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Finite-time blow-up of parabolic PDE with singular coefficients, method of the fundamental solutions

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Abstract
This article is dedicated to the nonlinear second-order partial differential equations of parabolic type with u^p - perturbation, we establish conditions on the nonlinear perturbation of the parabolic operator under which the solutions of initial value problems do not exist for all time that is the solutions blow up.

Keywords: Quasilinear parabolic equation, blow up, fundamental solution, evolution equation

Introduction

In Euclidean space $R^l, l>2$, let us consider second-order parabolic partial differential equation in the form

$$\frac{\partial}{\partial t}u=\left[\sum_{k,j=1,\dots,l}\nabla_k a_{kj}(t,x)\nabla_j-\sum_{k=1,\dots,l}b_k(t,x)\nabla_k\right]u+u^p \tag{1}$$

With the initial condition

$$u(0,x)=u_0(x),$$

Where $u(t,x)$; $(t,x)\in[0,\infty)\times R^l, l>2$ is unknown function [1-4, 7-11, 16]. We are assuming that $a_{kj}(t,x)$ is a measurable symmetric uniformly elliptic matrix of $l\times l$ dimension so that there are $\nu,\mu:\ 0<\nu\leq\mu<\infty$ such that the condition

$$\nu\sum_{i=1}^{l}\xi_i^2\leq\sum_{ij=1,\dots,l}a_{ij}(t,x)\xi_i\xi_j\leq\mu\sum_{i=1}^{l}\xi_i^2$$

Holds for all $x\in R^l, l\geq 3$. The function $b(t,x):[0,\infty)\times R^l$ a $R^l, l\geq 3$ is linear perturbation and $1<p<1+\frac{2}{l}, l\geq 3$. Here module of coefficients $|b|$ belongs to N_2 [1, 7, 9, 10, 16]. The best-known results for equations of this type can be found in the article [11], where the perturbation has to satisfy conditions for the existence of a solution that has been formulated in [10] by S.I. Kametaka and O. A. Oleinik on page 588, namely that sum

$$\sum_{k=1,\dots,l}|b_k(t,x)|\in L^\infty$$

Or in other words coefficients must be bounded functions or more general belong to L^p functional classes, in the presented work perturbation can belong to class Nash N_2, which is wider than L^p.

Since [10, 11] substantial progress has been made in the understanding of the classes of perturbations of parabolic operators, especially in the case of linear equations with time-independent coefficients [1, 7]. In this article, we consider the equation (1) under wider conditions on the perturbation than [10, 11], for example, the function b can be such that $b^2 \notin L^{1+\varepsilon}_{loc}(R^l), \quad l > 2$ [1]. To show the boundary of the new theory, let us consider the following example [1, 16]:

$$\frac{\partial}{\partial t}u = \sum_{k,j=1,\dots,l} \nabla_k a_{kj}(x)\nabla_j u - \sum_{k=1,\dots,l} b_k(t,x)\nabla_k u$$

Under the following condition on linear perturbation

$$\int_{R_+} \left\langle b(t,\cdot)\cdot a^{-1}(t,\cdot)\cdot b(t,\cdot)\left|\varphi(t,\cdot)\right|^2 \right\rangle dt \le$$
$$\le C\int_{R_+} \left\langle a(t,\cdot)\cdot\nabla\varphi(t,\cdot),\nabla\varphi(t,\cdot)\right\rangle dt + M\int_{R_+}\left\langle \varphi(t,\cdot),\varphi(t,\cdot)\right\rangle dt,$$

Where there are constants $C < 4$ and $M < \infty$. For instance, b can be a vector function that satisfies the following condition

$$\sum_{k=1,\dots,l} b_k^2(t,x) \le \nu^2 C\left(\frac{l-2}{2}\right)^2 \frac{1}{|x|^2} + M\frac{1}{|t|}\left(\ln\left(e+\frac{1}{|t|}\right)\right)^{-\frac{3}{2}}.$$

Let the matrix a_{ij} be diagonal then the differential operator be

$$-\Delta + b\cdot\nabla, \text{ here } b = \frac{l-2}{2}\sqrt{\beta}\frac{x}{|x|^2}, \ 0<\beta<4.$$

These conditions are more general than the sufficient conditions that were formulated in [10] by S.I. Kametaka OA. Oleinik, and in [11] by V. V. Chistyakov.
Since the results of V. V. Chistyakov are founded on classical results of S. I. Pokhozhaev [8] and S.I. Kametaka O. A. Oleinik [10, 11], let us consider the simplest example in which the sufficient conditions [11] are not satisfied, however, the solution exists. Let us consider the elliptic equation

$$a \circ d^2 u \equiv \sum_{i,j=1}^{l} a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u = 0,$$

Where the matrix a is $a_{ij} = \delta_{ij} + b\frac{x_i x_j}{|x|^2}, \quad b = -1 + \frac{l-1}{1-\chi}, \quad \chi < 1, \quad l \ge 3$ [1, 16].

We calculate

$$\nabla a = b(l-1)\frac{x}{|x|^2}, \ \left(a_{ij}\right)^{-1} = \delta_{ij} - \frac{b}{b+1}\frac{x_i x_j}{|x|^2}, \ \nabla a \circ a^{-1} \circ \nabla a = (1+b)^{-1}\left(\frac{l-1}{|x|}\right)^2$$

Then we obtain inequality with the best constant

$$\left\langle \nabla\varphi \circ a \circ \nabla\varphi \right\rangle \ge (1+b)\frac{l-2}{2}\left\|\frac{\varphi}{|x|}\right\|_2^2 \quad \forall\varphi \in W_1^2(R^l), l \ge 3$$

So, if $\beta = 4\left(1+\frac{\chi}{l-2}\right)^2$ then $\nabla a \circ a^{-1} \circ \nabla a \in PK_\beta(A)$ with the constant $c(\beta) = 0$, for $\beta < 4$ it is necessary $\chi \in (-2(l-2), 0)$, where we denote the functional class $PK_\beta(A)$ according to formula

$$PK_\beta(A)=\left\{ f\in L^1_{loc}(R^l,d^lx):\left|\langle h\ f\ h\rangle\right|\le\beta\langle\nabla h\circ a\circ\nabla h\rangle+c(\beta)\|h\|_2^2\ \forall h\in C_0^\infty\right\}$$

Where $\beta>0$, $c(\beta)\in R^1$.We assume that the initial condition is $u(|x|=1)=1$. As the solutions, we can consider two functions: the first is $u\equiv 1$ - tautological constant and the second is $u=|x|^\chi$ [16]. If parameter $\chi=-\frac{l-2}{s}$ then $\beta=4\left(1+\frac{\chi}{l-2}\right)^2$ and $\beta\le 4$ for $p>s$ in the ball $K_1(0)$ function $u=|x|^\chi\in L^p(K_1(0))$ on another hand must hold the following estimation

$$\left\|\exp(-t\Lambda_p)\right\|_{p\to s}\le C\exp\left(\frac{c(\beta)t}{\sqrt{\beta}}\right)t^{\frac{-(s-p)l}{2ps}},\ \frac{2}{2-\sqrt{\beta}}<p<s\le\infty$$

Where semigroup $\exp(-t\Lambda_p)$ is generated by a linear operator

$\Lambda_p=A+b(l-1)\frac{x}{|x|^2}g\nabla$. That means $|x|^\chi\in L^{\frac{pl}{l-2}}(K_1(0))$

But it is impossible because $|x|^\chi\notin L_{loc}^{\frac{pl}{l-2}}K_1(0)$ so the function $|x|^\chi$ cannot be a solution and there is only one trivial solution [16]. If $\beta>4$ then the equation $a\circ d^2u=0$ always has two bounded solutions. Parallel with this equation, we can consider a Cauchy problem for a parabolic equation with the same differential operator. Let us assume that the linear operator $-\Lambda_p\supset\nabla a\nabla-b\nabla$ defines over $D(A_p)$ generates holomorphic semigroup in

$L^p\left(R^l,d^lx\right)$-space. Let $b\circ a^{-1}\circ b\in PK_\beta(A)$, we denote $b_n=\chi_n b$

Where χ_n is an indicator of?

$\left\{x\in R^l:\ \left(b\circ a^{-1}\circ b\right)(x)\le n\right\}$ and $\lim\limits_{n\to\infty}\exp(-t\Lambda_p(b_n))=\exp(-t\Lambda_p(b))$

Uniformly at $t\in[0,1]$. If $\beta<1$, $p\in\left[\frac{2}{2-\sqrt{\beta}},\infty\right[$ then there is C_0 - contraction semi group, which is generated by the operator $A+b\nabla$ and the estimates

$$\left\|\exp(-t\Lambda_p)\right\|_{p\to p}\le\exp\left(\frac{c(\beta)t}{p-1}\right),$$

$$\left\|\exp(-t\Lambda_p)\right\|_{p\to s}\le C\exp\left(\frac{c(\beta)t}{\sqrt{\beta}}\right)t^{\frac{-(s-p)l}{2ps}},\ \frac{2}{2-\sqrt{\beta}}<p<s\le\infty$$

Hold for $1\le\beta<4$, $p<s\in\left[\frac{2}{2-\sqrt{\beta}},\infty\right[$, operator sum $A+b\nabla$ cannot be defined correctly, however, semigroup exists and can be defined as a limit $\exp(-t\Lambda_p(b))\equiv\lim\limits_{n\to\infty}\exp(-t\Lambda_p(b_n)),\quad t\ge 0$ in this case it is a definition of the semigroup [16].
So, new results in the theory of semigroups bring about a new understanding of the conditions under which we consider partial differential equations of the parabolic type. Thus, arose a problem of rectification of the conditions on the coefficients of PDE under which these equations will have a solution in certain functional spaces on more general than in [11]. Let us consider a parabolic equation with the Laplace operator and with the critical case of perturbation u^p, in the form

$$\frac{\partial}{\partial t}u=\Delta u+u^p.$$

If $p > 1 + \frac{2}{l}$ and an initial value $u_0 \neq 0, u_0 \geq 0$ is sufficiently small then there is a bounded solution on any bounded interval of time; if $1 < p < 1 + \frac{2}{l}$ then for any initial value $u_0(x)$ the solution blows up on bounded time interval i.e. there is a moment of time $\tilde{t}$ such that $u(t, x) \xrightarrow{t \to \tilde{t}} \infty$, the global solution does not exist for a time $\tilde{t}$ [11].
The main result of this article can be formulated as follows.
The solution $u(t,x)$ to the Cauchy problem for the equation (1) with the initial condition $u(0,x) = u_0 \neq 0, \quad u_0 \geq 0$ blows up, when $1 < p \leq \frac{2+l}{l}$ and $|b| \in PK(\beta)$

2. The parabolic Laplace equation with u^p perturbation

In this item, we are going to illustrate methods of the perturbation theory on the example of a parabolic equation with operator Laplace. Let us consider the following Cauchy problem

$$\frac{\partial}{\partial t} u = a\,\Delta u + u^p, \qquad u(0,x) = u_0(x) \tag{2}$$

The fundamental solution of $\frac{\partial}{\partial t} u = a\Delta u$ for $t > 0,\ x \in R^l$ is

$$p(t,x) = (4\pi a t)^{-\frac{l}{2}} \exp\left(-\frac{|x|^2}{4\pi a t} \right) \text{for all } t > 0,\ x \in R^l.$$

It is easy to show that the integral equation

$$u(t,x) = \langle p(t, x - \cdot) u_0(\cdot) \rangle + \int_0^t \langle p(t-s, x-\cdot) u^p(s,\cdot) \rangle ds$$

Is equivalent to the Cauchy problem for the differential equation in the sense that every solution to the integral equation is also a solution to the differential equation, and, vice versa, every solution to the differential equation is also a solution to the integral equation?

Theorem. Assuming that $u_0 \neq 0, u_0 \geq 0$, $u \in L^1(R^l)$ and $1 < p \leq \frac{2+l}{l}$, any nonnegative solution $u(t,x)$, to (2) blows up, i.e., there is a moment of time $t_0 > 0$ such that $u(t,x) = \infty, \quad t \geq t_0,\ x \in R^l$. Proof. Indeed, let $u(t,x); \quad (t,x) \in [0,\infty) \times R^l$ be the nonnegative weak solution then we have an equality

$$u(t,x) = \langle p(t, x - \cdot) u_0(\cdot) \rangle + \int_0^t \langle p(t-s, x-\cdot) u^p(s,\cdot) \rangle ds \qquad \forall (t,x) \in [0,\infty) \times R^l.$$

We are choosing $t_0 > 0$ such that $p(t_0, 0) \leq 1$ and we obtain

$$u(t+t_0,x) = \langle p(t, x - \cdot) u_0(t_0,\cdot) \rangle + \int_0^t \langle p(t-s, x-\cdot) u^p(s+t_0,\cdot) \rangle ds$$

Then

$$u(t+t_0,x) \geq C\, p(t+t_0,x) + \int_0^t \langle p(t-s, x-\cdot) u^p(s+t_0,\cdot) \rangle ds$$ And

$$u(t+t_0,x)\ge C\,p\big(t+\tau,x\big)+\int_0^t\Big\langle p\big(t-s,x-\cdot\big)u^p(s+t_0,\cdot)\Big\rangle ds\,,$$

Thus it is enough to show that the solution

$$u(t,x)=C\,p\big(t+\tau,x\big)+\int_0^t\Big\langle p\big(t-s,x-\cdot\big)u^p(s,\cdot)\Big\rangle ds$$

«blows up».

We multiply the last equality by $p\big(t,x\big)$ and integrate over all space

$$\Big\langle p\big(t,x\big)u(t,x)\Big\rangle\ge C\,p\big(2t+\tau,0\big)+\int_0^t\Big\langle p\big(2t-s,\cdot\big)u^p(s,\cdot)\Big\rangle ds\ge$$

$$\ge C\big(4\pi a\big)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi a}\right)\big(2t+\tau\big)^{-\frac{l}{2}}+\int_0^t\left(\frac{s}{2t-s}\right)^{\frac{l}{2}}\Big\langle p\big(s,\cdot\big)\big(u(s,\cdot)\big)^p\Big\rangle ds.$$

$$\ge C\big(4\pi a\big)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi a}\right)\big(2t+\tau\big)^{-\frac{l}{2}}+\int_0^t\left(\frac{s}{2t-s}\right)^{\frac{l}{2}}\Big\langle p\big(s,\cdot\big)u(s,\cdot)\Big\rangle^p ds.$$

Let us denote $\varphi(t)=\Big\langle p\big(t,x\big)u(t,x)\Big\rangle$ then we have

$$\varphi(t)\ge C\big(4\pi a\big)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi a}\right)\big(2t+\tau\big)^{-\frac{l}{2}}+\int_0^t\left(\frac{s}{2t}\right)^{\frac{l}{2}}\big(\varphi(s)\big)^p ds$$

Assume that $\phi\big(t\big)=t^{\frac{l}{2}}\varphi(t)$, we obtain

$$\phi(t)\ge C\big(4\pi a\big)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi a}\right)\left(\frac{\varepsilon}{2\varepsilon+\tau}\right)^{\frac{l}{2}}+\int_\varepsilon^t\left(\frac{s}{2}\right)^{\frac{l}{2}}\left(\frac{\phi(s)}{s^{\frac{l}{2}}}\right)^p ds$$

$$\phi(t)\ge C\big(4\pi a\big)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi a}\right)\left(\frac{\varepsilon}{2\varepsilon+\tau}\right)^{\frac{l}{2}}+\left(\frac{1}{2}\right)^{\frac{l}{2}}\int_\varepsilon^t s^{-\frac{l}{2}(p-1)}\big(\phi(s)\big)^p ds\,.$$

The last inequality is equivalent to the differential problem:

$$\frac{d\phi(t)}{\big(\phi(t)\big)^p dt}=\left(\frac{1}{2}\right)^{\frac{l}{2}}t^{-\frac{l}{2}(p-1)}$$

$$\phi(\varepsilon)=C\big(4\pi a\big)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi a}\right)\left(\frac{\varepsilon}{2\varepsilon+\tau}\right)^{\frac{l}{2}}.$$

So, for $1 < p$ and $\frac{l}{2}(p-1) \le 1$ the solution blows up in the bounded time interval.

3. Nonlinear parabolic equation with singular coefficients, the general case

We consider the Cauchy problem for the following parabolic equation:

$$\frac{\partial}{\partial t}u = [\nabla_k a_{kj}(t,x)\nabla_j - b_k(t,x)\nabla_k]u + u^p, \quad u(0,x) = u_0(x).$$

If the elliptic matrix depends on time, fundamental solutions to the equation

$\frac{\partial}{\partial t}u = [\nabla_k a_{kj}(t,x)\nabla_j - b_k(t,x)\nabla_k]u$ can be written in the form

$$p_1(t,x;\tau,z) = p_0(t,x-z;\tau,z) + \int_\tau^t d\eta \int p_0(t,x-y;\eta,y)F(\eta,y;\tau,z)dy$$

Where

$$p_0(t,x;\tau,y) = (2\pi)^{-l}\int \exp\left(ix\eta - \int_\tau^t a(\gamma,y)\eta^2 d\gamma\right)d\eta$$

Are fundamental solutions to the equation?

$[\partial_t - \nabla_k a_{kj}(t,y)\nabla_j]u(t,x) = 0$, here $F(\eta,y;\tau,z)$ is the heat kernel density.

Definition. *Let χ be positive constant, function $f \in L^1_{loc}(R^{l+1})$ belongs to Nash class N^χ_φ if exists $h > 0$ such that $n_\varphi(f;\chi,h) \equiv (n_\varphi{}^+ + n_\varphi{}^-)(f;\chi,h) < \infty$, where*

$$n_\varphi{}^+(f;\chi,h) \equiv ess\sup_{z,s}\int_s^{s+h}\left\langle\left(\frac{1}{4\pi\chi(\tau-s)}\right)^{-\frac{l}{2}}\exp\left(-\frac{|\cdot - z|^2}{4\chi(\tau-s)}\right)|f(\tau,\cdot)|\right\rangle\frac{d\tau}{\varphi(\tau-s)}$$

,

$$n_\varphi{}^-(f;\chi,h) \equiv ess\sup_{z,t}\int_{t-h}^{t}\left\langle\left(\frac{1}{4\pi\chi(t-\tau)}\right)^{-\frac{l}{2}}\exp\left(-\frac{|\cdot - z|^2}{4\chi(t-\tau)}\right)|f(\tau,\cdot)|\right\rangle\frac{d\tau}{\varphi(t-\tau)},$$

and function $\varphi: R_+ \to R_+$ satisfies the conditions: $\varphi(0) = 0$, exists number $C > 0$ such that integrals $\int_0^C \frac{\varphi(\tau)}{\tau}d\tau$,

$\int_0^C \frac{d\tau}{\varphi(\tau)}$, $\int_0^C [\varphi'(\tau)]_+ d\tau$ are bounded. Function $f \in L^1_{loc}(R^{l+1})$ belongs to the Nash class N^χ [16] if there is a

Limit $\lim_{h\to 0} n_1(f;\chi,h) \equiv n_1{}^+(f;\chi,h) + n_1{}^-(f;\chi,h) = 0$.

Lemma 1. *Let $|b| \in N_2$ and the matrix a is uniformly elliptic then there are constants c, k such that*

$$\int_0^t \left\| Be^{-sA}\varphi \right\|_{L^1} ds \le cm(b^2;kt)\left\|\varphi\right\|_{L^1} \quad \forall \varphi \in L^1,$$

Where

$$m(b^2;kt) = ess \sup_{x\in R^l} \int_0^t \left(e^{t\Delta} b^2(x)\right)^{\frac{1}{2}} \frac{dt}{\sqrt{t}} \text{ and } D(B) \supset D(A_1),$$

and we have inequality for an operator norm

$$\left\| B\left(\lambda + A_1\right)^{-1} \right\|_{1\to 1} \le c(\lambda), \text{ where } \lim_{\lambda\to\infty} c(\lambda) = 0 \text{ for arbitrary } \lambda > 0.$$

Applying the Jean-Marie Duhamel principle to the differential operator $\partial_t + A(t,x) + B(t,x)$, we obtain equality

$$U_{t,s} = P_{t,s} - \int_s^t U_{t,\tau} B(\tau) P_{\tau,s} d\tau,$$

Where
$U_{t,s}$ is a propagator of the differential operator $\partial_t + A(t,x) + B(t,x)$, and $P_{t,s}$ is a propagator of the operator $\partial_t + A(t,x)$. Since $\left\|P_{t,s}\right\|_{1\to 1} \le 1$ for all t,s such that $t - s \le k$ for

$$\beta(\tau) = \sup_{\|\psi\|_1 = 1, s\ge 0} \int_s^{s+\tau} \left\| B(t) P_{t,s} \psi \right\|_1 dt < 1$$

Using the Fubini theorem, we are obtaining

$$\beta(\tau) \le \sup_{x\in R^l, s\ge 0} \int_s^{s+\tau} \left\langle \left| \left(b(t,\cdot)\cdot \nabla p(t,\cdot;s,x)\right| \right\rangle dt,$$ and then we have inequality $\left\|U_{t,s}\right\|_{1\to 1} \le \left(1 - \beta(\tau)\right)^{-1}$.

Let $p(t,x;s,y)$ be the heat kernel of the linear operator $\partial_t - A(t,x)$. The heat kernel of the heat equation $\partial_t - \Delta$ is the Gaussian density $G\left(t,x\right)$. Under certain conditions, the operator $\partial_t - A(t,x)$ can be considered as a perturbation of the operator $\partial_t - \Delta$ so for the heat kernel $p(t,x;s,y)$ the Gaussian estimations hold. Similarly, the operator $\partial_t - A(t,x) + B(t,x)$ can be considered as a perturbation of the operator $\partial_t - A(t,x)$ and so for the fundamental solution $q(t,x;s,y)$ of the equation.

$$\frac{\partial}{\partial t} u = [\nabla_k a_{kj}(t,x) \nabla_j - b_k(t,x) \nabla_k] u$$

Hold the next estimations

$const\ p(t,x;s,y) \le q(t,x;s,y) \le Const\ p(t,x;s,y)$. So, for arbitrary $x, y \in R^l, t - s \le T \in R,\ l > 2$ we obtain the following estimations

$$cG_{\delta}\left(t-s,x-y\right)=c\left(4\pi\delta\left(t-s\right)\right)^{\frac{-l}{2}}\exp\left(\frac{-\left|x-y\right|^{2}}{4\left(t-s\right)\delta}\right)\leq$$

$$\leq q(t,x;s,y)\leq C\left(4\pi\alpha\left(t-s\right)\right)^{\frac{-l}{2}}\exp\left(\frac{-\left|x-y\right|^{2}}{4\left(t-s\right)\alpha}\right)=$$

$$=CG_{\alpha}\left(t-s,x-y\right).$$

Let $q(t,x;s,y)$ be a fundamental solution of the equation

$$\frac{\partial}{\partial t}u=[\nabla_{k}a_{kj}(t,x)\nabla_{j}-b_{k}(t,x)\nabla_{k}]u$$

for $t>0,\ x\in R^{l}$ then we can write integral tautology $u(t,x)=\left\langle q(t,x;0,\cdot)u_{0}(\cdot)\right\rangle+\int\limits_{0}^{t}\left\langle q(t,x;s,\cdot)u^{p}(s,\cdot)\right\rangle ds$.

Moreover, the perturbation $b(t,x)\nabla$ must satisfy the following integral estimation

$$\int\limits_{s}^{t}\left\langle\left(4\pi\alpha\left(\tau-s\right)\right)^{\frac{-l}{2}}\exp\left(\frac{-\left|x-g\right|^{2}}{4\left(\tau-s\right)\alpha}\right)\left|b(\tau,g)\nabla p(\tau,g\,s,y)\right|\right\rangle d\tau\leq$$

$$\leq c_{0}\tilde{C}\left(4\pi\alpha\left(t-s\right)\right)^{\frac{-l}{2}}\exp\left(\frac{-\left|x-y\right|^{2}}{4\left(t-s\right)\alpha}\right),$$

Where constant can be obtained define according to the formula

$$\tilde{C}=n_{\varphi}(v;\alpha_{1},r)\left(\max_{\tau\in(0,r)}\varphi(\tau)+\int\limits_{0}^{r}\left(\left|\varphi'(\tau)\right|+\frac{\varphi(\tau)}{\tau}\right)dt\right)\ \text{and}\ n_{\varphi}(v;\alpha_{1},r)=\max_{\gamma\in\left\{\alpha,2\alpha,\frac{\alpha c}{\alpha-c}\right\}}n_{\varphi}(v;\gamma,r),$$

a $n_{\varphi}(v;\gamma,r):=n^{+}_{\varphi}(v;\gamma,r)+n^{-}_{\varphi}(v;\gamma,r)<\infty$,

Where

$$n^{-}_{\varphi}(v;\gamma,r)=ess\sup_{x,t}\int\limits_{t-r}^{t}\left\langle\left(4\pi\gamma\left(t-\tau\right)\right)^{\frac{-l}{2}}\exp\left(\frac{-\left|x-g\right|^{2}}{4\left(t-\tau\right)\gamma}\right)\left|v\left(\tau,g\right)\right|\right\rangle\frac{d\tau}{\varphi\left(t-\tau\right)}$$ And

$$n^{+}_{\varphi}(v;\gamma,r)=ess\sup_{x,t}\int\limits_{t}^{t+r}\left\langle\left(4\pi\gamma\left(\tau-t\right)\right)^{\frac{-l}{2}}\exp\left(\frac{-\left|x-g\right|^{2}}{4\left(\tau-t\right)\gamma}\right)\left|v\left(\tau,g\right)\right|\right\rangle\frac{d\tau}{\varphi\left(\tau-t\right)}.$$

Theorem. ***Let*** $|b|\in N_{2}$ ***and*** $1<p\leq\frac{2+l}{l}$ ***then the solution*** $u(t,x);\quad(t,x)\in[0,\infty)\times R^{l}$ ***to the Cauchy problem***

$$\frac{\partial}{\partial t}u=[\nabla_{k}a_{kj}(t,x)\nabla_{j}-b_{k}(t,x)\nabla_{k}]u+u^{p},$$

$u(0,x)=u_0\neq 0,\quad u_0\geq 0$ Blows up, that means there is a moment of time $t_0>0$ such that $u(t,x)=\infty,\quad t\geq t_0,\ x\in R^l$.

Proof. Assuming that function $u(t,x)\quad \forall(t,x)\in[0,\infty)\times R^l$ is a weak solution to the Cauchy problem for the parabolic nonlinear equation with the initial condition $u(0,x)=u_0\neq 0,\quad u_0\geq 0$, we write

$$u(t,x)=\langle q(t,x;0,\cdot)u_0(\cdot)\rangle+\int_0^t\langle q(t,x;s,\cdot)u^p(s,\cdot)\rangle ds \qquad \forall(t,x)\in[0,\infty)\times R^l.$$

The function $\breve{u}(t,x)=\langle q(t,x;0,\cdot)u_0(\cdot)\rangle \quad \forall(t,x)\in[0,\infty)\times R^l$ is a solution to the Cauchy problem

$$\frac{\partial}{\partial t}\breve{u}=[\sum_{k,j=1,\ldots,l}\nabla_k a_{kj}(t,x)\nabla_j-b_k(t,x)\nabla_k]\breve{u},$$ and function
$$\breve{u}(0,x)=u_0(x)\geq 0$$

$$\tilde{u}(t,x)=\int_0^t\langle q(t,x;s,\cdot)u^p(s,\cdot)\rangle ds \qquad \forall(t,x)\in[0,\infty)\times R^l$$

is a solution to following the Cauchy problem

$$\frac{\partial}{\partial t}\tilde{u}=[\sum_{k,j=1,\ldots,l}\nabla_k a_{kj}(t,x)\nabla_j-b_k(t,x)\nabla_k]\tilde{u}+\tilde{u}^p,$$
$$\tilde{u}(0,x)=0.$$

Let us choose a moment of time $t_0>0$ such that $q(t_0,0,0,0)\leq 1$ and

$$u(t+t_0,x)=\langle q(t,x,0,\cdot)u_0(t_0,\cdot)\rangle+\int_0^t\langle q(t,x,s,\cdot)u^p(s+t_0,\cdot)\rangle ds$$

so we have to show that the solution to the integral equation

$$u(t,x)=C\,q(t,x,\tau,0)+\int_0^t\langle q(t,x,s,\cdot)u^p(s,\cdot)\rangle ds$$

Blows up.

We multiply the last equality by $q(t,x,0,0)$ and integrate over all space

$$\langle q(t,x,0,0)u(t,x)\rangle_x\geq cG_\delta(2t+\tau,0)+\int_0^t\left\langle\langle q(t,x,0,0)q(t,x,s,\cdot)u^p(s,\cdot)\rangle_\cdot\right\rangle_x ds\geq$$

$$\geq cG_\delta(2t+\tau,0)+C_l\int_0^t\left\langle\langle G_\delta(t,x)G_\delta(t-s,\cdot)u^p(s,\cdot)\rangle_\cdot\right\rangle_x ds=$$

$$\geq cG_\delta(2t+\tau,0)+C_l\int_0^t\langle G_\delta(2t-s,\cdot)u^p(s,\cdot)\rangle ds\geq$$

$$\geq c\left(4\pi\delta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi\delta}\right)\left(2t+\tau\right)^{-\frac{l}{2}}+C_l\int_0^t\left(\frac{s}{2t-s}\right)^{\frac{l}{2}}\left\langle G_\delta\left(s,\cdot\right)\left(u(s,\cdot)\right)^p\right\rangle ds.$$

$$\geq c\left(4\pi\delta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi\delta}\right)\left(2t+\tau\right)^{-\frac{l}{2}}+C_l\int_0^t\left(\frac{s}{2t-s}\right)^{\frac{l}{2}}\left\langle G_\delta\left(s,\cdot\right)\left(u(s,\cdot)\right)\right\rangle^p ds.$$

Next, applying Gaussian upper estimation, we have

$$K_l\left\langle G_\delta\left(t,\cdot\right)u(t,\cdot)\right\rangle \geq$$

$$\geq c\left(4\pi\delta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi\delta}\right)\left(2t+\tau\right)^{-\frac{l}{2}}+C_l\int_0^t\left(\frac{s}{2t-s}\right)^{\frac{l}{2}}\left\langle G_\delta\left(s,\cdot\right)\left(u(s,\cdot)\right)\right\rangle^p ds.$$

Let us denote

$$\varphi(t)=\left\langle G_\delta\left(t,\cdot\right)u(t,\cdot)\right\rangle=\left\langle\left(4\pi\delta t\right)^{\frac{-l}{2}}\exp\left(\frac{-\left|\cdot\right|^2}{4\delta t}\right)u(t,\cdot)\right\rangle,$$

Then

$$\varphi(t)\geq c\left(4\pi\delta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi\delta}\right)\left(2t+\tau\right)^{-\frac{l}{2}}+C_l\int_0^t\left(\frac{s}{2t}\right)^{\frac{l}{2}}\left(\varphi(s)\right)^p ds$$

Take $\phi\left(t\right)=t^{\frac{l}{2}}\varphi(t)$, we obtain

$$\phi(t)\geq c\left(4\pi\delta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi\delta}\right)\left(\frac{\varepsilon}{2\varepsilon+\tau}\right)^{\frac{l}{2}}+\int_\varepsilon^t\left(\frac{s}{2}\right)^{\frac{l}{2}}\left(\frac{\phi(s)}{s^{\frac{l}{2}}}\right)^p ds$$

$$\phi(t)\geq c\left(4\pi\delta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi\delta}\right)\left(\frac{\varepsilon}{2\varepsilon+\tau}\right)^{\frac{l}{2}}+\left(\frac{1}{2}\right)^{\frac{l}{2}}\int_\varepsilon^t s^{-\frac{l}{2}(p-1)}\left(\phi(s)\right)^p ds$$

Thus, we have obtained the following differential problem

$$\frac{d\phi(t)}{\left(\phi(t)\right)^p dt}=\left(\frac{1}{2}\right)^{\frac{l}{2}}t^{-\frac{l}{2}(p-1)}$$

$$\phi(\varepsilon)=c\left(4\pi\delta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4\pi\delta}\right)\left(\frac{\varepsilon}{2\varepsilon+\tau}\right)^{\frac{l}{2}}.$$

We can consider the value $\phi(\varepsilon)$ as the initial value of the ordinary differential equation, the solution of this equation blows up when $1 < p$ and $\frac{l}{2}(p-1) \le 1$ so the solution of the Cauchy problem for the parabolic equation blows up on bounded time interval.

Conclusion

In this equation, we have established the condition on the perturbation under which the Cauchy problem for the parabolic partial differential equation with u^p - perturbation blows up for $1 < p \le \frac{2+l}{l}$.

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