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AS-overgroup of clasical groups

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Abstract
Suppose that G has been recognised as tensor product or tensor induced. An AS-overgroup C of G can be constructed in time O(n2 log q), and G can be standardised in time O(n3 log q).

Keywords: Matrix groups, conjugacy, algorithms

1. Introduction
1.1 Introductory Material
Let K: = GL (n, q) and G and H are given as matrix groups. Eick and H¨ofling [Eick and H¨ofling 03] developed an algorithm to determine the conjugacy of irreducible soluble subgroups of GL (n, q). They represent G and H as polycyclic groups and hence compute Aut(G) and an explicit isomorphism between G and H. These are combined to determine the existence of an element of GL (n, q) that conjugates G to H. This technique is effective, but it is limited by the time requirements of computing automorphism groups and is only applicable to irreducible groups. Our algorithm uses Aschbacher's theorem [Aschbacher 84] to reduce the time spent searching for a conjugating element: its primary goal is to prove that G and H are conjugate, although we present some ideas on how to prove that they are not. It is applicable to geometric subgroups of GL(n, q). The approach is to use the geometries described in Aschbacher's theorem to find $A,B \in$ GL(n, q) such that GA and HB are contained in a given maximal subgroup $C \leq$ GL(n, q). Standard conjugacy techniques (for permutation groups) are then used to try to find an element of C that conjugates GA to HB. Whilst there is not always a guarantee that such an element exists, experiments show that generally one does; for some of the Aschbacher classes, we prove that one can be found whenever G and H are conjugate in the general linear group. The development of this algorithm was motivated by the observation that determining the conjugacy of subgroups of GL(6, 3) often required several days of computing time. Although the methods described in this paper will not always succeed either in finding a conjugating element or in proving that G and H are not conjugate, they are useful because they can often solve the conjugacy problem inside general linear groups that were far too large for previous approaches. The timings data in Section 6 demonstrates this.
An implementation of this algorithm will be released with Version 2.11 of Magma [Bosma *et al*. 97]. At present the algorithm only works to determine conjugacy under the general linear group. There are several directions in which it could be generalised. The most obvious is to make it work inside any classical matrix group. The biggest problem will be the construction of the relevant maximal subgroups, but recent work of Holt and the author [Holt and Roney-Dougal] gives generating matrices for most of these groups in the linear, symplectic, and unitary cases. The algorithm could perhaps be made faster by making certain sections of it recursive. Aschbacher's theorem is used to find a maximal subgroup $C \leq$ GL(n, q) and two matrices $A,B \in$ GL(n, q) such that $GA,HB \leq C$. For many of the Aschbacher classes, it should be possible to recursively apply Aschbacher's theorem to part or all of the group C, to construct $A\_,B\_ \in C$ and a maximal subgroup $C\_ \leq C$ such that $GAA\_,HBB\_ \leq C\_$, and then to search $C\_$ for a conjugating element. We write $H \sim_K G$ to denote that H is conjugate to G under K. The cost of this recursive approach is that it seems intuitively less likely that $GAA\_ \sim C\_ HBB\_$ than that $GA \sim C HB$; however, the time gains of computing conjugacy inside a smaller group, with a smaller degree permutation representation, would probably outweigh this. In Section 2 we make some key definitions, state Aschbacher's theorem, and prove a few elementary results. In Section 3 we give an overview of our algorithm for determining the conjugacy of geometric matrix groups, and then in Section 4 we describe how it works in each geometric Aschbacher class.

In Section 5 we discuss the accuracy and reliability of the algorithm and conclude in
Section 6 with timings data.

2. Preliminary Results

We now recall some basic mathematical definitions, prove a few fundamental lemmas, and state Aschbacher's theorem. Let $G \leq \mathrm{GL}(n, q)$ be given, and set $V := \mathrm{F}(n)q$. Then G is *reducible* if it stabilises a proper nontrivial subspace of V, and is *irreducible* otherwise. If the image of G under the natural embedding into $\mathrm{GL}(n, \mathrm{F})$ is irreducible for all field extensions F of Fq, then G is *absolutely irreducible*. If G is irreducible and preserves a direct sum decomposition $V = V1 \oplus \cdots \oplus Vt$ with $t > 1$, then G is *imprimitive*. Theorem 2.1. (Aschbacher's Theorem [Aschbacher 84].) *Let* $G \leq \mathrm{GL}(n, q)$ *be given, let* $q = pe$, *let* $V := \mathrm{F}nq$ *and let* $Z := Z(\mathrm{GL}(n, q))$. *Then one of the following holds:*

1. *G is reducible.*
2. *G is imprimitive.*
3. *G can be embedded in* $\Gamma\mathrm{L}(n/s, qs)$ *for some prime s dividing n.*
4. *G preserves a tensor product* $V = V1 \otimes V2$, *where* $\dim V1 = \dim V2$.
5. *A conjugate of G is a subgroup of* $\mathrm{GL}(n, pf)Z$, *where e/f is prime.*
6. *The dimension* $n = rm$, *where r is prime. If r is odd or* $n = 2$, *then r divides* $q - 1$, *and G normalises an extraspecial r-group. Otherwise,* 4 *divides* $q - 1$, *and G normalises a* 2*-group of symplectic type.*
7. *G preserves a tensor induced decomposition* $V = V1 \otimes \cdots \otimes Vt$ *with* $t > 1$.
8. *G lies between a classical group and its normaliser in* $\mathrm{GL}(n, q)$, *or preserves a classical form up to scalar multiplication.*
9. *For some nonabelian simple group T, the group* $G/(G \cap Z)$ *is almost simple with socle T. In this* Roney-Dougal:

case the normal subgroup $(G \cap Z).T$ *acts absolutely irreducibly, preserves no nondegenerate classical form, is not a subfield group, and does not contain* $\mathrm{SL}(n, q)$. The original theorem describes subgroups of all classical groups: see [Aschbacher 84]. We follow the notation of [Kleidman and Liebeck 90] when naming classical groups. In particular $\mathrm{O}\_(n, q)$, where _ is +, −, or omitted, denotes the largest subgroup of $\mathrm{GL}(n, q)$ to preserve a quadratic form of type _. The groups $\mathrm{GSp}(n, q)$ and $\mathrm{GO}\_(n, q)$ are the normalisers in $\mathrm{GL}(n, q)$ of $\mathrm{Sp}(n, q)$ and $\mathrm{O}\_(n, q)$. A group $G \leq \mathrm{GL}(n, q)$ *lies in class Ci* if the *i*th condition of the theorem holds, and G is *geometric* if $G \in Ci$ for some $i \leq 8$. The class Ci is *recognisable* for G if there exist algorithms to recognise that $G \in Ci$. Let G be any geometric group other than a member of $C8$ that does not fix a classical form (up to scalar multiplication), then, G can be recognised as being a member of at least one Aschbacher class: more details will be given later. A matrix group G is *AS-maximal* if G is a maximal member of an Aschbacher class. Aschbacher proved a theorem that may be informally stated by saying that, with the exception of reducible AS-maximals that are conjugate under the duality automorphism, the geometric AS-maximals of a given type are all conjugate under the general linear group [Aschbacher 84, Theorem BΔ].
An *AS-overgroup* for a geometric group G is an AS maximal that preserves a structure of the same type as G: constructions for canonical AS-overgroups will be given later. If G has been conjugated into a given AS overgroup, then G has been *standardised* (with respect to that AS-overgroup).
We finish with some algorithmic preliminaries. We assume that integer operations require constant time. We also assume that primitive polynomials are known for all finite fields that we encounter and that elements of F*pe* are stored as polynomials of degree $e - 1$ over F*p*. Thus, field operations require time $O(\log q)$, and elements of $\mathrm{GL}(n, q)$ are constructed in $O(n2 \log q)$. We assume that matrix multiplication is $O(n3 \log q)$ and that primitive field elements are known. We will not assume the availability of discrete logs. By *constructing* a group we mean producing a set of generating elements for the group: usually these will be a collection of matrices. Lemma 2.2. *Given a primitive element* $z \in \mathrm{F} * q$, *the groups* $\mathrm{GL}(n, q)$, $\mathrm{SL}(n, q)$ *and* $\mathrm{Sp}(n, q)$ *can be constructed in time* $O(n2 \log q)$. *The groups* $\mathrm{GU}(n, q)$ *and* $\mathrm{SU}(n, q)$ *can be constructed in time* $O(n2 \log q + \log 2 q)$. *Proof:* Pairs of generating matrices are known for $\mathrm{GL}(n, q)$, $\mathrm{SL}(n, q)$, $\mathrm{Sp}(n, q)$, $\mathrm{GU}(n, q)$, and $\mathrm{SU}(n, q)$ [Taylor 87]. In the linear and symplectic cases, all coefficients lie in the set $S := \{0, \pm 1, \pm z \pm 1\}$. All coefficients in the unitary case lie in $T := S \cup \{\pm z \pm p, \pm(1 + zp - 1) - 1\}$. The set S can be constructed in $O(\log q)$, and T can be constructed in $O(\log 2 q)$.
If $D = (dij)$ $n \times n$ is diagonal, we write $D = \mathrm{Diag}[d11, d22, \ldots, dnn]$. If $dij = 0$ unless $j = n - i + 1$, we write $D = \mathrm{Anti\,Diag}[d1n, d2(n-1), \ldots, dn1]$. When generated as in Lemma 2.2, $\mathrm{Sp}(d, q)$ preserves a form $\mathrm{Anti\,Diag}[1, \ldots, 1, -1, \ldots, -1]$, and $\mathrm{GU}(d, q)$ preserves a form $\mathrm{Anti\,Diag}[1, \ldots, 1]$. For odd q we assume that $\mathrm{SO}(2m+1, q)$ preserves a form with matrix $I2m+1$. When q is odd, we assume that $\mathrm{SO}\pm(2m, q)$ preserves an orthogonal form with matrix $\mathrm{Diag}[z, 1, \ldots, 1]$ or $I2m$, depending on whether $(q - 1)n/4$ is even or odd. For even q we assume that the orthogonal groups of + and – type preserve the form given by Magma. Lemma 2.3. *There is a Las Vegas* $O(\log 3 q)$ *algorithm that, with probability of success* 1/2, *finds* $a, b \in \mathrm{F}q$ *such that* $a2 + b2 = z$. *Proof:* Search $\mathrm{F} * q$ for an element b such that $z - b2$ is a square. At least half of the field elements are squares, and each test of squareness costs $O(\log 3 q)$ [Lidl and Niederreiter 83].
Lemma 2.4. *For* $\_ \in \{+, -, \circ\}$, *the groups* $\Omega\_(n, q)$, $\mathrm{SO}\_(n, q)$, $\mathrm{O}\_(n, q)$, *and* $\mathrm{GO}\_(n, q)$ *may be constructed in time* $O(n3 \log q + \log 3 q)$. *Proof:* Let $S := \{0, 1, z, v, v\}$, where $v\_ = \mathrm{F} * q2$. The set S can be constructed in time $O(\log 2 q)$. In [Rylands and Taylor 98] small sets of generating matrices are given for $\Omega\_(n, q)$, which can be constructed in time $O(n2 \log q)$, given S.
Let $S\_$ extend $\Omega\_(n, q)$ to $\mathrm{SO}\_(n, q)$, if these groups are not equal. Let Rs be a reflection in a vector of square norm and Rn be a reflection in a vector of nonsquare norm: Rs and Rn can be constructed in time $O(n2 \log q)$. By [Kleidman and Liebeck 90, Sections 2.6–2.8], we may take $S\_ := -I$ if n is even, q is odd, and the discriminant of the form is nonsquare; $S\_ := RsRn$ if n is odd or n is even, q is odd, and the discriminant is square; or $S\_ := Rs$ 154 Experimental Mathematics, Vol. 13 (2004), No. 2 if n and q are both even. Thus, we construct $S\_$ in time $O(n3 \log q)$.
Let $T\_$ extend $\mathrm{SO}\_(n, q)$ to $\mathrm{O}\_(n, q)$, if these groups are not equal. By [Kleidman and Liebeck 90, Sections 2.6–2.8], we may take $T\_ := Rs$ if n is even and q is odd, and $T\circ := -I$ if q is odd, in time $O(n2 \log q)$. Let $D\_$ extend $\mathrm{O}\_(n, q)$ to $\mathrm{GO}\_(n, q)$. Assume that the quadratic form has matrix Anti

Diag [1,..., 1] in type + and either the identity or Diag[*z*, 1,..., 1] in type −: a matrix conjugating our original group to one preserving this form can be constructed in time $O(n3 \log q)$ [Holt and Roney-Dougal]. Then $D := zIn$ if n is odd or q is even. If q is odd, then $D+ :=$ Diag[*z*,..., *z*, 1,..., 1] and $D- :=$ Diag[*P*,..., *P*] or Diag [Anti Diag [*z*, 1], *P*,..., *P*], depending on whether the discriminant of the form is square or nonsquare, where *a* and *b* are as in Lemma 2.3 and $P := \_ a\ b\ b\ -a\ \_$

Lemma 2.5. *Given a set S of generating matrices for* $G \leq \mathrm{GL}(n, q)$ *and a set T of generating permutations for* $H \leq \mathrm{Sym}(d)$, *a set of generating matrices for* $G \_ H \leq \mathrm{GL}(nd, q)$ *can be constructed in time* $O((|S| + |T|)(nd)2 \log q)$.

Proof: For each generating matrix $X \in S$, define a matrix Diag[*X*, *In*,..., *In*]. For each $Y \in T$, define an $nd \times nd$ block matrix whose (i, j)th block is *In*, if *Y* maps $i \rightarrow j$, and 0 otherwise. The group $G \_ H$ is generated by these $(|S| + |T|)$ matrices. Let $G \leq \mathrm{GL}(n, q)$ and $H \leq \mathrm{GL}(m, q)$. By $G \otimes H$ we mean a group isomorphic to $(G \times H)/(x, x-1)$: $x \in G \cap H \cap Z(\mathrm{GL}(nm, q))\_$. Note that if *G* and *H* are absolutely irreducible, then this reduces to the standard central product $G \circ H$. The group $G \otimes H$ has a natural action on $\mathrm{F}(n)\, q \otimes \mathrm{F}(m) q$. By *H* Tens Wr *K*, where $H \leq \mathrm{GL}(n, q)$ and $K \leq \mathrm{Sym}(t)$ is transitive, we mean the subgroup of $\mathrm{GL}(nt, q)$ given by $(H \otimes \cdots \otimes H)$: *K*. The group *K* permutes the factors in the central product.

Lemma 2.6. *Let* $G := S\_ \leq \mathrm{GL}(n, q)$ *and* $H := T \_ \leq \mathrm{GL}(m, q)$. *The group* $G \otimes H \leq \mathrm{GL}(mn, q)$ *can be constructed in* $O((|S| + |T|)(mn)2 \log q)$, *and G* Tens Wr Sym(*t*) *can be constructed in* $O(|S|n2t \log q)$. *Proof:* The group $G \otimes H$ is generated by the Kronecker products of elements of *S* with $1H$ and of $1G$ with elements of *T*. Given $X \leq G$ and $Y \leq H$, the Kronecker product $X \otimes Y$ has $XijYkl$ in position $((i - 1)m + k, (j - 1)m + l)$. Each matrix is therefore written down in time $O((mn)2 \log q)$. The final claim is from [Holt and Roney-Dougal].

3. Algorithmic Overview

We give a description of the algorithm for geometric groups, which is then specialised for each Aschbacher class. Is GL Conjugate (G, H)

1. Input: $G, H \leq \mathrm{GL}(n, q)$.
2. If $G = H$, return true. If not then compute several group-theoretic invariants of *G* and *H*. If these are different, return false.
3. Replace *G* and *H* by random $\mathrm{GL}(n, q)$-conjugates.
4. For each $Ci \_= C9$ to which *G* can be recognised as belonging:
   (a) Identify a structure *S* that *G* preserves, construct an AS-overgroup *C* for *G*, and find $A \in \mathrm{GL}(n, q)$ that standardises *G*.
   (b) If *H* can be shown not to preserve a structure isomorphic to *S*, then return false.
   (c) Form a faithful representation ρ of *C*.
   (d) For at least one structure isomorphic to *S* that is preserved by *H* do:
   1. Find $B \in \mathrm{GL}(n, q)$ which standardises *H*.
   2. Use an existing conjugacy algorithm for permutation groups to decide whether there exists an $X\rho \in C\rho$ such that $(GA)\rho X\rho = (HB)\rho$.
   3. iii. If so, return true, $AXB-1$. If not, and $i = 6$, then return false.
   (e) If $i \in \{1, 8\}$ return false.

4.5 Classical Groups Several methods exist for finding a classical form preserved by an absolutely irreducible group *G*: see [Holt and Rees 94] for instance. Since these methods allow one to specify the type of form that is being sought (symplectic, unitary, or orthogonal), the conjugacy algorithm may return false at Step 4(b). The loop in Step 4(d) is run only once, since the form preserved by an absolutely irreducible group is unique, up to scalar multiplication. Note that the full normaliser of $\mathrm{Sp}(n, q)$ and $\mathrm{SO}\pm(n, q)$ does not fix a form, even up to multiplication by scalars.

We remark that groups containing $\mathrm{SL}(n, q)$ are normal in $\mathrm{GL}(n, q)$, thus two groups of this type are conjugate if and only if they are equal. This possibility will therefore be dealt with at Step 2 of the algorithm.

In the following proposition, the matrix *DS* is diagonal and acts as *z* on the first $n/2$ basis vectors and as 1 on the remainder.

Proposition 4.10. *Let* $Z := Z(\mathrm{GL}(n, q2))$, *then* $N\mathrm{GL}(n,q2)(\mathrm{SU}(n, q)) = Z$, $\mathrm{GU}(n, q)\_$, $\mathrm{GSp}(n, q) = Z(\mathrm{GL}(n, q))$, $\mathrm{Sp}(n, q)$,*DS*_, *and* $N\mathrm{GL}(n,q)(\mathrm{SO}\_(n, q)) = \mathrm{GO}\_(n, q)$.

Proof: This follows from various results in [Kleidman and Liebeck 90, Section 4.8].

Theorem 4.11. *Suppose that a group G has been recognised as C*8. *An AS-overgroup C of G can be constructed in time* $O(n3 \log q + \log 3\ q)$, *and G can be standardised in time* $O(n3 \log q)$. *Furthermore, if* $H \sim \mathrm{GL}(n,q)$ *G, and A,B are standardising matrices, then GA* $\sim C$ *HB. Proof:* By Lemmas 2.2 and 2.4 the classical group can be constructed in time $O(n3 \log q + \log 3\ q)$. In each case the normaliser is generated by the classical group and at most two other matrices, each of which can be written down in time $O(n2 \log q)$. The standardisation function is described in [Holt and Roney-Dougal]. It is shown there to have complexity $O(n3 \log q)$. For the final claim, we may suppose without loss of generality that $A = B = 1$, so that $G, H \leq C$. Let $X \in \mathrm{GL}(n, q)$ satisfy $GX = H$. Then *H* preserves the same form as *CX*, but since *H* is absolutely irreducible, it must preserve a unique form of any given type. Thus, *CX* preserves the same form as *C*, so $X \in N\mathrm{GL}\ (n,q)(C) = C$, and the result follows.

Roney-Dougal: Conjugacy of Subgroups of the General Linear Group 159 4.6 Tensor Product, Subfield, and Tensor Induced Groups We consider these families together, as each of their recognition algorithms also returns a standardising matrix. Definition 4.12. If the group $G \leq \mathrm{GL}(n, q)$ preserves a decomposition $V = V1 \otimes V2$, then *G* is a *tensor product group*.

Suppose that $n = ms$ for $s > 1$. If $G \leq \mathrm{GL}(n, q)$ preserves a decomposition $V = V1 \otimes \cdots \otimes Vs$ with $\dim(Vi) = m$ for $1 \leq i \leq s$, then *G* is *tensor induced*. A group $G \leq \mathrm{GL}(n, q)$ is *subfield* if there exists a subfield $\mathrm{F}q0 \subset \mathrm{F}q$ such that a conjugate of *G* may be embedded in $\mathrm{GL}(n, q0)Z$. The AS-maximals in *C*4 are $\mathrm{GL}(n1, q) \circ \mathrm{GL}(n2, q)$, with $n1 < \sqrt{n}$. Recognition algorithms for absolutely irreducible tensor product groups are given in [Leedham-Green and O'Brien 97a, Leedham-Green and O'Brien 97b]. In *C*7, the AS-maximals are $\mathrm{GL}(m, q)$Tens Wr Sym(*s*). A recognition algorithm for absolutely irreducible *C*7 groups is given in [Leedham- Green and O'Brien 02]. Both of these recognition algorithms allow the user to specify the degrees of the tensor factors, so if factors of the correct degree cannot be found, we return false in Step 4(b). However, in each case there may be several different decompositions

preserved by the group, so the loop in Step 4(d) is repeated up to 20 times, with *H* replaced by a random conjugate each time.

Main Theorem. *Suppose that G has been recognised as tensor product or tensor induced. An AS-overgroup C of G can be constructed in time O*(*n*2 log *q*)*, and G can be standardised in time O*(*n*3 log *q*).

Proof: The construction claims follow from Lemma 2.6, and the standardisation claims are clear. The AS-Maximals in *C*5 are GL(*n*, *q*0)*Z*, where F*q*0 has prime index in F*q*. In [Glasby and Howlett 97] an algorithm is given which determines whether an absolutely irreducible group *G* is conjugate to a subgroup of GL(*n*, *q*0); this is extended to a general subfield group in [Glasby *et al*.]. In both cases, the degree of the subfield representation may be specified by the user, so if matching fields are not found, then the algorithm returns false at Step 4(b). The loop in Step 4(d) is run up to 20 times, with *H* replaced by a random conjugate each time. Lemma 4.14. *Suppose that G* ≤ GL(*n*, *q*) *has been recognised as subfield. Given a primitive element of* F*q*0*, an AS-overgroup C of G can be created, and G can be standardised, in time O*(*n*3 log *q* + log2 *q*). *Proof:* This is clear.

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