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Estimates of the zygmund type of mixed fractional integrals of riemann-liouville and derivatives of marchaud

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Abstract
Zygmund type estimates are obtained for the mixed continuity modulus of some mixed fractional integrals. is known, the Riemann-Liouville fractional integration operator establishes an isomorphism between Hölder spaces for functions one variables. We study mixed Riemann-Liouville fractional integration operators and mixed fractional derivative in Marchaud form of function of two variables.

Keywords: functions of two variables, fractional derivative of Marchaud form, mixed fractional derivative, mixed fractional integral, mixed continuity modulus

1. Introduction

One of the most important problems in the theory of integral operators in space is the problem of elucidating the dependence of the smoothness of the image on the smoothness of the preimage. The solution to such a problem plays an important role in the solvability of integral equations, their stability and so on. The concept of smoothness can be formulated in a variety of terms. One of the ways of sufficiently fine-grabbing the smoothness of functions is the notion of generalized Hölderness, formulated in terms of the behavior of the modulus of continuity. Thus one of the important questions in the theory of operators is as follows: Let be A an operator acting in a Banach spaces X and let be the modulus of continuity $\omega(f,h)=\sup\limits_{|t|\le h}\|f(x+h)-f(x)\|_X$ of X. H of X. How can the behavior of the modulus of continuity be characterized $\omega(A\varphi,h)$ if the behavior of the modulus of continuity of function $\omega(\varphi;h)$: $\omega(\varphi;h)\le C\psi(h)$ for all is known $\varphi\in X$, where is $\psi(x)$ a given continuous function, $\psi(0)=0$. A similar problem can be considered completely solved for different spaces and also for the Hölder spaces of functions of one variable and weights, when fractional integration and fractional differentiation operators [1]. The assertion for multidimensional case for a mixed fractional Riemann-Liouville integral was studied [2, 15]. When mixed fractional derivatives form Marchaud

$$\left(D^{\alpha,\beta}_{a+,c+}\varphi\right)(x,y)=\frac{(x-a)^{-\alpha}(y-c)^{-\beta}}{\Gamma(1-\alpha)\Gamma(1-\beta)}\varphi(x,y)+$$

$$+\frac{\alpha\beta}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_a^x\int_c^y\frac{\varphi(x,y)-\varphi(t,s)}{(x-t)^{1+\alpha}(y-s)^{1+\beta}}dtds,\ x>a,\ y>c \qquad (1)$$

were not studied. This paper is devoted to the study of the properties for functions of two variables. Consider the operator (1) in a rectangle $Q=\{(x,y):a<x<b,\ c<y<d\}$.

2. Preliminary

Definition 1. Let given bounded on [a, b] function $\varphi(x)$. Under modulus of continuity $\varphi(x)$ understood expression

$$\sup_{h\in[0,\delta]}|\varphi(x+h)-\varphi(x)|=\omega(\varphi;\ \delta),\quad 0<\delta\le b-a.$$

1)

Definition 2. We denote by Φ^1 function class $\omega(\delta)\in(0,b-a]$ and satisfying the conditions

1) $\omega(\delta)>0$ in (0, *b–a*], $\lim\limits_{\delta\to 0}\omega(\delta)=0$;
2) $\omega(\delta)\uparrow$ in (0, *b–a*];
3) $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$.

Below in the estimates we need inequalities:

1) if $\omega(\varphi;h)$ is modulus continuity, then we have : $x_2\omega(\varphi;\ x_1)\le Cx_1\omega(\varphi;\ x_2),\ x_2\le x_1$; (2)
2) if $\lambda\le 1$ then $\left|x_1^\lambda-x_2^\lambda\right|\le C(x_1-x_2)x_2^{\lambda-1},\ \ x_1\ge x_2>0$; (3)
3) if $\lambda\ge 0$ then $\left|x_1^\lambda-x_2^\lambda\right|\le C(x_1-x_2)x_1^{\lambda-1},\ \ x_1\ge x_2>0$. (4)

For a continuous function $\varphi(x,y)$ on R^2 we introduce the notation

$$\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)=\varphi(x+h,y)-\varphi(x,y),\quad \left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)=\varphi(x,y+\eta)-\varphi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)=\varphi(x+h,y+\eta)-\varphi(x+h,y)-\varphi(x,y+\eta)+\varphi(x,y),$$

so that

$$\varphi(x+h,y+\eta)=\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)+\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)+\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)+\varphi(x,y). \tag{5}$$

Everywhere in the sequel by C,C_1,C_2 etc we denote positive constants which may different values in different occurences and even in the same line.

Now we introduce the following characteristics:

1) Private modules of continuity

$$\overset{1,0}{\omega}(\varphi;\ \delta,0)=\sup_y\sup_{0\le h\le\delta}\left|\left(\overset{1,0}{\Delta}_h\varphi\right)(x,y)\right| \text{ and } \overset{0,1}{\omega}(\varphi;\ 0,\sigma)=\sup_x\sup_{0\le\eta\le\sigma}\left|\left(\overset{0,1}{\Delta}_\eta\varphi\right)(x,y)\right|;$$

2) Mixed modulus continuity of order 1.1

$$\overset{1,1}{\omega}(\varphi;\ \delta,\sigma)=\sup_{x,y}\sup_{\substack{0\le h\le\delta\\ 0\le\eta\le\sigma}}\left|\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)\right|, \text{ where } 0<\delta\le b,\ 0<\sigma\le d.$$

It follows from the definition $\overset{1,1}{\omega}(\varphi;\delta,\sigma)$ that this function belongs in each variable Φ^1. In addition, we note that there is an inequality

$$\overset{1,1}{\omega}(\varphi;\delta,\sigma)\le 2\min\left\{\overset{1,0}{\omega}(\varphi;\delta,0),\ \overset{0,1}{\omega}(\varphi;0,\sigma)\right\}. \tag{6}$$

Definition 3. We denote by $\Phi^{1,1}$ the class of functions of two variables $\omega(\delta,\sigma)$ satisfying conditions:

1) $\omega(\delta,\sigma)$ in δ for any fixed σ;
2) $\omega(\delta,\sigma)$ in σ for any fixed δ.

We call this class the class of mixed modulus of continuity of the first order of continuity functions of two variables.

The following statements are known (see [1, p. 249-253]). We use the schemes of the proofs to make the presentation easier for two-dimensional case.

The following two theorems give estimates which might be called Zygmund types by analogy with the Zygmund estimate known in the theory of singular integrals and estimating the continuity modulus $\omega(H\varphi,h)$ of a conjugate function $H\varphi$ via the continuity modulus $\omega(\varphi,h)$ of a function $\varphi(x)$ itself.

Consider the one-dimensional fractional Riemann-Liouville integral

$$\left(I_{a+}^\alpha\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\varphi(t)dt,\ x>a,\ 0<\alpha<1. \tag{7}$$

Theorem 1. [1]. Let $\varphi(x)$ be continuous on $[a,b]$ and let $\varphi(a)=0$. For a fractional integral $I_{a+}^\alpha\varphi,\ 0<\alpha<1$, the estimate

$$\omega\left(I_{a+}^{\alpha}\varphi, h\right) \le Ch\int_{h}^{b-a}\frac{\omega(\varphi,t)}{t^{2-\alpha}}dt \tag{8}$$

is valid.

Proof. Representing (7) as

$$\left(I_{a+}^{\alpha}\varphi\right)(x) = \frac{\varphi(a)}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}dt + \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{\varphi(t)-\varphi(a)}{(x-t)^{1-\alpha}}dt = \Delta_1(x)+\Delta_2(x).$$

Let $h>0;\ x, x+h\in[a,b]$. We have

$$\Delta_2(x+h)-\Delta_2(x) = \frac{\varphi(x)-\varphi(a)}{\Gamma(1+\alpha)}\left[(x+h-a)^{\alpha}-(x-a)^{\alpha}\right] + \frac{1}{\Gamma(\alpha)}\int_{0}^{h}\frac{\varphi(x+t)-\varphi(t)}{(h-t)^{1-\alpha}}dt +$$

$$+\frac{1}{\Gamma(\alpha)}\int_{0}^{x-a}\left[\varphi(x-t)-\varphi(t)\right]\left[(h+t)^{\alpha-1}-t^{\alpha-1}\right]dt = I_1+I_2+I_3.$$

We have: $|I_1| \le C\omega(\varphi,x)\left|(x+h-a)^{\alpha}-(x-a)^{\alpha}\right|$. In the case $x-a\le h$ we have $|I_1|\le Ch^{\alpha}\omega(\varphi,h)$. Let $x-a\ge h$ then

$$|I_1| \le C\omega(\varphi,x-a)(x-a)^{\alpha}\left[\left(1+\frac{h}{x-a}\right)^{\alpha}-1\right] \le Ch(x-a)^{\alpha-1}\omega(\varphi,x-a). \tag{9}$$

Since

$$C(x-a)^{\alpha-1}\omega(\varphi,x-a) \le \omega(\varphi,x-a)\int_{x-a}^{b-a}t^{\alpha-2}dt \le \int_{x-a}^{b-a}\frac{\omega(\varphi,t)}{t^{2-\alpha}}dt \le \int_{h}^{b-a}\frac{\omega(\varphi,t)}{t^{2-\alpha}}dt.$$

It follows from (9) that $|I_1| \le Ch\int_{h}^{b-a}\frac{\omega(\varphi,t)}{t^{2-\alpha}}dt$.

Further $|I_2| \le \int_{0}^{h}\frac{\omega(\varphi,t)dt}{(h-t)^{1-\alpha}} = h^{\alpha}\int_{0}^{1}\frac{\omega(\varphi,h\xi)}{(1-\xi)^{1-\alpha}}d\xi \le Ch^{\alpha}\omega(\varphi,h)$ with $C=\int_{0}^{1}(1-\xi)^{\alpha-1}d\xi$.

To estimate I_3 we distinguish the case 1) $x-a\ge h$ and 2) $x-a\le h$. In the first case

$$|I_3| \le C\left\{\int_{0}^{h}\omega(\varphi,t)\left[t^{\alpha-1}-(h+t)^{\alpha-1}\right]dt + \int_{h}^{x-a}\omega(\varphi,t)\left[t^{\alpha-1}-(h+t)^{\alpha-1}\right]dt\right\} \le \le C_2\left[h^{\alpha}\omega(\varphi,h)+h\int_{h}^{b-a}\frac{\omega(\varphi,t)}{t^{2-\alpha}}dt\right]$$

Obviously in the second case $|I_3|\le C_1h^{\alpha}\omega(\varphi,h)$.

Estimates for I_1, I_2, I_3 lead to (8) if we take into account the fact that $h^{\alpha}\omega(\varphi,h)$ is dominated by the right-hand side of (8). The latter is easily obtained in view of the monotonicity of the function $\omega(\varphi,t)$.

The Marchaud fractional differentiation operator has a form

$$\left(\mathrm{D}_{a+}^{\alpha}\varphi\right)(x) = \frac{\varphi(x)}{(x-a)^{\alpha}\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)}\int_{a}^{x}\frac{\varphi(x)-\varphi(t)}{(x-t)^{1+\alpha}}dt \tag{10}$$

where $0<\alpha<1$.

Theorem 2. Let $\varphi(x)$ be continuous on $[a,b]$ and $\varphi(a)=0$. Then its fractional derivative $\mathrm{D}_{a+}^{\alpha}\varphi,\ 0<\alpha<1$ admits the estimate

$$\omega\left(\mathrm{D}_{a+}^{\alpha}\varphi, h\right) \le C\int_{0}^{h}\frac{\omega(\varphi,t)}{t^{1+\alpha}}dt \tag{11}$$

provided that the integral on the right-hand side converges.

Proof. We present (10) as

$$\left(\mathrm{D}_{a+}^{\alpha}\varphi\right)(x)=\frac{\varphi(a)}{(x-a)^{\alpha}\Gamma(1-\alpha)}+\frac{\varphi(x)-\varphi(a)}{(x-a)^{\alpha}\Gamma(1-\alpha)}+\frac{\alpha}{\Gamma(1-\alpha)}\int_{a}^{x}\frac{\varphi(x)-\varphi(t)}{(x-t)^{1+\alpha}}dt=$$

$$=\frac{\varphi(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}+F_{1}(x)+F_{2}(x).$$

We begin by noting that the function $F_{1}(x)=\dfrac{\varphi(x)-\varphi(a)}{(x-a)^{\alpha}},\ 0<\alpha<1$ admits the estimate

$$\omega(F_{1},h)\leq C\int_{0}^{h}\frac{\omega(\varphi,t)}{t^{1+\alpha}}dt. \tag{12}$$

Let us prove (12). Taking $h>0$ we have

$$F_{1}(x+h)-F_{1}(x)=[\varphi(x)-\varphi(a)]\left[(x+h-a)^{-\alpha}-(x-a)^{-\alpha}\right]+\frac{\varphi(x+h)-\varphi(x)}{(x+h-a)^{\alpha}}=A_{1}+A_{2}.$$

Hence

$$|A_{2}|\leq(x+h-a)^{-\alpha}\omega(\varphi,h)\leq h^{-\alpha}\omega(\varphi,h)\leq C\int_{0}^{h}\frac{\omega(\varphi,t)}{t^{1+\alpha}}dt$$

here in the last inequality we made use of the fact that the function $\dfrac{\omega(\varphi,t)}{t}$ almost decreases. Further again taking this decreasing into account for A_{1} when $x-a\leq h$ we have

$$|A_{1}|\leq(x-a)^{-\alpha}\omega(\varphi,x-a)\leq C\int_{0}^{x-a}t^{-1-\alpha}\omega(\varphi,t)dt\leq C\int_{0}^{h}\frac{\omega(\varphi,t)}{t^{1+\alpha}}dt.$$

When $x-a\geq h$ the mean value theorem yields the estimate

$$|A_{1}|\leq Ch(x-a)^{-1-\alpha}\omega(\varphi,x-a)\leq C\frac{\omega(\varphi,h)}{h^{\alpha}}\leq C\int_{0}^{h}\frac{\omega(\varphi,t)}{t^{1+\alpha}}dt.$$

Gathering estimates for A_{1} and A_{2} we obtain the inequality (12).

To prove the theorem it's sufficient in view of (12) to consider only the second summand in the expression (10) for the Marchaud fractional derivative. For this function we have

$$F_{2}(x+h)-F_{2}(x)=\int_{a}^{x}\frac{\varphi(x+h)-\varphi(x)}{(x+h-t)^{1+\alpha}}dt+\int_{x-a}^{x+h-a}\frac{\varphi(x+h)-\varphi(t)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\int_{a}^{x}(f(x)-f(t))\left[(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right]dt=J_{1}+J_{2}+J_{3}.$$

If $x-a\leq h$ then $|J_{1}|=\displaystyle\int_{0}^{x-a}\frac{\varphi(x+h)-\varphi(x)}{(h+t)^{1+\alpha}}dt\leq\int_{0}^{x-a}\frac{\omega(\varphi,h)}{(h+t)^{1+\alpha}}dt\leq\int_{0}^{h}\frac{\omega(\varphi,h)}{(h+t)^{1+\alpha}}dt\leq C\frac{\omega(\varphi,h)}{h^{\alpha}}.$

If $x-a\geq h$ we have

$$|J_{1}|\leq\int_{0}^{x-a}\frac{\omega(\varphi,h)}{(h+t)^{1+\alpha}}dt\leq\int_{0}^{h}\frac{\omega(\varphi,h)}{(h+t)^{1+\alpha}}dt+\int_{h}^{x-a}\frac{\omega(\varphi,h)}{(h+t)^{1+\alpha}}dt\leq C_{1}\frac{\omega(\varphi,h)}{h^{\alpha}}+\int_{h}^{\infty}\frac{\omega(\varphi,h)}{(h+t)^{1+\alpha}}dt\leq C_{2}\frac{\omega(\varphi,h)}{h^{\alpha}}.$$

As for J_{2} we have $|J_{2}|\leq\displaystyle\int_{x-a}^{x+h-a}(x+h-t)^{-1-\alpha}\omega(\varphi,x+h-t)dt\leq C\int_{x-a}^{x+h-a}t^{-1-\alpha}\omega(\varphi,t)dt$. If $x-a\leq h$ then

$$|J_{2}|\leq C\int_{x-a}^{2h}t^{-1-\alpha}\omega(\varphi,t)dt\leq C\int_{0}^{2h}t^{-1-\alpha}\omega(\varphi,t)dt\leq C_{1}\int_{0}^{h}t^{-1-\alpha}\omega(\varphi,t)dt.$$

If $x-a\geq h$, after the substitution $t=\xi+x-a$ and taking into account the (almost) decreasing nature of the function $t^{-1}\omega(\varphi,t)$, we have

$$|J_2| \le \int_0^h \frac{\omega(\varphi, x-a+\xi)}{(x-a+\xi)^{1+\alpha}} d\xi \le C \frac{\omega(\varphi,h)}{h} \int_0^h \frac{d\xi}{(x-a+\xi)^{\alpha}} \le C \frac{\omega(\varphi,h)}{h^{\alpha}}.$$

Now let's estimate J_3. If $x-a \le h$ we have

$$|J_3| \le \int_0^h \omega(\varphi,t) \frac{(t+h)^{1+\alpha} - t^{1+\alpha}}{t^{1+\alpha}(t+h)^{1+\alpha}} dt \le C \int_0^h \frac{\omega(\varphi,t)}{t^{1+\alpha}} \frac{hdt}{t+h} \le C \int_0^h \frac{\omega(\varphi,t)}{t^{1+\alpha}} dt.$$

If $x-a \ge h$:

$$|J_3| \le \int_0^h |\varphi(x) - \varphi(x-t)| \left| (h+t)^{-\alpha-1} - t^{-1-\alpha} \right| dt + \int_h^{x-a} |\varphi(x) - \varphi(x-t)| \left| (h+t)^{-\alpha-1} - t^{-1-\alpha} \right| dt \le$$

$$\le C_1 \int_0^h \frac{\omega(\varphi,t)}{t^{1+\alpha}} dt + C_2 h \int_h^{x-a} \frac{\omega(\varphi,t)}{t^{2+\alpha}} dt \le C_1 \int_0^h \frac{\omega(\varphi,t)}{t^{1+\alpha}} dt + C_2 h \int_h^{\infty} \frac{\omega(\varphi,h)}{t^{2+\alpha}} dt \le C \left[\int_0^h \frac{\omega(\varphi,t)}{t^{1+\alpha}} dt + \frac{\omega(\varphi,h)}{h^{\alpha}} \right].$$

Gathering estimates for J_1, J_2, J_3 we arrive at (11). The theorem is this proved.

3. Main result

We consider the mixed fractional Riemann-Liouville integral

$$\left(I^{\alpha,\beta}_{a+,c+}\varphi\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y \frac{\varphi(t,s)dtds}{(x-t)^{1-\alpha}(y-s)^{1-\beta}}, \quad x>a,\ y>c,\ 0<\alpha,\beta<1 \tag{13}$$

Theorem 3. Let $\varphi(x,y) \in C(Q)$ and $\varphi(x,y)\big|_{x=a,y=c} = 0$. Then for the mixed fractional integral (13) we have estimates of the Zygmund type

$$\overset{1,1}{\omega}\left(I^{\alpha,\beta}_{a+,c+}\varphi; h,\eta\right) \le C_1 h\eta \int_h^{b-a} \int_\eta^{d-c} \frac{\overset{1,1}{\omega}(\varphi;t,s)}{t^{2-\alpha}s^{2-\beta}} dtds \tag{14}$$

$$\overset{1,0}{\omega}\left(I^{\alpha,\beta}_{a+,c+}\varphi; h,0\right) \le C_2 h \int_h^{b-a} t^{\alpha-2}\, \overset{1,0}{\omega}(\varphi;t,0) dt, \quad \overset{0,1}{\omega}\left(I^{\alpha,\beta}_{a+,c+}\varphi; 0,\eta\right) \le C_3 \eta \int_\eta^{d-c} s^{\beta-2}\, \overset{0,1}{\omega}(\varphi;0,s) ds. \tag{15}$$

We will not prove this theorem. Its proof can be found from [2], [3], [7], [13] and [14].

Theorem 4. Let $\varphi(x,y)$ be continuous on Q and $\varphi(x,y)\big|_{x=a,y=c} = 0$. Then for mixed fractional derivative $D^{\alpha,\beta}_{a+,c+}\varphi,\ 0<\alpha,\beta<1$ we have estimates of the Zygmund type

$$\overset{1,1}{\omega}\left(D^{\alpha,\beta}_{a+,c+}\varphi; h,\eta\right) \le C_1 \int_0^h \int_0^\eta \frac{\overset{1,1}{\omega}(\varphi;t,s)}{t^{1+\alpha}s^{1+\beta}} dtds \tag{16}$$

$$\overset{1,0}{\omega}\left(D^{\alpha,\beta}_{a+,c+}\varphi; h,0\right) \le C_2 \int_0^h t^{-\alpha-1}\, \overset{1,0}{\omega}(\varphi;t,0) dt, \quad \overset{0,1}{\omega}\left(D^{\alpha,\beta}_{a+,c+}\varphi; 0,\eta\right) \le C_3 \int_0^\eta s^{-\beta-1}\, \overset{0,1}{\omega}(\varphi;0,s) ds. \tag{17}$$

Proof. Using the identity (5), we represent the derivative (1)

$$\left(D^{\alpha,\beta}_{a+,c+}\varphi\right)(x,y) = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \left[\frac{\varphi(a,c)}{(x-a)^{\alpha}(y-c)^{\beta}} + \frac{\psi_1(x)}{(y-c)^{\beta}} + \frac{\psi_2(y)}{(x-a)^{\alpha}} + \psi(x,y) \right]$$

where

$$\psi_1(x) = \frac{\varphi(x,c) - \varphi(a,c)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{\varphi(x,c) - \varphi(t,c)}{(x-t)^{\alpha+1}} dt, \quad \psi_2(y) = \frac{\varphi(a,y) - \varphi(a,c)}{(y-c)^{\beta}} + \beta \int_c^y \frac{\varphi(a,y) - \varphi(a,s)}{(y-s)^{1+\beta}} ds,$$

$$\psi(x,y) = \frac{\left(\overset{1,1}{\Delta}_{x-a,y-c}\,\varphi\right)(a,c)}{(x-a)^{\alpha}(y-c)^{\beta}} + \frac{\alpha}{(y-c)^{\beta}} \int_a^x \frac{\left(\overset{1,1}{\Delta}_{x-t,y-c}\,\varphi\right)(t,c)}{(x-t)^{1+\alpha}} dt + \frac{\beta}{(x-a)^{\alpha}} \int_c^y \frac{\left(\overset{1,1}{\Delta}_{x-a,y-s}\,\varphi\right)(a,s)ds}{(y-s)^{1+\beta}} +$$

$$+\alpha\beta\int_a^x\int_c^y \frac{\left(\overset{1,1}{\Delta}_{x-t,y-s}\varphi\right)(t,s)\,dtds}{(x-t)^{1+\alpha}(y-s)^{1+\beta}}.$$

According to the theorem $\varphi(x,y)\big|_{x=a,y=c}=0$. Then $\psi_1(x)=0$ and $\psi_2(y)=0$. We have

$$\left(D^{\alpha,\beta}_{a+,c+}\varphi\right)(x,y)=\frac{\psi(x,y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}=f(x,y)=f_1(x,y)+f_2(x,y)+f_3(x,y)+f_4(x,y).$$

We estimate each term separately.

Let $h>0, x, x+h\in[a,b]$. We consider the differences:

$$|f_1(x+h,y)-f_1(x,y)|\le\frac{\left|\left(\overset{1,1}{\Delta}_{h,y-c}\varphi\right)(a,c)\right|}{(y-c)^{\beta}(x+h-a)^{\alpha}}+\frac{\left|\left(\overset{1,1}{\Delta}_{h,y-c}\varphi\right)(a,c)\right|}{(y-c)^{\beta}}\left|(x+h-a)^{\alpha}-(x-a)^{\alpha}\right|\le$$

$$\le C_1\left(\frac{\overset{1,1}{\omega}(\varphi;h,y-c)}{(y-c)^{\beta}(x+h-a)^{\alpha}}+\frac{\overset{1,1}{\omega}(\varphi;h,y-c)}{(y-c)^{\beta}}\left|(x+h-a)^{\alpha}-(x-a)^{\alpha}\right|\right),$$

$$|f_2(x+h,y)-f_2(x,y)|\le\int_a^x\frac{\left|\left(\overset{1,1}{\Delta}_{h,y-s}\varphi\right)(x,c)\right|dt}{(x+h-t)^{1+\alpha}(y-c)^{\beta}}+\int_{x-a}^{x+h-a}\frac{\left|\left(\overset{1,1}{\Delta}_{x+h-t,y-s}\varphi\right)(t,c)\right|}{(x+h-t)^{1+\alpha}(y-c)^{\beta}}dt+$$

$$+(y-c)^{-\beta}\int_a^x\left|\left(\overset{1,1}{\Delta}_{x-t,y-s}\varphi\right)(t,c)\right|\left|(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right|dt\le\frac{C_2}{(y-c)^{\beta}}\left(\int_a^x\frac{\overset{1,1}{\omega}(\varphi;h,y-c)}{(x+h-t)^{1+\alpha}}dt+\right.$$

$$\left.+\int_{x-a}^{x+h-a}\frac{\overset{1,1}{\omega}(\varphi;x+h-t,y-c)}{(x+h-t)^{1+\alpha}}dt+\int_a^x\overset{1,1}{\omega}(\varphi;x-t,y-c)\left|(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right|dt\right),$$

$$|f_3(x+h,y)-f_3(x,y)|\le\int_c^y\frac{\left|\left(\overset{1,1}{\Delta}_{h,y-s}\varphi\right)(a,s)\right|ds}{(x+h-a)^{\alpha}(y-s)^{1+\beta}}+\left|(x+h-a)^{-\alpha}-(x-a)^{-\alpha}\right|\left|\int_c^y\frac{\left|\left(\overset{1,1}{\Delta}_{x-a,y-s}\varphi\right)(a,s)\right|ds}{(y-s)^{1+\beta}}\right|\le$$

$$\le C_3\left((x+h-a)^{-\alpha}\int_c^y\frac{\overset{1,1}{\omega}(\varphi;h,y-s)ds}{(y-s)^{1+\beta}}+\left|(x+h-a)^{-\alpha}-(x-a)^{-\alpha}\right|\left|\int_c^y\frac{\overset{1,1}{\omega}(\varphi;x-a,y-s)ds}{(y-s)^{1+\beta}}\right|\right),$$

$$|f_4(x+h,y)-f_4(x,y)|\le\int_c^y\int_a^x\frac{\left|\left(\overset{1,1}{\Delta}_{h,y-s}\varphi\right)(x,s)\right|dtds}{(x+h-t)^{1+\alpha}(y-s)^{1+\beta}}+\int_c^y\int_{x-a}^{x+h-a}\frac{\left|\left(\overset{1,1}{\Delta}_{x+h-t,y-s}\varphi\right)(t,s)\right|}{(x+h-t)^{1+\alpha}(y-s)^{1+\beta}}dtds+$$

$$+\int_c^y\int_a^x\frac{\left|\left(\overset{1,1}{\Delta}_{x-t,y-s}\varphi\right)(t,s)\right|}{(y-s)^{1+\beta}}\left|(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right|dtds\le C_4\left(\int_c^y\int_a^x\frac{\overset{1,1}{\omega}(\varphi;h,y-s)dtds}{(x+h-t)^{1+\alpha}(y-c)^{1+\beta}}+\right.$$

$$\left.+\int_c^y\int_{x-a}^{x+h-a}\frac{\overset{1,1}{\omega}(\varphi;x+h-t,y-s)}{(x+h-t)^{1+\alpha}(y-c)^{1+\beta}}dtds+\int_c^y\int_a^x\frac{\overset{1,1}{\omega}(\varphi;x-t,y-s)}{(y-s)^{1+\beta}}\left|(x+h-t)^{-1-\alpha}-(x-t)^{-1-\alpha}\right|dt\right).$$

Using estimations A_1, A_2, J_1, J_2, J_3 of the proof of Theorem 2 and inequalities (2), (6), it's easily possible to receive an estimation

$$\overset{1,0}{\omega}(f;h,0)\le C_2\int_0^h t^{-\alpha-1}\overset{1,0}{\omega}(\varphi;t,0)dt$$

The estimate

$$\overset{0,1}{\omega}(f;0,\eta)\le C_3\int_0^\eta s^{-\beta-1}\overset{0,1}{\omega}(\varphi;0,s)ds$$

is symmetrical obtained.

Let $h,\eta >;\ x,x+h\in[a,b],\ y,y+\eta\in[c,\ d]$. We consider the differences

$$\left(\overset{1,1}{\Delta}_{h,\eta} f_1\right)(x,y)=\frac{\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)}{(x+h-a)^\alpha(y+\eta-c)^\beta}+\frac{\left(\overset{1,1}{\Delta}_{h,y-c}\varphi\right)(x,c)}{(x+h-a)^\alpha}\left[\frac{1}{(y-c)^\beta}-\frac{1}{(y+\eta-c)^\beta}\right]+$$

$$+\frac{\left(\overset{1,1}{\Delta}_{x-a,\eta}\varphi\right)(a,y)}{(y+\eta-c)^\beta}\left[\frac{1}{(x-a)^\alpha}-\frac{1}{(x+h-a)^\alpha}\right]+$$

$$+\left(\overset{1,1}{\Delta}_{x-a,y-c}\varphi\right)(a,c)\left[\frac{1}{(x-a)^\alpha}-\frac{1}{(x+h-a)^\alpha}\right]\left[\frac{1}{(y-c)^\beta}-\frac{1}{(y+\eta-c)^\beta}\right]$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} f_2\right)(x,y)=\frac{\beta}{(x+h-a)^\alpha}\int_{y-c}^{y+\eta-c}\frac{\left(\overset{1,1}{\Delta}_{h,\ y+\eta-s}\varphi\right)(x,s)}{(y+\eta-s)^{1+\beta}}ds+\frac{\beta}{(x+h-a)^\alpha}\int_0^y\frac{\left(\overset{1,1}{\Delta}_{h,\ \eta}\varphi\right)(x,y)}{(y+\eta-s)^{1+\beta}}ds+$$

$$+\beta\left[\frac{1}{(x-a)^\alpha}-\frac{1}{(x+h-a)^\alpha}\right]\int_{y-c}^{y+\eta-c}\frac{\left(\overset{1,1}{\Delta}_{x-a,\ y+\eta-s}\varphi\right)(a,s)}{(y+\eta-s)^{1+\beta}}ds+$$

$$+\frac{\beta}{(x+h-a)^\alpha}\int_c^y\left(\overset{1,1}{\Delta}_{h,\ y-s}\varphi\right)(x,s)\left[\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right]ds+$$

$$+\beta\left[\frac{1}{(x-a)^\alpha}-\frac{1}{(x+h-a)^\alpha}\right]\int_c^y\frac{\left(\overset{1,1}{\Delta}_{x-a,\eta}\varphi\right)(a,y)}{(y+\eta-s)^{1+\beta}}ds+$$

$$+\beta\left[\frac{1}{(x-a)^\alpha}-\frac{1}{(x+h-a)^\alpha}\right]\int_c^y\left(\overset{1,1}{\Delta}_{x-a,y-s}\varphi\right)(a,s)\left[\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right]ds$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} f_3\right)(x,y)=\frac{\alpha}{(y+\eta-c)^\beta}\int_{x-a}^{x+h-a}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\eta}\varphi\right)(t,y)}{(x+h-t)^{1+\alpha}}dt+\frac{\alpha}{(y+\eta-c)^\beta}\int_a^x\frac{\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\alpha\left[\frac{1}{(y-c)^\beta}-\frac{1}{(y+\eta-c)^\beta}\right]\int_{x-a}^{x+h-a}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y}\varphi\right)(t,c)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\frac{\alpha}{(y+\eta-c)^\beta}\int_a^x\left(\overset{1,1}{\Delta}_{x-t,\eta}\varphi\right)(t,y)\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dt+$$

$$+\alpha\left[\frac{1}{(y-c)^\beta}-\frac{1}{(y+\eta-c)^\beta}\right]\int_a^x\frac{\left(\overset{1,1}{\Delta}_{h,y-c}\varphi\right)(x,c)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\alpha\left[\frac{1}{(y-c)^\beta}-\frac{1}{(y+\eta-c)^\beta}\right]\int_a^x\left(\overset{1,1}{\Delta}_{x-t,y-c}\varphi\right)(t,c)\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dt$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} f_4\right)(x,y)=\int_a^x\int_c^y \frac{\left(\overset{1,1}{\Delta}_{h,\eta}\varphi\right)(x,y)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+\int_a^x\int_{y-c}^{y+\eta-c}\frac{\left(\overset{1,1}{\Delta}_{h,y+\eta-s}\varphi\right)(x,s)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+$$

$$+\int_a^x\int_c^y\frac{\left(\overset{1,1}{\Delta}_{h,y-s}\varphi\right)(x,s)}{(x+h-t)^{1+\alpha}}\left[\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right]dtds+\int_{x-a}^{x+h-a}\int_c^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\eta}\varphi\right)(t,y)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+$$

$$+\int_{x-a}^{x+h-a}\int_{y-c}^{y+\eta-c}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y+\eta-s}\varphi\right)(t,s)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+\int_{x-a}^{x+h-a}\int_c^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,y-s}\varphi\right)(t,s)}{(x+h-t)^{1+\alpha}}\left[\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right]dtds+$$

$$+\int_a^x\int_c^y\frac{\left(\overset{1,1}{\Delta}_{x-t,\eta}\varphi\right)(t,y)}{(y+\eta-s)^{1+\beta}}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dtds+$$

$$+\int_a^x\int_{y-c}^{y+\eta-c}\frac{\left(\overset{1,1}{\Delta}_{x-t,y+\eta-s}\varphi\right)(t,s)}{(y+\eta-s)^{1+\beta}}\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]dtds+$$

$$+\int_a^x\int_c^y\left(\overset{1,1}{\Delta}_{x-t,y-s}\varphi\right)(t,s)\left[\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right]\left[\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right]dtds.$$

The validity of these representations may be checked directly. We have

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f_1\right)(x,y)\right|\le C_1\left(\frac{\overset{1,1}{\omega}(\varphi;h,\eta)}{(x+h-a)^{\alpha}(y+\eta-c)^{\beta}}+\frac{\overset{1,1}{\omega}(\varphi;h,y-c)}{(x+h-a)^{\alpha}}\left|\frac{1}{(y-c)^{\beta}}-\frac{1}{(y+\eta-c)^{\beta}}\right|+\right.$$

$$+\frac{\overset{1,1}{\omega}(\varphi;x-a,\eta)}{(y+\eta-c)^{\beta}}\left|\frac{1}{(x-a)^{\alpha}}-\frac{1}{(x+h-a)^{\alpha}}\right|+$$

$$\left.+\overset{1,1}{\omega}(\varphi;x-a,y-c)\left|\frac{1}{(x-a)^{\alpha}}-\frac{1}{(x+h-a)^{\alpha}}\right|\left|\frac{1}{(y-c)^{\beta}}-\frac{1}{(y+\eta-c)^{\beta}}\right|\right)$$

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f_2\right)(x,y)\right|\le C_2\left(\frac{1}{(x+h-a)^{\alpha}}\int_{y-c}^{y+\eta-c}\frac{\overset{1,1}{\omega}(\varphi;h,y+\eta-s)}{(y+\eta-s)^{1+\beta}}ds+\frac{1}{(x+h-a)^{\alpha}}\int_0^y\frac{\overset{1,1}{\omega}(\varphi;h,\eta)}{(y+\eta-s)^{1+\beta}}ds+\right.$$

$$+\left|\frac{1}{(x-a)^{\alpha}}-\frac{1}{(x+h-a)^{\alpha}}\right|\int_{y-c}^{y+\eta-c}\frac{\overset{1,1}{\omega}(\varphi;x-a,y+\eta-s)}{(y+\eta-s)^{1+\beta}}ds+$$

$$+\frac{1}{(x+h-a)^{\alpha}}\int_c^y\overset{1,1}{\omega}(\varphi;h,y-s)\left|\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right|ds+$$

$$+\left|\frac{1}{(x-a)^{\alpha}}-\frac{1}{(x+h-a)^{\alpha}}\right|\int_c^y\frac{\overset{1,1}{\omega}(\varphi;x-a,\eta)}{(y+\eta-s)^{1+\beta}}ds+$$

$$\left.+\left|\frac{1}{(x-a)^{\alpha}}-\frac{1}{(x+h-a)^{\alpha}}\right|\int_c^y\overset{1,1}{\omega}(\varphi;x-a,y-s)\left|\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right|ds\right)$$

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f_3\right)(x,y)\right|\le C_3\left(\frac{1}{(y+\eta-c)^{\beta}}\int_{x-a}^{x+h-a}\frac{\overset{1,1}{\omega}(\varphi;x+h-t,\eta)}{(x+h-t)^{1+\alpha}}dt+\frac{1}{(y+\eta-c)^{\beta}}\int_a^x\frac{\overset{1,1}{\omega}(\varphi;h,\eta)}{(x+h-t)^{1+\alpha}}dt+\right.$$

$$+\left|\frac{1}{(y-c)^{\beta}}-\frac{1}{(y+\eta-c)^{\beta}}\right|\int_{x-a}^{x+h-a}\frac{\overset{1,1}{\omega}(\varphi;x+h-t,y-s)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\frac{1}{(y+\eta-c)^{\beta}}\int_a^x \overset{1,1}{\omega}(\varphi;x-t,\eta)\left|\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right|dt+$$

$$+\left|\frac{1}{(y-c)^{\beta}}-\frac{1}{(y+\eta-c)^{\beta}}\right|\int_a^x \frac{\overset{1,1}{\omega}(\varphi;h,y-c)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\left|\frac{1}{(y-c)^{\beta}}-\frac{1}{(y+\eta-c)^{\beta}}\right|\int_a^x \overset{1,1}{\omega}(\varphi;x-t,y-c)\left|\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right|dt \Bigg)$$

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta} f_4\right)(x,y)\right|\le C_4\Bigg(\int_a^x\int_c^y \frac{\overset{1,1}{\omega}(\varphi;h,\eta)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+\int_a^x\int_{y-c}^{y+\eta-c}\frac{\overset{1,1}{\omega}(\varphi;h,y+\eta-s)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+$$

$$+\int_a^x\int_c^y \frac{\overset{1,1}{\omega}(\varphi;h,y-s)}{(x+h-t)^{1+\alpha}}\left|\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right|dtds+\int_{x-a}^{x+h-a}\int_c^y \frac{\overset{1,1}{\omega}(\varphi;x+h-t,\eta)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+$$

$$+\int_{x-a}^{x+h-a}\int_{y-c}^{y+\eta-c}\frac{\overset{1,1}{\omega}(\varphi;x+h-t,y+\eta-s)dtds}{(x+h-t)^{1+\alpha}(y+\eta-s)^{1+\beta}}+$$

$$+\int_{x-a}^{x+h-a}\int_c^y \frac{\overset{1,1}{\omega}(\varphi;x+h-t,y-s)}{(x+h-t)^{1+\alpha}}\left|\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right|dtds+$$

$$+\int_a^x\int_c^y \frac{\overset{1,1}{\omega}(\varphi;x-t,\eta)}{(y+\eta-s)^{1+\beta}}\left|\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right|dtds+$$

$$+\int_a^x\int_{y-c}^{y+\eta-c}\frac{\overset{1,1}{\omega}(\varphi;x-t,y+\eta-s)}{(y+\eta-s)^{1+\beta}}\left|\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right|dtds+$$

$$+\int_a^x\int_c^y \overset{1,1}{\omega}(\varphi;x-t,y-s)\left|\frac{1}{(x-t)^{1+\alpha}}-\frac{1}{(x+h-t)^{1+\alpha}}\right|\left|\frac{1}{(y-s)^{1+\beta}}-\frac{1}{(y+\eta-s)^{1+\beta}}\right|dtds\Bigg).$$

After which every term is estimated in the standard way and we get

$$\overset{1,1}{\omega}(f;h,\eta)\le C_1\int_0^h\int_0^\eta \frac{\overset{1,1}{\omega}(\varphi;t,s)}{t^{1+\alpha}s^{1+\beta}}dtds$$

This completes the proof.

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