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## Investigations on special polynomials including Apostol-type and Humbert-type polynomials

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### Abstract

In this article we are interested by a special family of numbers and polynomials introduced and studied by Ozdemir *et al.* We give explicit formula and apply the obtained to Apostol-Bernoulli polynomials and numbers.

**Keywords:** Bell polynomials, apostol-bernoulli polynomials, euler and genocchi polynomials, modified humber polynomials

### 1. Introduction

The special numbers and polynomials play an important role in many areas of Mathematics and applied Mathematics. Numerous works were published in this wide topic [2, 3, 8]. In the present paper we are interested by the family of polynomials  $G_n(x, y; k, m, d)$  introduced by Ozdemir *et al.* [11] defined by means of the generating function

$$F(t, x, y; k, m, d) = \frac{1 - x^k - y^m}{1 - x^k e^t - y^m e^{(m+d)t}}$$

$$= \sum_{n \geq 0} \frac{G_n(x, y; k, m, d)}{n!} \left( \frac{t}{1 - x^k - y^m} \right)^n.$$

Polynomials  $G_n(x, y; k, m, d)$  are a generalization of numerous families of polynomials and numbers including Apostol-Bernoulli, Apostol-Euler polynomials and numbers, as well as classical Bernoulli, Euler and Genocchi polynomials and numbers. The Humbert polynomial  $H_{n,m}^{(\alpha)}(x)$  [10] appear in the series expansion

$$\left( \frac{1}{1 - mxt + t^m} \right)^\alpha = \sum_{n \geq 0} H_{n,m}^{(\alpha)}(x) t^n.$$

Other generalizations of Humbert polynomials are given [4, 5]. Modified Humbert polynomials are introduced in [11] by the generating function

$$\frac{1}{1 - ape^t + e^{at}} = \sum_{n \geq 0} Y_n(\rho, a) \frac{t^n}{n!},$$

Which admit the following generalization

$$\left( \frac{1}{1 - ape^t + e^{at}} \right)^\alpha = \sum_{n \geq 0} Y_n^{(\alpha)}(\rho, a) \frac{t^n}{n!}.$$

The Fibonacci-type polynomials in two variables  $F_n(x, y; k, m, d)$  has been recently defined by Ozdemir-Simsek [12] as follows

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$$H(t, x, y; k, m, d) = \frac{1}{1 - x^k t - y^m t^{m+d}}$$

$$= \sum_{n \geq 0} F_n(x, y; k, m, d) t^n$$

If we take  $e^t$  instead of  $t$  in the last identity, one obtains the following family of rational functions  $F_n^{(e)}(x, y; k, m, d)$  given by the generating function

$$\frac{1}{1 - x^k - y^m e^{(m+d)t}} = \sum_{n \geq 0} F_n^{(e)}(x, y; k, m, d) \frac{t^n}{n!}$$

One can see [9] for further information about the notion of generating function of a sequence of functions. Thereafter the connection between  $F(t, x, y; k, m, d)$  and  $H(t, x, y; k, m, d)$  is given by

$$F(t, x, y; k, m, d) = (1 - x^k - y^m) \text{Hoe}^t.$$

In this paper we complete the work of Ozdemir *et al.* by the explicit formulae of  $F_n^{(e)}(x, y; k, m, d)$  and polynomials  $F_n(x, y; k, m, d)$ . The obtained results are applied to compute explicit formulae of modified Humbert polynomials and other well-known families of numbers and polynomials.

### 2. Bell polynomials

The exponential partial Bell polynomial  $B_{n,k} := B_{n,k}(x_1, x_2, \dots)$  of the sequence  $(x_j)_{j \in \mathbb{N}}$  is defined by the exponential generating function

$$\frac{1}{k!} \left( \sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots) \frac{t^n}{n!}$$

For which the explicit formula is

$$B_{n,k} = \frac{n!}{k!} \sum_{\pi_n(k)} \binom{k}{k_1, \dots, k_{n-k+1}} \prod_{r=1}^{n-k+1} \left( \frac{x_r}{r!} \right)^{k_r}$$

Where  $\pi_n(k)$  is the set of all partitions  $k_1 + 2k_2 + \dots + nk_n = n$  of  $n$  such that  $k_1 + k_2 + \dots + k_n = k$  we recall that the exponential partial Bell polynomial of the sum  $(x_j + x'_j)_{j \in \mathbb{N}}$  is given by the following identity [2]:

$$B_{n,k}(\dots, x_i + x'_i, \dots) = \sum_{r \leq j, v \leq n} \binom{n}{v} B_{v,r}(x_1, x_2, \dots) B_{n-v, k-r}(x'_1, x'_2, \dots).$$

Some particular values  $B_{n,k}^{[2]}$  are

$$B_{n,k}(1, 1, \dots) = S(n, k)$$

$$B_{n,k}(1!, 2!, \dots) = \frac{n!}{k!} \binom{n-1}{k-1}$$

$$B_{n,k}(1, 2, \dots) = \binom{n}{k} k^{nyk}$$

By convention, we consider  $B_{0,0} = 1$  and  $B_{n,0} = 0$  for  $n \geq 1$ . The stirling number of second kind  $S(n, k)$  is given by the generating function

$$\frac{1}{k!} (e^t - 1)^k = \sum_{n=1}^{\infty} S(n, k) \frac{t^n}{n!}$$

and its explicit formula is

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$$

Let  $f(t) = \sum_{n=0}^{\infty} a_n t^n$  and  $g(t) = \sum_{n=0}^{\infty} b_n t^n$  be two generating functions. The series expansion of  $f \circ g$  is given by the expression <sup>[5]</sup>

$$f \circ g(t) = f(b_0) + \sum_{n \geq 1} \sum_{k=1}^n B_{n,k}(1!b_1, 2!b_2, \dots) f^{(k)}(b_0) \frac{t^n}{n!}.$$

If  $b_0 = 0$ , it is obvious that

$$f \circ g(t) = a_0 + \sum_{n \geq 1} \sum_{k=1}^n k! a_k B_{n,k}(1!b_1, 2!b_2, \dots) \frac{t^n}{n!}.$$

Hence for  $f(t) = e^t$  we have

$$e^{g(t)} = 1 + \sum_{n \geq 1} \sum_{k=1}^n B_{n,k}(1!b_1, 2!b_2, \dots) \frac{t^n}{n!}.$$

For  $\alpha$  a complex number,  $f(t) = t^\alpha$  and  $(\alpha)_k = \alpha(\alpha - 1) \dots (\alpha - k + 1)$ , if  $b_0 \neq 0$ , the identity in <sup>[6]</sup> becomes

$$g^\alpha(t) = b_0^\alpha + \sum_{n \geq 1} \sum_{k=1}^n (\alpha)_k b_0^{\alpha-k} B_{n,k}(1!b_1, 2!b_2, \dots) \frac{t^n}{n!},$$

Which implies that (see <sup>[7]</sup>)

$$g^{-1}(t) = b_0^{-1} + \sum_{n \geq 1} \sum_{k=1}^n (-1)_k k! b_0^{1-k} B_{n,k}(1!b_1, 2!b_2, \dots) \frac{t^n}{n!}.$$

### 3. Results

Let  $\alpha_k$  be the sequence defined by  $\alpha_0 = 1 - x^k - y^m$  and  $\alpha_n = -x^k - (m + d)^n y^m$ ; according to this sequence the following lemma holds true

**Lemma 1.** We have

$$B_{n,j}(\alpha_1, \alpha_2, \dots) = \sum_{r: j, v \leq n} \binom{n}{v} (m + d)^{n-v} S(v, r) S(n - v, j - r) (-x^k)^r (-y^m)^{j-r}$$

**Proof.** We have

$$B_{n,j}(\alpha_1, \alpha_2, \dots) = \sum_{r: j, v \leq n} \binom{n}{v} B_{v,r}(-x^k, -x^k, \dots) B_{n-v, j-r}(-(m + d)y^m, -(m + d)^2 y^m, \dots).$$

But

$$B_{v,r}(-x^k, -x^k, \dots) = (-x^k)^r S(v, r)$$

and

$$B_{n-v, j-r}(-(m + d)y^m, -(m + d)^2 y^m, \dots) = (-y^m)^{j-r} (m + d)^{n-v} S(n - v, j - r).$$

The explicit formula of  $F_n^{(e)}(x, y; k, m, d)$  is given by the following theorem

**Lemma 2.** We have  $F_0^{(e)}(x, y; k, m, d) = 1 - x^k - y^m$  and

$$F_n^{(e)}(x, y; k, m, d) = \sum_{j=1}^n (-1)^j j! \frac{B_{n,j}(\alpha_1, \alpha_2, \dots)}{(1 - x^k - y^m)^{j+1}}.$$

**Proof.** Since

$$1 - x^k e^t - y^m e^{(d+m)t} = \sum_{n \geq 0} \alpha_n \frac{t^n}{n!}.$$

Thus

$$\frac{1}{1 - x^k - y^m e^{(m+d)t}} = (1 - x^k - y^m)^{-1} + \sum_{n \geq 1} \sum_{j=1}^n (-1)^j j! \frac{B_{n,j}(\alpha_1, \alpha_2, \dots)}{(1 - x^k - y^m)^{j+1}} \frac{t^n}{n!}.$$

Furthermore the desired result follows. We have already proved the following main results.

**Theorem 1.** For  $n \geq 1$  we have

$$F_n^{(e)}(x, y; k, m, d) = \sum_{j=1}^n \sum_{r \leq j, v \leq n} \binom{n}{v} (-1)^j j! (m + d)^{n-v} S(v, r) S(n - v, j - r) \frac{(-x^k)^r (-y^m)^{j-r}}{(1 - x^k - y^m)^{j+1}}$$

and

$$G_n(x, y; k, m, d) = \sum_{j=1}^n \sum_{r \leq j, v \leq n} \binom{n}{v} (-1)^j j! (m + d)^{n-v} S(v, r) S(n - v, j - r) (-x^k)^r (-y^m)^{j-r} (1 - x^k - y^m)^{n-j}.$$

### 4. Applications

The main result can be applied to finitely many special cases. We restrict our study on Apostol-Bernoulli polynomials and numbers, Euler and Genocchi polynomials and modified Humbert polynomials and numbers.

#### 4.1 Apostol-Bernoulli polynomials and numbers

Apostol-Bernoulli polynomials  $^{[1]}B_n(x, \alpha)$  are given by means of the following generating function

$$F_{AB}(t, x, \alpha) = \frac{te^{xt}}{\alpha e^t - 1} = \sum_{n \geq 0} B_n(x, \alpha) \frac{t^n}{n!}.$$

The comparison with  $F_n(x, y; k, m, d)$  is possible by taking  $y = 0, m \neq 0$  to obtain  $te^{xt}F = (x^k - 1)F_{AB}$ . Thereafter (see <sup>[11]</sup> Theorem 3.4.)

$$B_n(x, x^k) = -n \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{x^{n-l-1}}{(1 - x^k)^{l+1}} G_1(x, 0, k, m, d).$$

Since  $F_0(x, 0; k, m, d) = 1$  and

$$G_1(x, 0; k, m, d) = \sum_{j=1}^1 j! S(1, j) x^{jk} (1 - x)^{1-j},$$

The following theorem holds true.

**Theorem 2.** For  $n \geq 2$  we have

$$B_n(x, x^k) = -\frac{nx^{n-1}}{1 - x^k} - n \sum_{l=1}^{n-1} \sum_{j=1}^l \binom{n-1}{l} j! S(l, j) \frac{x^{n-1+jk-1}}{(1 - x^k)^{j+1}}.$$

For  $x = 0$  and  $\alpha \neq 1$ , we get Apostol-Bernoulli numbers  $B_n(\alpha) = B_n(0, \alpha)$ . Ozdemir *et al.* [11, Theorem3.3] proved that

$$B_n(x^k + y) = \frac{n}{(1 - x^k - y)^n} G_{n-1}(x, y; k, 1, 0).$$

It obvious to remark that

$$G_n(x, y; k, 1, 0) = \sum_{j=1}^n \sum_{r \leq j, v \leq n} \binom{n}{v} (-1)^j j! S(v, r) S(n - v, j - r) (-x^k)^r (-y)^{j-r} (1 - x^k - y)^{n-j},$$

And the following theorem is immediate.

**Theorem 3.** We have  $B_1(x^k + y) = \frac{1}{1 - x^k - y}$  and for  $n \geq 2$ ;

$$E_n(x^k + y) = -n \sum_{j=1}^{n-1} \sum_{r \leq j, v \leq n-1} \binom{n-1}{v} (-1)^j j! S(v, r) S(n-v, j-r) \frac{(-x^k)^r (-y)^{j-r}}{(1-x^k-y)^{j+1}}.$$

**4.2 Euler and Genocchi polynomials**

The Euler polynomials of higher order  $E_n^{(h)}(x)$  are defined by means of the following generating function

$$F_{Eh}(x, t; h) = \left( \frac{2e^{xt}}{e^t + 1} \right)^h = \sum_{n \geq 0} E_n^{(h)}(x) \frac{t^n}{n!},$$

So that, obviously;  $E_n^{(1)}(x) = E_n(x)$  and  $E_n^{(1)}(0) = E_n$ . Ozdemir *et al.* [11, Theorem 3.6] proved that

$$E_n(x) = \sum_{l=0}^n \binom{n}{l} \frac{x^{n-l}}{2^l} G_1(-1, 0; 1, m, d).$$

Consequently

**Theorem 4.** We have

$$E_n(x) = x^n + \sum_{l=0}^n \sum_{j=0}^{l-1} \binom{n}{l} \frac{j!}{2^j} S(l, j) x^{n-l}.$$

The result is deduced from the identity

$$G_1(-1, 0; 1, m, d) = \sum_{j=0}^{l-1} j! 2^{l-j} S(l, j).$$

Genocchi polynomials  $G_n(x)$  [13, 8] are defined by the generating function

$$F_G(x, t) = \frac{2te^{xt}}{e^t + 1} = \sum_{n \geq 0} G_n(x) \frac{t^n}{n!}.$$

Their connection to polynomials  $G_1(x, y; k, m, d)$  is given by (see [11, Theorem 3.7])

$$G_n(x) = \sum_{l=0}^n \binom{n-1}{l} \frac{x^{n-l-1}}{2^l} G_1(-1, 0; 1, m, d).$$

Thereafter the following theorem holds true

**Theorem 5.** We have

$$G_n(x) = x^n + \sum_{l=1}^n \sum_{j=1}^{l-1} \binom{n-1}{l} \frac{j!}{2^j} S(l, j) x^{n-l-1}.$$

**4.3 Modified Humbert polynomials and numbers**

The modification of the generating function of Humbert polynomials goes alone to define some other families of special polynomials and numbers. In this subsection we investigate the polynomials  $P_n^{(\alpha)}(x, \mu, a)$  defined by

$$P^\alpha(t, x; \mu, a) = \left( \frac{e^{xt}}{1 - a\mu e^t + e^{at}} \right)^\alpha = \sum_{n \geq 0} P_n^{(\alpha)}(x, \mu, a) \frac{t^n}{n!},$$

and the numbers  $Y_n^{(\alpha)}(\mu, a) = P_n^{(\alpha)}(0, \mu, a)$ . For  $\alpha = 1$  we can write  $P_n(x, \mu, a)$  and  $Y_n(\mu, a)$  and we have

$$P(t, x; \mu, a) = \frac{e^{xt}}{1 - a\mu e^t + e^{at}} = \sum_{n \geq 0} P_n(x, \mu, a) \frac{t^n}{n!}.$$

One remarks that

$$P^\alpha(t, x; \mu, a) = \left( \frac{e^{xt}}{2 - a\mu} \right)^\alpha F^\alpha(t, a\mu; -1, 1, 1, a - 1).$$

Consequently the connection between the polynomials  $P_n^{(\alpha)}(x, \mu, a)$  and  $G_n^{(\alpha)}(a\mu, -1, 1, 1, a - 1)$  is given by the relation

$$P_n^{(\alpha)}(x, \mu, a) = \frac{1}{(2 - a\mu)^n} \sum_{l=0}^n \binom{n}{l} G_l^{(\alpha)}(a\mu, -1, 1, 1, a - 1)(ax)^{n-l}.$$

Polynomials  $G_n^{(\alpha)}(x, y; k, m, d)$  are generated by  $F^\alpha(t, x, y; k, m, d)$  and their explicit formula is given by the following theorem.

**Theorem 6.** We have  $G_0^{(\alpha)}(a\mu, -1, 1, 1, a - 1) = 1$  and

$$G_n^{(\alpha)}(a\mu, -1, 1, 1, a - 1) = \sum_{j=1}^n \sum_{r \leq j, v \leq n} \binom{n}{v} (-a)_j (2 - a\mu)^{n-j} a^{n-v} S(v, r) S(n - v, j - r) (-a\mu)^r.$$

**Proof**

It is obvious that

$$F^\alpha(t, a\mu, -1, 1, 1, a - 1) = (2 - a\mu)^n (1 - a\mu e^t + e^{at})^{-n}.$$

We have

$$(1 - a\mu e^t + e^{at})^{-n} = (2 - a\mu)^{-n} + \sum_{n \geq 1} \sum_{j=1}^n (-a)_j (2 - a\mu)^{-n-j} B_{nj}(a_1, a_2, \dots).$$

Thus  $G_0^{(\alpha)}(a\mu, -1, 1, 1, a - 1) = 1$  and

$$G_n^{(\alpha)}(a\mu, -1, 1, 1, a - 1) = \sum_{j=1}^n (-a)_j (2 - a\mu)^{n-j} B_{nj}(a_1, a_2, \dots).$$

And the desired result follows. Consequently for  $\alpha = 1$  we have

$$P_n(x; \mu, a) = \frac{1}{(2 - a\mu)^n} \sum_{l=0}^n \binom{n}{l} G_l(a\mu, -1, 1, 1, a - 1)(ax)^{n-l},$$

with

$$G_l(a\mu, -1, 1, 1, a - 1) = \sum_{j=1}^l \sum_{r \leq j, v \leq l} \binom{l}{v} (-1)^j j! (2 - a\mu)^{l-j} a^{l-v} S(v, r) S(l - v, j - r) (-a\mu)^r.$$

The following corollary holds true.

**Corollary 1.** We have

$$P_n(x; \mu, a) = (2 - a\mu)^{-1} x^n + \sum_{l=1}^n \sum_{j=1}^l \sum_{r \leq j, v \leq l} \binom{n}{l} \binom{l}{v} (-1)^j j! a^{l-v} S(v, r) S(l - v, j - r) \frac{(-a\mu)^r x^{n-l}}{(2 - a\mu)^{j+1}}.$$

For computing the numbers  $Y_n(\mu, a)$  Ozdemir *et al.* [11] used the recurrence relation

$$Y_n(\mu, a) = \frac{1}{2 - a\mu} \sum_{j=0}^{n-1} (a\mu - a^{n-j}) Y_j(\mu, a).$$

An improvement of this result is established in the following corollary.

**Corollary 2.** We have  $Y_0(\mu, a) = (2 - a\mu)^{-1}$  and

$$Y_n(\mu, a) = \sum_{j=1}^n \sum_{r \leq j, v \leq n} \binom{n}{v} (-1)^j j! S(v, r) S(n - v, j - v) \frac{(-a\mu)^r a^{n-v}}{(2 - a\mu)^{j+1}}.$$

**5. Conclusion**

Bell polynomials are an important tool from analytic combinatorics for computing explicit formulas for several families of polynomials and numbers. In this work we used these polynomials, the notion of generating function of functions and algebraic operation on functions to evaluate polynomials recently introduced by Ozdemir *et al.* This allowed us to deduce the explicit form of Apostol-Bernoulli polynomials and numbers, Euler and Genocchi polynomials and modified Humbert polynomials and numbers.

**6. References**

1. Apostol TM. On the Lerch Zeta function. Pacific J Math 1951;1(2):161-167.

2. Comtet L. *Advanced Combinatorics: The art of finite and infinite expansions*. Reidel, Dordrecht and Boston 1974.
3. Djordjevic GB, Milovanovic GV. *Special Classes of Polynomials*. University of Nis; Leskovac 2014.
4. Goubi M. On a generalization of Catalan polynomials. *Facta Universitatis (Nis) Ser. Math. Inform* 2018;33(2):163-176.
5. Goubi M. Successive derivative of Fibonacci type polynomials of higher order in two variables. *Filomat* 2018;32(4):5149-5159.
6. Goubi M. On composition of generating functions. *CJMS* 2020;9(2):256-265.
7. Goubi M. Explicit formula of a new class of q-Hermite-based Apostol-type polynomials and generalization. *Notes Numb. Theory Discr. Math* 2020;26:93-102.
8. Goubi M. On a generalized family of Euler-Genocchi polynomials. *Integers* 2021;A48:21.
9. Goubi M. Generating functions for generalization Simsek numbers and their applications. *App. Anal. Discrete Math* 2021 (Available online).
10. Humbert P. Some extensions of Pincherles polynomials. *Proc. Edinburg Math. Soc* 1921;1(39):21-24.
11. Ozdemir G, Simsek Y, Milovanovic GV. Generating functions for Special Polynomials and Numbers Including Apostol-type and Humbert type Polynomials. *Mediterr. J Math* 2017;14(117):1-17.
12. Ozdemir G, Simsek Y. Generating functions for two variables related to a family of Fibonacci type polynomials and Numbers. *Filomat* 2016;30(4):969-975.
13. Ozden H, Simsek Y, Srivastava HM. A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. *Comput. Math. Appl* 2010;60:2779-2787.