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Multiple solutions for some Neumann-Steklov boundary value problems with ψ-laplacian

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Abstract
We study the existence of multiple solutions of the quasilinear equation $(\psi(u'(t)))' = f(t,u(t),u'(t)), t \in [0,T]$ submitted to nonlinear Neumann-Steklov boundary conditions, where $\psi:]-a,a[\to \mathbb{R}$, with $0 < a < +\infty$, is an increasing homeomorphism such that $\psi(0) = 0$. Combining some sign conditions and lower and upper solutions method, we obtain existence of two or three solutions.

Keywords: $\psi-$ Laplacian, L^1 $-$Carathéodory function, nonlinear Neumann-Steklov problem, periodic problem, lower and upper-solutions, sign conditions

1. Introduction

This work is devoted to the study of the existence of solutions of the the quasilinear equation

$$\left(\psi\left(u'(t)\right)\right)' = f\left(t,u(t),u'(t)\right), \forall\, t \in [0,T] \tag{1}$$

Submitted the nonlinear Neumann-Steklov boundary conditions

$$\psi\left(u'(0)\right) = h_0\left(u(0)\right), \psi(u'(T)) = h_T(u(T)) \tag{2}$$

Where $\psi:]-a,a[\to \mathbb{R}$ with $0 < a < +\infty$, is an increasing homeomorphism such that $\psi(0) = 0$, $h_0, h_T: \mathbb{R} \to \mathbb{R}$ and $f: [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Generally, in the lower and upper solutions method, to show existence of at least one solution of a problem, we need existence of at least one lower solution and at least one upper solution. In the case of the sign conditions method, we usually need two sign conditions to show existence of at least one solution of a problem.
In 2016, Goli and Adjé [10] proved existence of solutions of (1)-(2), when there exists only one sign condition and only one lower solution or only one upper solution.

We use the results proven by Goli and Adjé [10] to show \begin{itemize}

- Existence of at least two solutions of (1)-(2), when we have only one sign condition, one strict lower solution and one strict upper solution.
- Existence of at least three solutions of (1)-(2), when we have two sign conditions, one strict lower solution and one strict upper solution.
- For some problems with Neumann-Steklov boundary conditions, the existence of two real numbers a and b such that $a > b$, $h_0(a) \leq 0 \leq h_T(a), h_T(b) \leq 0 \leq h_0(b)$, $f(t,a,0) < 0$ and $f(t,b,0) > 0$, $\forall t \in [0,T]$, allows us to affirm the existence of 2 or 3 solutions.

In section 2, we give some preliminaries results
In section 3, combining some sign conditions and existence only one strict lower solution and one strict upper solution of problem (1)-(2), we prove existence of at least two or three solutions of problem (1)-(2). We show in this section that the existence of at least two or three solutions for certain forced relativistic pendulum equations with friction and Neumann-Steklov boundary conditions is guaranteed by the presence of one strict lower solution and one strict upper solution.

2. Preliminary

Definition 2.1: A solution of problem (1)-(2) is a function $u \in C^1([0,T])$ such that $\psi(u') \in C^1([0,T])$, $||u'||_\infty < a$ and satisfies (1)-(2).

Definition 2.2: A function $\delta \in C^1([0,T])$ is a lower-solution of the problem (1)-(2) if $||\delta'||_\infty < a$, $\psi(\delta') \in C^1([0,T])$,

$$\left(\psi(\delta'(t))\right)' \geq f\left(t, \delta(t), \delta'(t)\right), t \in [0,T], \quad (3)$$

$$\psi\left(\delta'(0)\right) \geq h_0\left(\delta(0)\right) \text{ and } \psi\left(\delta'(T)\right) \leq h_T\left(\delta(T)\right) \quad (4)$$

Definition 2.3: A function $\gamma \in C^1([0,T])$ is an upper-solution of the problem (1)-(2) if $||\gamma'||_\infty < a$, $\psi(\gamma') \in C^1([0,T])$,

$$\left(\psi(\gamma'(t))\right)' \leq f\left(t, \gamma(t), \gamma'(t)\right), t \in [0,T], \quad (5)$$

$$\psi\left(\gamma'(0)\right) \leq h_0\left(\gamma(0)\right) \text{ and } \psi\left(\gamma'(T)\right) \geq h_T\left(\gamma(T)\right) \quad (6)$$

Definition 2.4: A lower-solution δ of (1)-(2) is said to be strict if every solution u of (1)-(2) with $u(t) \geq \delta(t)$ on $[0,T]$ is such that $u(t) > \delta(t)$ on $[0,T]$.

Definition 2.5: A upper-solution γ of (1)-(2) is said to be strict if every solution u of (1)-(2) with $u(t) \leq \gamma(t)$ on $[0,T]$ is such that $u(t) < \gamma(t)$ on $[0,T]$.

Remark 2.1
- A lower solution of (1)-(2) is strict if the inequality (3) is strict for all $t \in [0,T]$;
- An upper solution of (1)-(2) is strict if the inequality (5) is strict for all $t \in [0,T]$.

Theorem 2.1 Assume that
- There exists δ a lower-solution of the problem (1)-(2).
- $\exists\ R > 0$ such that

$$u_L \geq R \text{ and } \left||u'|\right|_\infty < a \Rightarrow \int_0^T f\left(t, u(t), u'(t)\right)dt - h_T(u(T)) + h_0(u(0)) > 0 \quad (7)$$

Then the problem (1)-(2) admits at least one solution u such that $\delta(t) \leq u(t)$ for all $t \in [0,T]$.

Proof. See Theorem 4.2 and its proof in [10].

Theorem 2.2: Assume that
- There exists γ an upper-solution of the problem (1)-(2).
- $\exists\ R > 0$ such that

$$u_M \leq -R \text{ and } \left||u'|\right|_\infty < a \Rightarrow \int_0^T f\left(t, u(t), u'(t)\right)dt - h_T(u(T)) + h_0(u(0)) < 0 \quad (8)$$

Then the problem (1)-(2) admits at least one solution u such that $u(t) \leq \gamma(t)$ for all $t \in [0,T]$.

Proof. See [10].

Theorem 2.3 Assume that there exist a lower-solution δ and an upper-solution γ of (1)-(2) such that

$$\exists \tilde{t} \in [0,T] \text{ such that } \delta(\tilde{t}) > \gamma(\tilde{t});$$

Then the problem (1)-(2) admits at least one solution u, such that

$$\min\{\delta(t_u), \gamma(t_u)\} \leq u(t_u) \leq \max\{\delta(t_u), \gamma(t_u)\}$$

For some $t_u \in [0,T]$ and

$$\left||u|\right|_\infty \leq \max\left\{\left||\delta|\right|_\infty, \left||\gamma|\right|_\infty\right\} + aT.$$

Proof. See Theorem 4.1 and its proof in [11].

3. Neumann-Steklov problem
3.1 Existence of at least two solutions

Theorem 3.1: Assume that

- There exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2) such that $\exists\, t \in [0,T], \delta(t) > \gamma(t)$;
- $\exists\, R > 0$ such that

$$u_L \geq R \text{ and } \left\|u'\right\|_\infty < a \Rightarrow \int_0^T f\big(t,u(t),u'(t)\big)dt - h_T(u(T)) + h_0(u(0)) > 0 \quad (9)$$

Then the problem (1)-(2) admits at least two solutions u and w such that

- $\delta(t) < u(t) \text{ for all } t \in [0,T]$.
- $\gamma(t_w) \leq u(t_w) \leq \delta(t_w)$ for some $t_w \in [0,T]$.

Proof. By Theorem 2.1 and the fact that δ is strict, the problem (1)-(2) admits at least one solution u such that

$$\delta(t) < u(t) \text{ for all } t \in [0,T] \quad (10)$$

Using the Theorem 2.3., the problem (1)-(2) admits at least one solution w such that

$$\gamma(t_w) = \min\{\delta(t_w),\gamma(t_w)\} \leq w(t_w) \leq \max\{\delta(t_w),\gamma(t_w)\} = \delta(t_w) \quad (11)$$

Using (10) and (11), we have $u \neq w$.

Theorem 3.2: Assume that:

- There exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2) such that $\exists\, t \in [0,T], \delta(t) > \gamma(t)$;
- $\exists\, R > 0$ such that

$$u_M \leq -R \text{ and } \left\|u'\right\|_\infty < a \Rightarrow \int_0^T f\big(t,u(t),u'(t)\big)dt - h_T(u(T)) + h_0(u(0)) < 0. \quad (12)$$

Then the problem (1)-(2) admits at least two solutions u and w such that

- $v(t) < \gamma(t) \text{ for all } t \in [0,T]$;
- $\gamma(t_w) \leq u(t_w) \leq \delta(t_w)$ for some $t_w \in [0,T]$.

Proof. By Theorem 2.2 and the fact that γ is strict, the problem (1)-(2) admits at least one solution v such that

$$v(t) < \gamma(t) \text{ for all } t \in [0,T] \quad (13)$$

Using the Theorem 2.3., the problem (1)-(2) admits at least one solution w such that

$$\gamma(t_w) = \min\{\delta(t_w),\gamma(t_w)\} \leq w(t_w) \leq \max\{\delta(t_w),\gamma(t_w)\} = \delta(t_w) \quad (14)$$

Using (13) and (14), we have $v \neq w$.

Corollary 3.1 Assume that

$\lim_{u\to-\infty} f(t,u,v) = -\infty \text{ or } \lim_{u\to+\infty} f(t,u,v) = +\infty$ uniformly in $\{(t,v);\ (t,v) \in\}[0,T] \times [-a,a]$;

- There exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2).
- h_0 and h_T are bounded on $\mathbb{R}$.

Then the problem (1)-(2) has at least two solutions.

Corollary 3.2. Assume that:

- $\lim_{u\to-\infty} f(t,u,v) = -\infty \text{ or } \lim_{u\to+\infty} f(t,u,v) = +\infty$ uniformly in $\{(t,v);\ (t,v) \in [0,T] \times [-a,a]\}$;
- h_0 and h_T are bounded on $\mathbb{R}$;
- There exist $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that $\delta > \gamma$ and $f(t,\delta,0) < 0$ and $f(t,\gamma,0) > 0$, $\forall\, t \in [0,T]$.

Then the problem (1)-(2) has at least two solutions.

Example 3.1 Consider the problem

$$\left(\frac{u'(t)}{\sqrt{1-\left(u''(t)\right)^2}}\right)' = t^2 + (\mathrm{u(t)})^2 + -9 + t^4(\mathrm{u'(t)} + \cos(\mathrm{u(t)})), t \in [0,1],$$

$$\frac{u'(0)}{\sqrt{1-\left(u'(0)\right)^2}} = -\tanh\big(u(0)\big) \ and \ \frac{u'(1)}{\sqrt{1-\left(u'(1)\right)^2}} = \tanh(u(1)),$$

We can take $h_0(x) = -\tanh(x)$, $h_T(x) = \tanh(x)$, $\delta = 1$ and $\gamma = -10$.

We have: h_0 and h_T are bounded on $\mathbb{R}$,

$\lim_{u\to+\infty} f(t,u,v) = +\infty$ uniformly in $\{(t,v);\ (t,v) \in [0,T]\times[-a,a]\}$,

$$f(t,\delta,0) = f(t,1,0) = t^2 + (1)^2 - 9 + t^4(0+\cos(1))$$

$$= t^2 - 8 + t^4(0+\cos(1)) < 0,$$

$$f(t,\gamma,0) = f(t,-10,0) = t^2 + (-10)^2 - 9 + t^4(0+\cos(-10))$$

$$= t^2 + 91 + t^4(0+\cos(-10)) < 0,$$

$$h_0(1) < 0 < h_T(1) \ and \ h_0(-10) > 0 > h_T(-10).$$

Using Corollary 3.2., we deduce the existence of at least two solutions.

3.2 Existence of at least three solutions

Theorem 3.3. Assume that:

- There exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2) such that

$\exists\, t \in [0,T], \delta(t) > \gamma(t);$

- $\exists\, R > 0$ such that
-

$u_L \geq R \ and \ \|u'\|_\infty < a \Rightarrow \int_0^T f\big(t,u(t),u'(t)\big)dt - h_T(u(T)) + h_0(u(0)) > 0$ and $\exists\, R_1 > 0$ such that

$$u_M \leq -R_1 \ and \ \|u'\|_\infty < a \Rightarrow \int_0^T f\big(t,u(t),u'(t)\big)dt - h_T(u(T)) + h_0(u(0)) < 0.$$

Then the problem (1)-(2) admits at least two solutions u, v and w such that

- $\delta(t) < u(t) \ for\ all\ t \in [0,T];$
- $v(t) < \gamma(t) \ for\ all\ t \in [0,T];$
- $\gamma(t_w) \leq u(t_w) \leq \delta(t_w)$ for some $t_w \in [0,T]$.

Proof. By Theorem 2.1. and the fact that δ is strict, the problem (1)-(2) admits at least one solution u such that

$$\delta(t) < u(t) \ for\ all\ t \in [0,T] \tag{15}$$

By Theorem 2.5. and the fact that γ is strict, the problem (1)-(2) admits at least one solution v such that

$$v(t) < \gamma(t) \ for\ all\ t \in [0,T] \tag{16}$$

Using the Theorem 2.3., the problem (1)-(2) admits at least one solution w such that

$$\gamma(t_w) = \min\{\delta(t_w),\gamma(t_w)\} \leq w(t_w) \leq \max\{\delta(t_w),\gamma(t_w)\} = \delta(t_w) \tag{17}$$

Using (15), (16) and (17), we have $u \neq v$, $u \neq w$ and $v \neq w$.

Corollary 3.3 Assume that:

- $\lim_{u\to-\infty} f(t,u,v) = -\infty \ and \ \lim_{u\to+\infty} f(t,u,v) = +\infty$ uniformly in $\{(t,v);\ (t,v) \in\}[0,T]\times[-a,a];$
- h_0 and h_T are bounded on $\mathbb{R}$;
- There exist a strict lower-solution δ and a strict upper-solution γ of (1)-(2).

Then the problem (1)-(3) has at least three solutions.

Corollary 3.4 Assume that

- $\lim_{u\to-\infty} f(t,u,v) = -\infty$ *and* $\lim_{u\to+\infty} f(t,u,v) = +\infty$ uniformly in $\{(t,v);\ (t,v) \in [0,T] \times [-a,a]\}$;
- h_0 and h_T are bounded on $\mathbb{R}$;
- There exist $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that $\delta > \gamma$ and $f(t,\delta,0) < 0$ and $f(t,\gamma,0) > 0$, $\forall\, t \in [0,T]$.

Then the problem (1)-(2) has at least three solutions.

Example 3.2

$$\left(\frac{u'(t)}{\sqrt{1-(u''(t))^2}}\right)' = \frac{t}{3} + (\mathrm{u(t)})^3 - 12u(t) - 1 + \sin t^4\,(\mathrm{u'(t)} + \arctan(\mathrm{u(t)})), t \in [0,1],$$

$$\frac{u'(0)}{\sqrt{1-(u'(0))^2}} = -\arctan\left(\frac{1}{2}u(0)\right) \textit{ and } \frac{u'(1)}{\sqrt{1-(u'(1))^2}} = \arctan\left(\frac{1}{2}u(1)\right),$$

We can take $\delta = 2$ and $\gamma = -2$.

We have,

h_0 and h_T are bounded on $\mathbb{R}$,

$\lim_{u\to-\infty} f(t,u,v) = -\infty$ *and* $\lim_{u\to+\infty} f(t,u,v) = +\infty$ uniformly in $\{(t,v);\ (t,v) \in [0,T] \times [-a,a]\}$,

$$f(t,\delta,0) = f(t,2,0) = \frac{t}{3} - 17 + \sin t^4\,(0 + \arctan(2)) < 0,$$

$$f(t,\gamma,0) = f(t,-2,0) = \frac{t}{3} + 15 + \sin t^4\,(0 + \arctan(-2)) > 0,$$

$$h_0(2) = -\frac{\pi}{4} < 0 < \frac{\pi}{4} = h_T(2) \textit{ and } h_0(-2) = \frac{\pi}{4} > 0 > -\frac{\pi}{4} = h_T(-2).$$

Using Corollary 3.4., we deduce the existence of at least three solutions

4. Conclusion

In this work, we investigated the existence of multiple solutions for a class of quasilinear differential equations involving the ψ-Laplacian under nonlinear Neumann-Steklov boundary conditions. By combining sign conditions with the lower and upper solutions method, we established sufficient criteria for the existence of at least two or three solutions without requiring multiple lower or upper solutions. The results extend and refine earlier studies by showing that even a single strict lower and upper solution can guarantee multiplicity. These findings are practically relevant for nonlinear models arising in physics and mechanics, including relativistic pendulum-type equations. Future research may focus on uniqueness, stability analysis, and extensions to higher-dimensional or time-dependent problems.

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