Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2020; 1(2): 46-52

www.mathematicaljournal.com
Received: 18-04-2020
Accepted: 20-05-2020

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Investigations on special polynomials including Apostol-type and Humbert-type polynomials

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Abstract
In this article we are interested by a special family of numbers and polynomials introduced and studied by Ozdemir *et al*. We give explicit formula and apply the obtained to Apostol-Bernoulli polynomials and numbers.

Keywords: Bell polynomials, apostol-bernoulli polynomials, euler and genocchi polynomials, modified humbert polynomials

1. Introduction

The special numbers and polynomials play an important role in many areas of Mathematics and applied Mathematics. Numerous works were published in this wide topic [2, 3, 8]. In the present paper we are interested by the family of polynomials $G_n(x,y;k,m,d)$ introduced by Ozdemir *et al*. [11] defined by means of the generating function

$$F(t,x,y;k,m,d)=\frac{1-x^k-y^m}{1-x^ke^t-y^me^{(m+d)t}}$$

$$=\sum_{n\geq 0}\frac{G_n(x,y;k,m,d)}{n!}\left(\frac{t}{1-x^k-y^m}\right)^n.$$

Polynomials $G_n(x,y;k,m,d)$ are a generalization of numerous families of polynomials and numbers including Apostol-Bernoulli, Apostol-Euler polynomials and numbers, as well as classical Bernoulli, Euler dand Genocchi polynomials and numbers. The Humbert polynomial $H_{n,m}^{(\alpha)}(x)$ [10] appear in the series expansion

$$\left(\frac{1}{1-mxt+t^m}\right)^\alpha=\sum_{n\geq 0}H_{n,m}^{(\alpha)}(x)t^n.$$

Other generalizations of Humbert polynomials are given [4, 5]. Modified Humbert polynomials are introduced in [11] by the generating function

$$\frac{1}{1-a\rho e^t+e^{at}}=\sum_{n\geq 0}Y_n(\rho,a)\frac{t^n}{n!},$$

Which admit the following generalization

$$\left(\frac{1}{1-a\rho e^t+e^{at}}\right)^\alpha=\sum_{n\geq 0}Y_n^{(\alpha)}(\rho,a)\frac{t^n}{n!}.$$

The Fibonacci-type polynomials in two variables $F_n(x,y;k,m,d)$ has been recently defined by Ozdemir-Simsek [12] as follows

$$H(t,x,y;k,m,d) = \frac{1}{1-x^k t - y^m t^{m+d}} = \sum_{n\geq 0} F_n(x,y;k;m,d)t^n$$

If we take e^t instead of t in the last identity, one obtains the following family of rational functions $F_n^{(e)}(x,y;k,m,d)$ given by the generating function

$$\frac{1}{1-x^k-y^m e^{(m+d)t}} = \sum_{n\geq 0} F_n^{(e)}(x,y;k,m,d)\frac{t^n}{n!}.$$

One can see [9] for further information about the notion of generating function of a sequence of functions. Thereafter the connection between $F(t,x,y;k,m,d)$ and $H(t,x,y;k,m,d)$ is given by

$$F(t,x,y;k,m,d) = (1-x^k-y^m)Hoe^t.$$

In this paper we complete the work of Ozdemir *et al*. by the explicit formulae of $F_n^{(e)}(x,y;k,m,d)$ and polynomials $F_n(x,y;k,m,d)$. The obtained results are applied to compute explicit formulae of modified Humbert polynomials and other well-known families of numbers and polynomials.

2. Bell polynomials

The exponential partial Bell polynomial [2] $B_{n,k} := B_{n,k}(x_1,x_2,\dots)$ of the sequence $(x_j)_{j\in N}$ is defined by the exponential generating function

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1,x_2,\dots)\frac{t^n}{n!},$$

For which the explicit formula is

$$B_{n,k} = \frac{n!}{k!}\sum_{\pi_n(k)} \binom{k}{k_1,\dots k_{n-k+1}} \prod_{r=1}^{n-k+1} \left(\frac{x_r}{r!}\right)^{k_r}.$$

Where $\pi_n(k)$ is the set of all partitions $k_1+2k_2+\cdots+nk_n = n$ of n such that $k_1+k_2+\cdots+k_n = k$ we recall that the exponential partial Bell polynomial of the sum $(x_j+x_j^{'})_{j\in N}$ is given by the following identity [2]:

$$B_{n,k}(\dots,x_i+x_j^{'}\dots) = \sum_{r\leq j,v\leq n} \binom{n}{v} B_{v,r}(x_1,x_2,\dots)B_{n-v,k-r}(x_1^{'},x_2^{'},\dots).$$

Some particular values $B_{n,k}$ [2] are

$$B_{n,k}(1,1,\dots) = S(n,k)$$

$$B_{n,k}(1!,2!,\dots) = \frac{n!}{k!}\binom{n-1}{k-1}$$

$$B_{n,k}(1,2,\dots) = \binom{n}{k} k^{nyk}.$$

By convention, we consider $B_{0,0}=1$ and $B_{n,0}=0$ for $n\geq 1$. The stirling number of second kind $S(n,k)$ is given by the generating function

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=1}^{\infty} S(n,k)\frac{t^n}{n!}$$

and its explicit formula is

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n.$$

Let $f(t)=\sum_{n=0}^{\infty}a_n t^n$ and $g(t)=\sum_{n=0}^{\infty}b_n t^n$ be two generating functions. The series expansion of fog is given by the expression [5]

$$fog(t)=f(b_0)+\sum_{n\geq 1}\sum_{k=1}^{n}B_{n,k}(1!b_1,2!b_2,\dots)f^{(k)}(b_0)\frac{t^n}{n!}.$$

If $b_0=0$, it is obvious that

$$fog(t)=a_0+\sum_{n\geq 1}\sum_{k=1}^{n}k!\,a_k B_{n,k}(1!b_1,2!b_2\dots)\frac{t^n}{n!}.$$

Hence for $f(t)=e^t$ we have

$$e^{g(t)}=1+\sum_{n\geq 1}\sum_{k=1}^{n}B_{n,k}(1!b_1,2!\,b_2\dots)\frac{t^n}{n!}.$$

For α a complex number, $f(t)=t^{\alpha}$ and $(\alpha)_k=\alpha(\alpha-1)\dots(\alpha-k+1)$, if $b_0\neq 0$, the identity in [6] becomes

$$g^{\alpha}(t)=b_0^{\alpha}+\sum_{n\geq 1}\sum_{k=1}^{n}(\alpha)_k b_0^{\alpha-k}B_{n,k}(1!b_1,2!b_2,\dots)\frac{t^n}{n!},$$

Which implies that (see [7])

$$g^{-1}(t)=b_0^{-1}+\sum_{n\geq 1}\sum_{k=1}^{n}(-1)_k k!\,b_0^{1-k}B_{n,k}(1!b_1,2!b_2,\dots)\frac{t^n}{n!}.$$

3. Results

Let α_k be the sequence defined by $\alpha_0=1-x^k-y^m$ and $\alpha_n=-x^k-(m+d)^n y^m$; according to this sequence the following lemma holds true

Lemma 1. We have

$$B_{n,j}(\alpha_1,\alpha_2,\dots)=\sum_{r\leq j,v\leq n}\binom{n}{v}(m+d)^{n-v}S(v,r)S(n-v,j-r)\left(-x^k\right)^r(-y^m)^{j-r}$$

Proof. We have

$$B_{n,j}(\alpha_1,\alpha_2,\dots)=\sum_{r\leq j,v\leq n}\binom{n}{v}B_{v,r}(-x^k,-x^k,\dots)B_{n-v,j-r}(-(m+d)y^m,-(m+d)^2y^m,\dots).$$

But

$$B_{v,r}(-x^k,-x^k,\dots)=(-x^k)^r S(v,r)$$

and

$$B_{n-v,j-r}(-(m+d)y^m,-(m+d)^2y^m,\dots)=(-y^m)^{j-r}(m+d)^{n-v}S(n-v,j-r).$$

The explicit formula of $F_n^{(e)}(x,y;k,m,d)$ is given by the following theorem

Lemma 2. We have $F_0^{(e)}(x,y;k,m,d)=1-x^k-y^m$ and

$$F_n^{(e)}(x,y;k,m,d)=\sum_{j=1}^{n}(-1)^j j!\frac{B_{n,j}(\alpha_1,\alpha_2,\dots)}{(1-x^k-y^m)^{j+1}}.$$

Proof. Since

$$1-x^k e^t-y^m e^{(d+m)t}=\sum_{n\geq 0}\alpha_n\frac{t^n}{n!}.$$

Thus

$$\frac{1}{1-x^k-y^m e^{(m+d)t}} = (1-x^k-y^m)^{-1} + \sum_{n\geq 1}\sum_{j=1}^{n} (-1)^j j! \frac{B_{n,j}(\alpha_1,\alpha_2,\dots)}{(1-x^k-y^m)^{j+1}}\frac{t^n}{n!}.$$

Furthermore the desired result follows. We have already proved the following main results.

Theorem 1. For $n \geq 1$ we have

$$F_n^{(e)}(x,y;k,m,d) = \sum_{j=1}^{n} \sum_{r\leq j, v\leq n} \binom{n}{v} (-1)^j j! (m+d)^{n-v} S(v,r) S(n-v,j-r) \frac{(-x^k)^r(-y^m)^{j-r}}{(1-x^k-y^m)^{j+1}}$$

and

$$G_n(x,y;k,m,d) = \sum_{j=1}^{n} \sum_{r\leq j, v\leq n} \binom{n}{v} (-1)^j j! (m+d)^{n-v} S(v,r) S(n-v,j-r)(-x^k)^r(-y^m)^{j-r}(1-x^k-y^m)^{n-j}.$$

4. Applications

The main result can be applied to finitely many special cases. We restrict our study on Apostol-Bernoulli polynomials and numbers, Euler and Genocchi polynomials and modified Humbert polynomials and numbers.

4.1 Apostol-Bernoulli polynomials and numbers

Apostol-Bernoulli polynomials [1] $B_n(x,\alpha)$ are given by means of the following generating function

$$F_{AB}(t,x,\alpha) = \frac{te^{xt}}{\alpha e^t - 1} = \sum_{n\geq 0} B_n(x,\alpha)\frac{t^n}{n!}.$$

The comparison with $F_n(x,y;k,m,d)$ is possible by taking $y=0, m\neq 0$ to obtain $te^{xt}F = (x^k-1)F_{AB}$. Thereafter (see [11] Theorem 3.4.)

$$B_n(x,x^k) = -n\sum_{l=0}^{n}\binom{n-1}{l}\frac{x^{n-l-1}}{(1-x^k)^{l+1}} G_l(x,0,k,m,d).$$

Since $F_0(x,0;k,m,d)=1$ and

$$G_l(x,0;k,m,d) = \sum_{j=1}^{l} j!\, S(l,j) x^{jk}(1-x)^{l-j},$$

The following theorem holds true.

Theorem 2. For $n \geq 2$ we have

$$B_n(x,x^k) = -\frac{nx^{n-1}}{1-x^k} - n\sum_{l=1}^{n-1}\sum_{j=1}^{l}\binom{n-1}{l} j!\, S(l,j)\frac{x^{n-l+jk-1}}{(1-x^k)^{j+1}}.$$

For $x=0$ and $\alpha\neq 1$, we get Apostol-Bernoulli numbers $B_n(\alpha)=B_n(0,\alpha)$. Ozdemir *et al*. [11, Theorem3.3] proved that

$$B_n(x^k+y) = \frac{n}{(1-x^k-y)^n} G_{n-1}(x,y;k,1,0).$$

It obvious to remark that

$$G_n(x,y;k,1,0) = \sum_{j=1}^{n}\sum_{r\leq j, v\leq n}\binom{n}{v}(-1)^j j!\, S(v,r) S(n-v,j-r)(-x^k)^r(-y)^{j-r}(1-x^k-y)^{n-j},$$

And the following theorem is immediate.

Theorem 3. We have $B_1(x^k+y) = \frac{1}{1-x^k-y}$ and for $n\geq 2$;

$$B_n(x^k+y) = -n\sum_{j=1}^{n-1}\sum_{r\le j, v\le n-1}\binom{n-1}{v}(-1)^j j!\,S(v,r)S(n-v,j-r)\frac{(-x^k)^r(-y)^{j-r}}{(1-x^k-y)^{j+1}}.$$

4.2 Euler and Genocchi polynomials
The Euler polynomials of higher order $E_n^{(h)}(x)$ are defined by means of the following generating function

$$F_{Eh}(x,t;h) = \left(\frac{2e^{xt}}{e^t+1}\right)^h = \sum_{n\ge 0} E_n^{(h)}(x)\frac{t^n}{n!},$$

So that, obviously; $E_n^{(1)}(x) = E_n(x)$ and $E_n^{(1)}(0) = E_n$. Ozdemir *et al*. [11, Theorem 3.6] proved that

$$E_n(x) = \sum_{l=0}^{n}\binom{n}{l}\frac{x^{n-l}}{2^l}G_l(-1,0;1,m,d).$$

Consequently
Theorem 4. We have

$$E_n(x) = x^n + \sum_{l=0}^{n}\sum_{j=0}^{l}\binom{n}{l}\frac{j!}{2^j}S(l,j)x^{n-l}.$$

The result is deduced from the identity

$$G_l(-1,0;1,m,d) = \sum_{j=0}^{l} j!\,2^{l-j}S(l,j).$$

Genocchi polynomials $G_n(x)$ [13, 8] are defined by the generating function

$$F_G(x,t) = \frac{2te^{xt}}{e^t+1} = \sum_{n\ge 0} G_n(x)\frac{t^n}{n!}.$$

Their connection to polynomials $G_l(x,y;k,m,d)$ is given by (see [11, Theorem3.7])

$$G_n(x) = \sum_{l=0}^{n}\binom{n-1}{l}\frac{x^{n-l-1}}{2^l}G_l(-1,0;1,m,d).$$

Thereafter the following theorem holds true
Theorem 5. We have

$$G_n(x) = x^n + \sum_{l=1}^{n}\sum_{j=1}^{l}\binom{n-1}{l}\frac{j!}{2^j}S(l,j)x^{n-l-1}.$$

4.3 Modified Humbert polynomials and numbers
The modification of the generating function of Humbert polynomials goes alone to define some other families of special polynomials and numbers. In this subsection we investigate the polynomials $P_n^{(\alpha)}(x,\mu,a)$ defined by

$$P^{\alpha}(t,x;\mu,a) = \left(\frac{e^{xt}}{1-a\mu e^t+e^{at}}\right)^{\alpha} = \sum_{n\ge 0} P_n^{(\alpha)}(x,\mu,a)\frac{t^n}{n!},$$

and the numbers $Y_n^{(\alpha)}(\mu,a) = P_n^{(\alpha)}(0,\mu,a)$. For $\alpha = 1$ we can write $P_n(x,\mu,a)$ and $Y_n(\mu,a)$ and we have

$$P(t,x;\mu,a) = \frac{e^{xt}}{1-a\mu e^t+e^{at}} = \sum_{n\ge 0} P_n(x,\mu,a)\frac{t^n}{n!}.$$

One remarks that

$$P^{\alpha}(t,x;\mu,a) = \left(\frac{e^{xt}}{2-a\mu}\right)^{\alpha} F^{\alpha}(t,a\mu;-1,1,1,a-1).$$

Consequently the connection between the polynomials $P_n^{(\alpha)}(x,\mu,a)$ and $G_n^{(\alpha)}(a\mu,-1,1,1,a-1)$ is given by the relation

$$P_n^{(\alpha)}(x,\mu,a) = \frac{1}{(2-a\mu)^n}\sum_{l=0}^{n}\binom{n}{l}G_l^{(\alpha)}(a\mu,-1,1,1,a-1)(\alpha x)^{n-l}.$$

Polynomials $G_n^{(\alpha)}(x,y;k,m,d)$ are generated by $F^{\alpha}(t,x,y;k,m,d)$ and their explicit formula is given by the following theorem.

Theorem 6. We have $G_0^{(\alpha)}(a\mu,-1,1,1,a-1)=1$ and

$$G_n^{(\alpha)}(a\mu,-1,1,1,a-1) = \sum_{j=1}^{n}\sum_{r\le j,v\le n}\binom{n}{v}(-\alpha)_j(2-a\mu)^{n-j}a^{n-v}S(v,r)S(n-v,j-r)(-a\mu)^r.$$

Proof
It is obvious that

$$F^{\alpha}(t,a\mu,-1,1,1,a-1) = (2-a\mu)^{\alpha}(1-a\mu e^t+e^{at})^{-\alpha}.$$

We have

$$(1-a\mu e^t+e^{at})^{-\alpha} = (2-a\mu)^{-\alpha}+\sum_{n\ge1}\sum_{j=1}^{n}(-\alpha)_j(2-a\mu)^{-\alpha-j}B_{n,j}(\alpha_1,\alpha_2,\dots).$$

Thus $G_0^{(\alpha)}(a\mu,-1,1,1,a-1)=1$ and

$$G_n^{(\alpha)}(a\mu,-1,1,1,a-1) = \sum_{j=1}^{n}(-\alpha)_j(2-a\mu)^{n-j}B_{n,j}(\alpha_1,\alpha_2,\dots),$$

And the desired result follows. Consequently for $\alpha=1$ we have

$$P_n(x;\mu,a) = \frac{1}{(2-a\mu)^n}\sum_{l=0}^{n}\binom{n}{l}G_l(a\mu,-1,1,1,a-1)(\alpha x)^{n-l},$$

with

$$G_l(a\mu,-1,1,1,a-1) = \sum_{j=1}^{l}\sum_{r\le j,v\le l}\binom{l}{v}(-1)^j j!\,(2-a\mu)^{l-j}a^{l-v}S(v,r)S(n-v,j-r)(-a\mu)^r.$$

The following corollary holds true.
Corollary 1. We have

$$P_n(x;\mu,a) = (2-a\mu)^{-1}x^n+\sum_{l=1}^{n}\sum_{j=1}^{l}\sum_{r\le j,v\le l}\binom{n}{l}\binom{l}{v}(-1)^j j!\,a^{l-v}S(v,r)S(l-v,j-r)\frac{(-a\mu)^r x^{n-l}}{(2-a\mu)^{j+1}}.$$

For computing the numbers $Y_n(\mu,a)$ Ozdemir *et al.* [11] used the recurrence relation

$$Y_n(\mu,a) = \frac{1}{2-a\mu}\sum_{j=0}^{n-1}(a\mu-a^{n-j})Y_j(\mu,a).$$

An improvement of this result is established in the following corollary.

Corollary 2. We have $Y_0(\mu,a)=(2-a\mu)^{-1}$ and

$$Y_n(\mu,a) = \sum_{j=1}^{n}\sum_{r\le j,v\le n}\binom{n}{v}(-1)^j j!\,S(v,r)S(n-v,j-v)\frac{(-a\mu)^r a^{n-v}}{(2-a\mu)^{j+1}}.$$

5. Conclusion
Bell polynomials are an important tool from analytic combinatorics for computing explicit formulas for several families of polynomials and numbers. In this work we used these polynomials, the notion of generating function of functions and algebraic operation on functions to evaluate polynomials recently introduced by Ozdemir *et al*. This allowed us to deduce the explicit form of Apostol-Bernoulli polynomials and numbers, Euler and Genocchi polynomials and modified Humbert polynomials and numbers.

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