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Hardy-little wood-type theorem for mixed fractional integrals in weighted holder spaces

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Abstract
We study mixed Riemann-Liouville fractional integration operators and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. The obtained are results generalized to the case of Hölder spaces with power weight.

Keywords: Functions of two variables, fractional derivative of Marchaud form, mixed fractional derivative, weight, mixed fractional integral, Hölder space.

Introduction

In 1928 H.G. Hardy and J.E. little wood [1] (see [15], Theorem 3.1 and 3.2) showed that the fractional integral

$$\left(I_{0+}^{\alpha}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)dt}{(x-t)^{1-\alpha}},\quad 0<x<1$$

of order $\alpha\in(0,1)$ improves the Hölder behavior of its density exactly by the order α. More exactly, these operators establish an isomorphism between the spaces $H_0^{\lambda}([0,1])$ and $H_0^{\lambda+\alpha}([0,1])$ under the condition $\alpha+\lambda$. This result was extended in many directions: to the case of Hölder spaces with power weight [14] to the case of generalized Hölder spaces with characteristics from the Bari-Stechkin class [12, 13]; to the case of more general weights [15, 16], etc. Different proofs were suggested in [2, 3], where the case of complex fractional orders was also considered the shortest proof being given in [2].

In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann - Liouville integral was studied in [4, 11].

As is known, the Riemann-Liouville fractional integration operator establishes an isomorphism between weighted Hölder spaces for functions one variable. But for functions two variable were not studied. This paper is aimed to fill in this gap. We study mixed Riemann-Liouville fractional integration operators and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. The obtained results generalized to the case of Hölder spaces with power weight of functions of two variables.

Mixed Riemann-Liouville fractional integrals of order (α,β)

$$\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{\varphi(t,s)dtds}{(x-t)^{1-\alpha}(y-s)^{1-\beta}},\qquad(1)$$

$$x>0,y>0.$$

Mixed fractional derivatives form Marchaud of order (α,β)

$$\left(D_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{\varphi(x,y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}x^{-\alpha}y^{-\beta}+$$
$$+\frac{\alpha\beta}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y\frac{\varphi(x,y)-\varphi(t,s)}{(x-t)^{1+\alpha}(y-s)^{1+\beta}}dtds,\ x>0,y>0 \tag{2}$$

Consider the operators in a rectangle

$$Q=\{(x,y):0<x<b,0<y<d\}$$

For a continuous function $\varphi(x,y)$ on R^2 we introduce the notation

$$\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)=\varphi(x+h,y)-\varphi(x,y),$$
$$\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)=\varphi(x,y+\eta)-\varphi(x,y),$$
$$\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)=\varphi(x+h,y+\eta)-\varphi(x+h,y)-\varphi(x,y+\eta)+\varphi(x,y).$$

So that

$$\varphi(x+h,y+\eta)=\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)+\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)+$$
$$+\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)+\varphi(x,y). \tag{3}$$

Everywhere in the sequel by C,C_1,C_2 etc we denote positive constants which may have different values in different occurrences and even in the same line.

Definition 1. Let $\lambda,\gamma\in(0,1]$. We say that $\varphi(x,y)\in H^{\lambda,\gamma}(Q)$, if

$$\left|\varphi(x_1,y_1)-\varphi(x_2,y_2)\right|\le C_1|x_1-x_2|^\lambda+C_2|y_1-y_2|^\gamma \tag{4}$$

For all $(x_i,y_i)\in Q, i=1,2$.

Condition (4) is equivalent to the couple of the separate conditions

$$\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|\le C_1|h|^\lambda,\quad \left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|\le C_2|\eta|^\gamma$$

Uniform with respect to another variable.

By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $\varphi\in H^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.
Let $\lambda=0$ and\or $\gamma=0$. We put $H^{0,0}(Q)=L^\infty(Q)$ and

$$H^{\lambda,0}(Q)=\left\{\varphi\in L^\infty(Q):\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right|\le C_1|h|^\lambda\right\},\ \lambda\in(0,1]$$
$$H^{0,\gamma}(Q)=\left\{\varphi\in L^\infty(Q):\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|\le C_2|\eta|^\gamma\right\},\ \gamma\in(0,1].$$

Definition 2. We say that $\varphi(x,y)\in \tilde{H}^{\lambda,\gamma}(Q)$, where $\lambda,\gamma\in(0,1]$, if

$\varphi\in H^{\lambda,\gamma}(Q)$ And $\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C_3|h|^{\lambda}|\eta|^{\gamma}$.

We say that $\varphi\in\tilde{H}_0^{\lambda,\gamma}(Q)$, if $\varphi\in H^{\lambda,\gamma}(Q)$ any $\varphi(0,y)\equiv\varphi(x,0)\equiv 0$.

These spaces become Banach spaces under the standard definition of the norms:

$$\|\varphi\|_{H^{\lambda,\gamma}}:=\|\varphi\|_{C(Q)}+\sup_{\substack{x,x+h\in[0,b]\\ y\in[0,d]}}\frac{\left|\left(\overset{1,0}{\Delta}_{h}\varphi\right)(x,y)\right|}{|h|^{\lambda}}+\sup_{\substack{y,y+\eta\in[0,b]\\ x\in[0,b]}}\frac{\left|\left(\overset{0,1}{\Delta}_{\eta}\varphi\right)(x,y)\right|}{|\eta|^{\gamma}},$$

$$\|\varphi\|_{\tilde{H}^{\lambda,\gamma}}:=\|\varphi\|_{H^{\lambda,\gamma}}+\sup_{\substack{x,x+h\in[0,b]\\ y,\eta+y\in[0,d]}}\frac{\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|}{|h|^{\lambda}|\eta|^{\gamma}}.$$

Note that

$$\varphi(x,y)\in H^{\lambda,\gamma}(Q)\Rightarrow\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C_{\theta}|h|^{\theta\lambda}|\eta|^{(1-\theta)\gamma} \tag{5}$$

For any $\theta\in[0,1]$, where $C_{\theta}=2C_1^{\theta}C_2^{1-\theta}$.

So that

$$\bigcap_{0\le\theta\le1}\tilde{H}^{\lambda\theta,(1-\theta)\gamma}(Q)\hookleftarrow H^{\lambda,\gamma}(Q)\hookleftarrow\tilde{H}^{\lambda,\gamma}(Q) \tag{6}$$

Where $\hookleftarrow$ stands for the continuous embedding.

The norm for $\bigcap\limits_{0\le\theta\le1}\tilde{H}^{\lambda\theta,(1-\theta)\gamma}(Q)$ is introduced as the maximum in θ of norms for $\tilde{H}^{\lambda\theta,(1-\theta)\gamma}(Q)$. Since $\theta\in[0,1]$ is arbitrary, it is not hard to see that the inequality in (5) is equivalent (up to the constant factor *C*) to

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|\le C\min\left\{|h|^{\lambda},|\eta|^{\gamma}\right\}. \tag{7}$$

We will also make use of the following weighted spaces. Let $\rho(x,y)$ be a non-negative function on *Q* (we will only deal with degenerate weights

$\rho(x,y)=\rho_1(x)\rho_2(y)$).

Definition 3. By $H^{\lambda,\gamma}(Q,\rho)$ and $\tilde{H}^{\lambda,\gamma}(Q,\rho)$ we denote the spaces of functions $\varphi(x,y)$ such that $\rho\varphi\in H^{\lambda,\gamma}(Q)$ and $\rho\varphi\in\tilde{H}^{\lambda,\gamma}(Q)$ respectively, equipped with the norms

$\|\varphi\|_{H^{\lambda,\gamma}(Q,\rho)}=\|\rho\varphi\|_{H^{\lambda,\gamma}(Q)}$ and $\|\varphi\|_{\tilde{H}^{\lambda,\gamma}(Q,\rho)}=\|\rho\varphi\|_{\tilde{H}^{\lambda,\gamma}(Q)}$.

By $H_0^{\lambda,\gamma}(Q,\rho)$ and $\widetilde{H}_0^{\lambda,\gamma}(Q,\rho)$ we denote the corresponding subspaces of functions φ such that

$$\rho\varphi\big|_{x=0} = \rho\varphi\big|_{y=0} \equiv 0.$$

Below we follow some technical estimations suggested in [2] for the case of one-dimensional Riemann-Liouville fractional integrals. We denote

$$B(x,y;t,s) = \frac{\rho(x,y) - \rho(t,s)}{\rho(t,s)(x-t)^{\delta}(y-s)^{\sigma}} \tag{8}$$

Where

$$0 < \delta, \sigma < 2,\ a < t < x < b,\ 0 < s < y < d.$$

Main Result

The definition in the Marchaud form may be used for all $-1 < \alpha, \beta < 1$: if $\alpha, \beta > 0$ (2) gives the mixed fractional derivative, if $\alpha, \beta < 0$, it is mixed fractional integral.

Let $\rho(x,y) = \rho(x)\rho(y)$ be the weight function and put

$$\psi(x,y) = \rho(x,y)\varphi(x,y) \quad \varphi \in \widetilde{H}_0^{\lambda,\gamma}(Q;\rho)$$

Evidently $\psi \in \widetilde{H}^{\lambda,\gamma}(Q)$ and $\psi(x,y)\big|_{x=0,y=0} = 0$ It is easy to see that

$$\left(\rho J_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y) = \left(J_{0+,0+}^{\alpha,\beta}\psi\right)(x,y) + C\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y), \tag{9}$$

Where $-1 < \alpha, \beta < 1$, C=constant and

$$\left(K_{0+,0+}^{\alpha,\beta}\psi\right)(x,y) = \int_0^x \int_0^y B(x,y;t,s)\psi(t,s)\,dtds,$$

So that in (9) we have the mixed fractional integral if $0 \le \alpha, \beta < 1$ and the mixed fractional derivative if

$$-1 < \alpha, \beta \le 0.$$

The representation (9) for the fractional (integral) derivative shows that the estimate for the continuity modulus in the weighted case reduces to two simpler estimates:

1) The known non-weighted estimate of Hardy-Littlewood for mixed fractional integral (see [4]) and mixed fractional derivative (see [7]); in the case weighted estimated of Hardy-Littlewood for mixed fractional integral (see [4]);
2) The estimate of the second term in (9), which is the mixed fractional derivative. It is main part of the job.

Theorem 2. Let $0 < \alpha, \beta < 1,\ 0 < \lambda, \gamma < 1,\ \rho(x,y) = x^{\mu} y^{\nu}$

And let $\alpha + \lambda < 1$, $\beta + \gamma < 1$ and $|\psi(x,y)| \le Cx^{\lambda+\alpha} y^{\gamma+\beta}$

With $\mu < 1 + \lambda,\ \nu < 1 + \gamma$. Then the operator $K_{0+,0+}^{\alpha,\beta}$ is bounded from the space

$\widetilde{H}_0^{\lambda,\gamma}(\rho)$ into $\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(\rho)$.

Proof. We don't proof this theorem. The proof of this theorem can be seen in [4].

Theorem 3: Let $0<\alpha,\beta<1,\ 0<\lambda,\gamma<1,\ \rho(x,y)=x^{\mu}y^{\nu}$ and let $\alpha+\lambda<1,\ \beta+\gamma<1$

and $|\psi(x,y)|\leq Cx^{\lambda}y^{\gamma}$

With $\mu<1+\lambda,\ \nu<1+\gamma$. Then the operator $K_{0+,0+}^{-\alpha,-\beta}$ is bounded from the space $\widetilde{H}_0^{\lambda,\gamma}(\rho)$ into $\widetilde{H}_0^{\lambda-\alpha,\gamma-\beta}(\rho)$.

Proof. The proof of this theorem can be seen in [8].

Theorem 4. Let $\rho(x,y)$ have the form $\rho(x,y)=x^{\mu}y^{\nu}$ with $\mu<1+\lambda,\ \ \nu<1+\gamma$ and let $0<\alpha,\beta<1,\ 0<\lambda,\gamma<1,\ \lambda+\alpha<1,\ \gamma+\beta<1$. Then the operator $J_{0+,0+}^{\alpha,\beta}$ establishes an isomorphism between the spaces $\widetilde{H}_0^{\lambda,\gamma}(\rho)$ and $\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$.

Proof. We should consider, as usual the following three parts of the proof:

1. Action of the mixed fractional integral operator from the space $\widetilde{H}_0^{\lambda,\gamma}(\rho)$ to the space $\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$;
2. Action of the mixed fractional differentiation operator from the space $\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$ to the space $\widetilde{H}_0^{\lambda,\gamma}(\rho)$;
3. The possibility to represent any function $f(x,y)\in\widetilde{H}_0^{\lambda+\alpha,\beta+\gamma}(\rho)$ as $\left(J_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)$ with the density in $\widetilde{H}_0^{\lambda,\gamma}(\rho)$.

Because of $|\psi(t,s)|\leq Ct^{\lambda+\alpha}s^{\gamma+\beta},\ \ |\psi(t,s)-\psi(x,0)|\leq C(t-x)^{\lambda+\alpha}s^{\gamma+\beta}$ the parts 1) -2) are covered by Theorems 2 and 3. The part 3) is treated in the standard way in case $0<\alpha,\beta<1$ by using the possibility of similar representation with the density from $L_{\overline{p}}\left(\mathrm{R}^2\right),\ \overline{p}=(p_1,p_2)$. See [15] Theorem 24.4.

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