rewrite the equation by shifting the index

$$a_{n+1} + 3a_n - 1 = 0, a_0 = 1$$

First we transform the equation to Ops transformation form

$$\begin{aligned} Ops(\{a_{n+1} + 3a_n - 1\}) &= 0 \\ &= Ops(\{a_{n+1}\}) + 3Ops(\{a_n\}) - Ops(\{1\}) \\ &= 0 \end{aligned}$$

Therefore

$$Ops(\{a_{n+1}\}) + 3Ops(\{a_n\}) = \left( \frac{Ops(\{a_n\}) - a_0}{x} \right) + 3Ops(\{a_n\})$$

$$Ops(\{a_n\}) - a_0 + 3xOps(\{a_n\}) = xOps(\{1\})$$

By substituting, and calculating the value of  $a_0$  and  $Ops(\{1\})$  we have

$$Ops(\{a_n\}) + 3xOps(\{a_n\}) = \left( \frac{1}{1-x} \right)$$

Therefore

$$Ops(\{a_n\}) = \left( \frac{1}{(1-x)(1+3x)} \right) = \frac{1}{4(1-x)} + \frac{3}{4(1+3x)} = Ops\left(\left\{ \frac{1}{4} + \frac{3}{4}(-3)^n \right\}\right)$$

Eliminating the  $Ops$  we now have

$$a_n = \left( \frac{1}{4} + \frac{3}{4}(-3)^n \right)$$

which is the solution of above recurrence relation.

#### 4. Conclusion

From this article, we conclude that Ops transformation is mapping

$$Ops: \mathbb{R}^{\mathbb{N}} \times \mathbb{R} \rightarrow \mathbb{R}^*$$

by

$$Ops(\{a_n\})(x) = \sum_{n=0}^{\infty} a_n x^n$$

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