E-ISSN: 2709-9407
P-ISSN: 2709-9393
Impact Factor (RJIF): 5.94
JMPES 2025; 6(2): 771-777

www.mathematicaljournal.com
Received: 21-10-2025
Accepted: 25-11-2025

PVH S Sai Kumar
Department of Mathematics, Dayalbagh Educational Institute, Dayalbagh, Agra, Uttar Pradesh, India

Corresponding Author:
PVH S Sai Kumar
Department of Mathematics, Dayalbagh Educational Institute, Dayalbagh, Agra, Uttar Pradesh, India

Neutral differential equations of even order with several delays and a damping term: Oscillatory behavior

PVH S Sai Kumar

DOI: https://www.doi.org/10.22271/math.2025.v6.i2e.275

Abstract
The higher-order delay differential equations are used in the describing of many natural phenomena. This paper presents a study on the oscillatory behavior of solutions of even order neutral differential equations with several delays of the form

$$(a(t)((m(t)x(t)+p(t)x(\tau(t)))^{n-1})^{\gamma})' + r(t)((m(t)x(t)+p(t)x(\tau(t)))^{n-1})^{\gamma} + \sum_{i=1}^{n} q_i(t)x^{\gamma_i}(\sigma_i(t)) = 0 \qquad (1)$$

is considered. In this research, we applied three techniques – the comparison technique, the Riccati technique, and the integral averages technique to analyze and establish the sufficient conditions for the oscillatory behaviour of the equations under the condition that $R(t) = \int_{t_0}^{\infty}\left[\frac{1}{a(s)}\exp(-\int_{t_0}^{s}\frac{r(u)}{a(u)}du)ds\right]^{\frac{1}{\gamma}} = \infty$ as $t \to \infty$. Also, the results are an extension and simplification as well as improvement of the previous results.

Keywords: Oscillation, even-order, neutral differential equation, damping term

1. Introduction

In this paper we study the delay differential equations of even order with several delays and a Damping term of the type

$$(a(t)((m(t)x(t)+p(t)x(\tau(t)))^{n-1})^{\gamma})' + r(t)((m(t)x(t)+p(t)x(\tau(t)))^{n-1})^{\gamma} + \sum_{i=1}^{n} q_i(t)x^{\gamma_i}(\sigma_i(t)) = 0\,.$$

A delay differential equation of neutral type, is an equation in which the highest order derivative of the unknown function appears both with and without delay. Oscillation theory is one of the branches of qualitative theory that studies the qualitative properties of solutions of differential equations, such as stability, symmetry, oscillation, and others, without finding solutions.

During the last decades, a large amount of Research attention has been focused on the oscillation problem of different kinds of differential equations. One area of active Research in recent years is studying the sufficient conditions for the oscillation of delay differential equations see, (Abdulaziz khalid et.al. Bazighifan et.al. Graef, J.R et.al. Omar Bazighifan et.al [2, 3, 5, 12].

Also, there has been a growing interest, in the theory of oscillation in functional differential equations (FDEs) due to their numerous applications in various fields of science, and extensive research has been conducted on the oscillation of solutions to several kinds of FDE's we refer to (Erbe et.al., Xing, G et.al [4, 19].

Many Researchers have focused on understanding the oscillatory behavior of various differential equations of different orders and have provided advanced technologies for obtaining oscillation criteria for higher order differential equations and we refer to (John R. et.al. Li T. et.al, Tunc C. et.al. Osama Moaa, Zhang C. et.al.) [7, 8, 17, 13, 20].

Osama Moaaz and Asma Al-Jaser [14] studied asymptotic performance of nonoscillatory solutions to the functional differential equation of the form

$$(r(t)|x'(u)|^{m-1}x'(u))' + a(u)H(x'(\rho(u))) + q(u)F(x(\delta(u))) = 0$$

Under the condition that

$$\int_{u_1}^{u} \frac{dl}{r^{\frac{1}{m}}(l)} = \infty;$$

Sarah Aloraini et.al. [16] Studied the Asymptotic and Monotonic Properties of Solutions of Functional Differential Equations with Multiple Delays and a Damping Term described by

$$(a(t)(z'(t))^{\gamma})' + p(t)\left(z'(t)\right)^{\gamma} + \sum_{i=1}^{n} q_i(t)x^{\gamma}(g_i(t) = 0$$

Under the condition that

$$\mathrm{R(t)} = \int_{\mathrm{t_0}}^{\infty} \left[\frac{1}{\mathrm{a}^{\frac{1}{\gamma}}(s)} \exp(-\frac{1}{k}\int_{0}^{u} \frac{\mathrm{p(u)}}{\mathrm{a(u)}} \mathrm{du})\mathrm{ds}\right]^{\frac{1}{\gamma}} = \infty\,;$$

Osama Moaaz et.al. [11] Investigated the asymptotic and oscillatory properties of delay differential equations of even order

$$\left(m(u)(\left(x^{m-1}(u)\right)^{\gamma}\right)' + \sum_{i=1}^{n} h_i(u)\, f(x(\vartheta_i(u)) = 0$$

Under the assumption

$$\int_{u_0}^{\infty} \frac{1}{m^{\frac{1}{\alpha}}(s)} ds < \infty;$$

Maryam Al-Kandari [10] examined the Enhanced criteria for detecting oscillations in neutral delay Emden-Fowler differential equations

$$(a(t)[(y_1(t) + p(t)y_1(\delta_2(t)))']^{r_1})' + f\left(t, y_1\left(\vartheta, (y)\right)\right) = 0$$

Under the condition that

$$\psi(y_0) = \int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{r_1}}(t)} dt < \infty.$$

By a solution of equation (1) we mean a function$x(t) \in C([T_x, \infty))$, $T_x \geq t_0$ which has the properties $z(t) \in C'\left([T_x, \infty)\right), a(t)\left(z'''(t)\right)^{\gamma} \in C'\left(\left([T_x, \infty)\right)\right.$ and satisfies equation (1) on $([T_x, \infty))$.We consider only those solutions x of equation (1) which satisfy$\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$, and assume that the equation (1) possesses such solutions. As usual, a solution of equation (1) is called oscillatory if it has a zero on $[T, \infty)$ for all $T \geq T_x$; otherwise it is called nonoscillatory. If all solutions of a differential equations are oscillatory, then the equation itself is called oscillatory

The main motivation for this work is to contribute to the development of the oscillatory theory for the higher order neutral equations by finding sufficient conditions which guarantee that the solutions of this type of equations are oscillatory.

2. Method

In this paper we use few Lemmas and Inequality (15) which are helpful to prove our results by applying the Riccati Transformation technique.

3. Main Results and Discussion

We need the following in our discussion

(H_1): γ is a quotient of odd positive integers, n is even, $m(t)$ is a real valued continuous function.

(H_2): $a \in C'[t_0, \infty), (0, \infty)), a'(t) \geq 0, p, q_i, r \in C[t_0, \infty), (0, \infty)), \lim_{t\to\infty} p(t) = 0$ And

$q_i(t) > 0, \tau(t) \leq t,\ \tau'(t) \geq 0$ and $\sigma_i(t) \leq t, \lim_{t\to\infty} \sigma_i(t) = \infty.$

(H_3): We define

$$R(t) = \int_{t_0}^{\infty}\left[\frac{1}{a(s)}\exp\left(-\int_{t_0}^{s}\frac{r(u)}{a(u)}du\right)ds\right]^{\frac{1}{\gamma}} = \infty \text{ as } t \to \infty. \tag{2}$$

3.1 Lemma [1]. Suppose that$\sigma \in C^n([v_0,\mathbb{R}^+)$, where $\rho^{(n)}(v)$ has a constant sign and is non-zero on $[v_0,\infty)$. Additionally, suppose that there is $v_1 \geq v_0$ such that $\rho^{n-1)}(v)\rho^n(v) \leq 0$ for every$v \geq v_1$. If $\lim_{v\to\infty}\rho(v) \neq 0$, then for any $\delta \in (0,1)$, there is $v_\epsilon \in [v_1,\infty)$ such that $\rho(v) \geq \frac{\epsilon}{(n-1)!}v^{n-1}|\rho^{(n-1)}(v)|$, for $\rho \in [v_\epsilon,\infty)$.

3.2 Lemma [18] Let $\zeta \in C^n([t_0,\infty,\mathbb{R}^+)$. If $\xi^n(t)$ is eventually of one sign for all large t, then there exists a $t_x > t_1$, for some $t_1 > t_0$, and an integer l, $0 \leq l \leq n$ with $n+l$ even $\xi^n(t) \geq 0$ or $n+l$ odd for $\xi^n(t) \leq 0$ such that $l > 0$ implies that $\xi^k(t) > 0$ for $t > t_x$, $k = 0,1,\ldots\ldots\ldots i-1$, $l \leq n-1$, implies that $(-1)^{l+k}\xi^k(t) > 0$ for $t > t_x$, $k = l, l+1,\ldots\ldots\ldots n-1$.

3.3 Theorem: Assume that (2) holds. If there exists a nondecreasing function $\rho \in C'([t_0,\infty),(0,\infty))$ such that

$$\int_{t_0}^{\infty}\left(\rho(s)A(s) - \frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\left(\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right)^{\gamma+1}\right)ds = \infty \tag{3}$$

Then equation (1) is oscillatory.

Proof: Assume that (1) has a nonoscillatory solution $x(t)$. without loss of generality, we may assume that $x(t)$ is Eventually positive. It follows from (1), (H_2), (H_3) and by Lemma 3.2, we have

$z(t) > 0$, $z'(t) > 0$, $z^{(n-1)}(t) > 0$ and $z^n(t) < 0$.

Also, we may assume that there exists a $t_1 \in [t_0,\infty)$ such that $x(t) > 0, x(\tau(t)) > 0, x(\sigma_i(t)) > 0$ for all

$i = 1,2,3.$ and $t \in [t_1,\infty)$.

Set

$$z(t) = m(t)x(t) + p(t)x(\tau(t)) \tag{4}$$

$$m(t)x(t) \geq z(t) - p(t)x(\tau(t))$$

$$x(t) \geq \frac{1}{m(t)}[1-p(t)]z(t)$$

Hence $x(\sigma_i(t)) \geq \frac{1}{m(\sigma_i(t)}[1-p(\sigma_i(t))]z(\sigma_i(t)), i = 1,2,3.$

We note that $\sigma(t) = min\{\sigma_i(t), i = 1,2,3\}$

$$R(t) = \int_{t_0}^{t}\left[\frac{1}{a(s)}\exp\left(-\int_{t_0}^{s}\frac{r(u)}{a(u)}du\right)ds\right]^{\frac{1}{\gamma}} = \infty, F_+(t) = max\{F(t),0\},$$

From (1) we have

$$(a(t)((m(t)x(t)+p(t)x(\tau(t)))^{n-1})^{\gamma})' + r(t)((m(t)x(t)+p(t)x(\tau(t)))^{n-1})^{\gamma}$$

$$= -q_1(t)x^{\gamma_1}(\sigma_1(t)) - q_2(t)x^{\gamma_2}(\sigma_2(t)) - q_3(t)x^{\gamma_3}(\sigma_3(t)) \tag{5}$$

Define,

$$\omega(t) = \rho(t)\frac{a(t)(z^{n-1}(t))^{\gamma}}{z^{\gamma}(\sigma(t))}, t \geq t_1 \tag{6}$$

$$\omega'(t) = \rho'(t)\frac{a(t)(z^{n-1}(t))^{\gamma}}{z^{\gamma}(\sigma(t))} + \rho(t)\frac{\left(a(t)(z^{n-1}(t))^{\gamma}\right)'}{z^{\gamma}(\sigma(t))} - \gamma\rho(t)\frac{a(t)(z^{n-1}(t))^{\gamma}z'(\sigma(t))\sigma'(t)}{z^{\gamma+1}(\sigma(t))}$$

$$\omega'(t) = \rho'(t)\frac{a(t)(z^{n-1}(t))^{\gamma}}{z^{\gamma}(\sigma(t))} + \rho(t)\left[\frac{-r(t)(z^{n-1}(t))^{\gamma} - q_i(t)x^{\gamma_i}(\sigma_i(t))}{z^{\gamma}(\sigma(t))}\right] - \gamma\rho(t)\frac{a(t)(z^{n-1}(t))^{\gamma}z'(\sigma(t))\sigma'(t)}{z^{\gamma+1}(\sigma(t))}$$

$$\omega'(t) = \rho'(t)\frac{a(t)\left(z^{n-1}(t)\right)^{\gamma}}{z^{\gamma}\left(\sigma(t)\right)} - \rho(t)\frac{r(t)\left(z^{n-1}(t)\right)^{\gamma}}{z^{\gamma}\left(\sigma(t)\right)} - \rho(t)\frac{q_i(t)x^{\gamma_i}(\sigma_i(t))}{z^{\gamma}\left(\sigma(t)\right)} - \gamma\rho(t)\frac{a(t)\left(z^{n-1}(t)\right)^{\gamma}z'(\sigma(t))\sigma'(t)}{z^{\gamma+1}(\sigma(t))}$$

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{r(t)}{a(t)}\omega(t) - \rho(t)\frac{q_i(t)x^{\gamma_i}(\sigma_i(t))}{z^{\gamma}\left(\sigma(t)\right)} - \gamma\sigma'(t)\omega(t)\frac{z'(\sigma(t))}{z(\sigma(t))} \tag{7}$$

Since

$$x(\sigma_i(t)) \geq \frac{1}{m(\sigma_i(t))}\left[1 - p\left(\sigma_i(t)\right)\right]z(\sigma_i(t))$$

Therefore

$$x^{\gamma_i}(\sigma_i(t)) \geq \frac{1}{m^{\gamma_i}(\sigma_i(t))}\left[1 - p\left(\sigma_i(t)\right)\right]^{\gamma_i} z^{\gamma_i}\left(\sigma_i(t)\right), i = 1,2,3. \tag{8}$$

We know that $\lim_{t\to\infty} z'(t) \neq 0$. By virtue of Lemma 2.1 for every constant $\lambda \in (0,1)$ and for all large t, we get

$$z'(t) \geq \frac{\lambda}{(n-2)!}t^{n-2}\, z^{(n-1)}(t) \tag{9}$$

From (7), (8), and (9) and we obtain

$$\omega'(t) \leq \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) - \rho(t) - \gamma\sigma'(t)\frac{\lambda}{(n-2)!}\frac{\left(\sigma(t)\right)^{n-2}z^{(n-1)}(\sigma(t))}{z(\sigma(t))}\omega(t). \tag{10}$$

Here we note that

$$A_i(t) = q_i(t)\frac{1}{m^{\gamma_i}\left(\sigma_i(t)\right)}\left[1 - p\left(\sigma_i(t)\right)\right]^{\gamma_i}\, for\ all\ i = 1,2,3.$$

Therefore from (10) we get

$$\omega'(t) \leq \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) - \rho(t) - \frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(t)\left(\sigma(t)\right)^{n-2}}{\left(\rho(t)a(t)\right)^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}. \tag{11}$$

Since $z'(t) > 0$ and$\sigma(t) \leq \sigma_i(t)$, we have

$$z\left(\sigma_i(t)\right) \geq z\left(\sigma(t)\right), i = 1,2,3.$$

$$\omega'(t) \leq -\rho(t)\left[A_1\left(\sigma_1(t)\right)z^{\gamma_1-\gamma} + A_2(t)z^{\gamma_2-\gamma}\left(\sigma_2(t)\right) + A_3(t)z^{\gamma_3-\gamma}\left(\sigma_3(t)\right)\right] + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) - \frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(t)\left(\sigma(t)\right)^{n-2}}{\left(\rho(t)a(t)\right)^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t). \tag{12}$$

Since$z'(t) > 0$, there exists $t_2 \geq t_1$ and $M > 0$ such that $z(t) > M$.
Hence, we obtain

$$\omega'(t) \leq -\rho(t)\left[A_1\left(\sigma_1(t)\right)M^{\gamma_1-\gamma} + A_2(t)M^{\gamma_2-\gamma}\left(\sigma_2(t)\right) + A_3(t)M^{\gamma_3-\gamma}\left(\sigma_3(t)\right)\right] + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) - \frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(t)\left(\sigma(t)\right)^{n-2}}{\left(\rho(t)a(t)\right)^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t). \tag{13}$$

$$\omega'(t) \leq -\rho(t)A(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) - \frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(t)\left(\sigma(t)\right)^{n-2}}{\left(\rho(t)a(t)\right)^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t). \tag{14}$$

Where,

$$A(t) = A_1\left(\sigma_1(t)\right)M^{\gamma_1-\gamma} + A_2(t)M^{\gamma_2-\gamma}\left(\sigma_2(t)\right) + A_3(t)M^{\gamma_3-\gamma}\left(\sigma_3(t)\right)$$

Using the following inequality in (14),

$$Vu - Uu^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{V^{\gamma+1}}{U^{\gamma}}, \tag{15}$$

Where $U > 0, V \geq 0, \gamma > 0$ with

$$V = \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right), U = \frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(t)(\sigma(t))^{n-2}}{(\rho(t)a(t))^{\frac{1}{\gamma}}} \text{and } u(t) = \omega(t)$$

We get

$$\omega'(t) \leq -\left(\rho(t)A(t) - \frac{((n-2)!)^{\gamma}a(t)\rho(t)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(t)\sigma'(t))^{\gamma}}\left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)^{\gamma+1}\right) \tag{16}$$

Integrating (16) from t_1 to t, we obtain

$$\int_{t_1}^{t}\left(\rho(s)A(s) - \frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\left(\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right)^{\gamma+1}\right)ds \leq \omega(t_1) \tag{17}$$

This is a contradiction to (3) as $t \to \infty$. thus the proof is completed.

4. Philos -type oscillation Result

In this section, we apply the integral averaging technique to establish a Philos type oscillation criteria for equation (1).
Definition Let

$D = \{(t,s) \in R^2 : t \geq s \geq t_0\}$ and $D_0 = \{(t,s) \in R^2 : t > s \geq t_0\}$.

A kernel function $H \in C(D,R)$ is said to belong to the function class $\mathfrak{I}$, written as $H \in \mathfrak{I}$, if

(i) $H(t,s) = 0$ for $t \geq t_0, H(t,s) > 0, (t,s) \in D_0$;

(ii) $H(t,s)$ has a continuous and nonpositive partial derivative $\frac{\partial H}{\partial s}$ *on* D_0, and there exists a function $\rho \in C'([t_0,\infty),(0,\infty))$ and $h \in C(D_0,R)$ such that

$$\frac{\partial}{\partial s}H(t,s) + \left(\frac{\rho'(s)}{\rho(s)} - \frac{r(t)}{a(t)}\right)H(t,s) = h(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s). \tag{18}$$

4.1 Theorem: Assume that (2)) holds. If there exists a positive function $\rho, \in C'([t_0,\infty),R)$ such that

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(H(t,s)A(s) - \frac{h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\right)ds = \infty. \tag{19}$$

Then (1) is oscillatory.

Proof. Let x be a nonoscillatory solution of equation (1). Multiplying (14) by $H(t,s)$ and integrating the resulting inequality from t_1 to t, we find that

$$\int_{t_1}^{t}H(t,s)\rho(s)A(s)ds \leq \omega(t_1)H(t,t_1) + \int_{t_1}^{t}\left(\frac{\partial}{\partial s}(H(t,s) + \left(\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right)H(t,s)\right)\omega(s)ds - -\int_{t_1}^{t}\frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(s)(\sigma(s))^{n-2}}{(\rho(s)a(s))^{\frac{1}{\gamma}}}H(t,s)\omega^{\frac{\gamma+1}{\gamma}}ds.$$

From (18) we get

$$\int_{t_1}^{t}H(t,s)\rho(s)A(s)ds \leq \omega(t_1)H(t,t_1) + \int_{t_1}^{t}h(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)\omega(s)ds - -\int_{t_1}^{t}\frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(s)(\sigma(s))^{n-2}}{(\rho(s)a(s))^{\frac{1}{\gamma}}}H(t,s)\omega^{\frac{\gamma+1}{\gamma}}ds. \tag{20}$$

Using the Lemma (15) with

$$U = \frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(s)(\sigma(s))^{n-2}}{(\rho(s)a(s))^{\frac{1}{\gamma}}}H(t,s), V = h(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s), and\ u(s) = \omega(s)$$

We get

$$h(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)\omega(s) - \frac{\lambda}{(n-2)!}\frac{\gamma\sigma'(s)(\sigma(s))^{n-2}}{(\rho(s)a(s))^{\frac{1}{\gamma}}}H(t,s)\omega^{\frac{\gamma+1}{\gamma}}(s) \leq \frac{h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{((n-2)!)^{\gamma}\rho(s)a(s)}{\lambda\sigma^{n-2}(s)\sigma'(s)}$$

Which implies that

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(H(t,s)A(s)-\frac{h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\right)ds\leq\omega(t_1). \tag{21}$$

Which is a contradiction to (19). Thus, the theorem (2) is proved

4.2 Theorem: Assume that (2) holds. If there exists a function $\rho\in C'([t_0,\infty),(0,\infty))$ such that

$$\lim_{t\to\infty}sup\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(\rho(s)A(s)-\frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\left(\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right)^{\gamma+1}\right)ds=\infty. \tag{22}$$

Then (1) is oscillatory.

Proof. In this case we define the function $\omega(t)$ by (6) and proceeding as in the proof of Theorem 3.1 to obtain (16). From (16) we have for $t\geq t_1$

$$\int_{t_1}^{t}(t-s)^n\left(\rho(s)A(s)-\frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\left(\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right)^{\gamma+1}\right)ds\leq-\int_{t_1}^{t}(t-s)^n\,\omega'(s)ds. \tag{23}$$

Since

$$\int_{t_1}^{t}(t-s)^n\,\omega'(s)=n\int_{t_1}^{t}(t-s)^{n-1}\omega(s)ds-\omega(t_1)(t-t_1)^n \tag{24}$$

We get

$$\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n\left(\rho(s)A(s)-\frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\left(\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right)^{\gamma+1}\right)\leq\omega(t_1)\left(\frac{t-t_1}{t}\right)^n-\frac{n}{t^n}\int_{t_1}^{t}(t-s)^{n-1}\,\omega(s)ds, \tag{25}$$

Hence

$$\frac{1}{t^n}\int_{t_1}^{t}(t-s)^n\left(\rho(s)A(s)-\frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\left(\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right)^{\gamma+1}\right)\leq\left(\frac{t-t_1}{t}\right)^n\omega(t_1) \tag{26}$$

And so

$$\lim_{t\to\infty}sup\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(\rho(s)A(s)-\frac{((n-2)!)^{\gamma}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}(\lambda\sigma^{n-2}(s)\sigma'(s))^{\gamma}}\left(\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right)^{\gamma+1}\right)ds\leq\omega(t_1),$$

Which is a contradiction to (22), and this completes the proof.

5. Conclusion

The goal of this paper is to study the 'Oscillatory behavior of Even Order neutral differential equations with Several Delays of equation (1) by using Riccati Transformation technique. Further extension of these results can be used to study a class of system of Fractional order equations. Some research in this area is in progress.

The results of this study complement many of previously published findings in the literature. To our knowledge, this equation has not been studied by many researchers, so it would be a good idea to apply these results to nonlinear higher order NDE's with a p-Laplacian like operator in future.

References

1. Almutairi A. Oscillatory properties test for even-order differential equations of neutral type. European Journal of Pure and Applied Mathematics. 2023;16(4):2499–2508.
2. Alsharidi AK, Muhib A. Some new oscillation results for second-order differential equations with several delays. 2025;1–17.
3. Bazighifan O. An approach for studying asymptotic properties of solutions of neutral differential equations. Symmetry. 2020;12:555:1–8.
4. Erbe L, Hassan TS, Peterson A. Oscillation of second-order neutral delay differential equations. Advances in Dynamical Systems and Applications. 2008;3(1):53–71.
5. Graef JR, Grace SR, Tunç E. Oscillation of even-order advanced functional differential equations. Publicationes Mathematicae (Debrecen). 2018;93(3–4):445–455.
6. Graef JR, Grace SR, Tunç E. Oscillatory behaviour of even-order nonlinear differential equations with a sublinear neutral term. Opuscula Mathematica. 2019;39(1):39–47.
7. Graef JR, Grace SR, Jadlovská I, Tunç E. Some new oscillation results for higher-order nonlinear differential equations with a nonlinear neutral term. Mathematics. 2022;10:2997:1–11.

8. Li T, Baculikova B, Džurina J, Zhang C. Oscillation of fourth-order neutral differential equations with p-Laplacian-like operators. Boundary Value Problems. 2014;2014(1):1–9.
9. Alqahtani M, Masood F, Saad KM, Bazighifan O. On the oscillation criteria for neutral differential equations with several delays. Scientific Reports. 2025;15:34014:1–17.
10. Al-Kandari M. Enhanced criteria for detecting oscillations in neutral delay Emden–Fowler differential equations. Kuwait Journal of Science. 2023;50:443–447.
11. Moaaz O, Albalawai W. Asymptotic behavior of solutions of even-order differential equations with several delays. Fractal and Fractional. 2022;6:87:1–11.
12. Bazighifan O, Postolache M. Multiple techniques for studying asymptotic properties of a class of differential equations with variable coefficients. Symmetry. 2020;12:1112:1–11.
13. Moaaz O, Salah H, Al-Jaser A, Anis M, Elabbasy EM. Comparison theorems for oscillation of higher-order neutral delay differential equations. Symmetry. 2024;16(7):903:1–17.
14. Moaaz O, Al-Jaser A. Asymptotic performance of nonoscillatory solutions of functional differential equations involving a delayed damping term. AIMS Mathematics. 2025;10(11):26153–26167.
15. Althobati S, Bazighifan O, Yavuz M. Some important criteria for oscillation of nonlinear differential equations with middle term. Mathematics. 2021;9(4):1–9.
16. Aloraini S, Almohaimeed AS, Moaaz O. Investigation of the asymptotic and monotonic properties of solutions of functional differential equations with multiple delays and a damping term. European Journal of Pure and Applied Mathematics. 2025;18(4):1–15.
17. Tunç C, Bazighifan O. Some new oscillation criteria for fourth-order neutral differential equations with distributed delay. Electronic Journal of Mathematical Analysis and Applications. 2019;7(1):235–241.
18. Thandapani E, Padmavathi S, Pinelas S. Oscillation criteria for even-order nonlinear neutral differential equations of mixed type. Bulletin of Mathematical Analysis and Applications. 2014;6(1):9–22.
19. Xing G, Li T, Zhang C. Oscillation of higher-order quasilinear neutral differential equations. Advances in Differential Equations. 2011;2011(45):1–10.
20. Zhang C, Li T, Sun B, Thandapani E. On the oscillation of higher-order half-linear delay differential equations. Applied Mathematics Letters. 2011;24(9):1618–16