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A new approach to multiplication modules via e-small submodules

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Abstract
Let R be a commutative ring with identity and M an R-module. In this paper, we introduce and investigate the concept of e-small multiplication modules. An R-module M is said to be an e-small multiplication module if every e-small submodule N of M can be written as $N = IM$ for some ideal I of R. We explore several fundamental properties of these module and study their connections with classical multiplication and e-small multiplication module. Various examples and counterexample are presented to highlight the behavior and scope of this class. Furthermore, we show that the direct sum of e-small multiplication modules retains the e-small multiplication property.

Keywords: E-small submodule, multiplication module, small multiplication module, e-small multiplication module

1. Introduction

Throughout this paper, all rings are commutative ring with identity and all modules are unitary m0dules. Let R be a ring and M be an R-module. The notation $N \leq M$ means that N is a submodule of M. $\mathrm{Soc}(M)$ and $\mathrm{Rad}(M)$ will stand for the socle of M and the radical of M. A submodule $L \leq M$ is said to be essential in M, indicated by $L \trianglelefteq M$, if $L \cap N \neq 0$ for all non-zero submodule $N \leq M$. A module M is named singular if $M \cong \frac{N}{L}$ for specific module N and an essential submodule $L \trianglelefteq N$. As a dual to the concept of an essential submodule, a submodule N of M is named small in M, symbolized by $N \ll M$, if $M \neq N + K$ for any proper submodule K of M (see [2, 7, 12]). A module M is named hollow if each proper submodule of M is small in M, and it is named l0cal if it is h0llow and finitely generated see [12].
Zhou [13] introduced δ-small submodules, extended small submodules of a module M as follows. A submodule $N \leq M$ is named δ-small in M (indicated by $N \ll_\delta M$) if whenever $M = N + L$ with $\frac{M}{L}$ singular, one has $M = L$. See [6]. Evidently, any small submodule is δ-small in M. Following [14], a submodule N of M is called an essential-small (for short e-small) submodule of M, denoted by $N \ll_e M$ if, whenever $M = N + E$ with $E \trianglelefteq M$, implies $E = M$, in reality, the authors Koşar et. al. put an e-small submodule in place of a g-small submodule, denoted by $N \ll_g M$. Obviously, any small (δ-small) submodule of an R-module M is e-small (g-small), but the reverses are not true in general [10]. If M is a projective m0dule, then each e-small subm0dule N of M is δ-small [14]. If T is essential and maximal subm0dule of M then T is said to be a generalized maximal submodule of M. The intersection of all generalized maximal submoduls 0f M is named the generalized radical 0f M besides indicated by $\mathrm{Rad}_g(M)$ (see [5]). In [4], it is indicated by $\mathrm{Rad}_e(M)$. If M have no generalized maximal submodules, then the generalized radical of M is defined by $\mathrm{Rad}_e(M) = M$.
Let M be a module over a ring R. Then M is called multiplication in case for every submodule N of M, there exists an ideal I in R such that $N = IM$ [8]. Multiplication modules ant their generalizations too studied by several authors. For more information about these modules see [1, 11].
M. Irani *et. al*. in 2023 [9] introduced the idea of small multiplication modules as well as the idea of δ-small multiplication modules. These ideas as natural generalizations of multiplication modules. A module M is called small (δ-small) multiplication module if for every small (δ-small) submodule N of M there exists an ideal I of R such that $N = IM$. Here, we introduce the notion of e-small multiplication modules.
In sections 2, we focus on small multiplication modules. Here, we give some of examples and important results of these modules.

In sections 3, we introduced the concept of e-small multiplication modules. This notion is a generalization of small multiplication modules. Here, we proved that if M is g-small multiplication module and $N \ll_e M$, then M/N is e-small multiplication module.
We state the next lemma which is contained in [14, Prop. 2.3].

Lemma 1.1: Let N be a submodule of a module M. The foll0wing are equivalent.
(1) $N \ll_e M$,
(2) if $M = X + N$, then X is a direct summand of M with $\frac{M}{X}$ a semisimple module.

It is obvious that Lemma 1.1(2) is equal to if $M = X + N$, at that time $M = X \oplus Y$ for s0me semisimple submodule Y of M.
Clearly, M is projective semisimple module if and only if $M \ll_e M$. A ring R is local if ${}_RR$ (or R_R) is a local module.
The following lemma follows from [14, Prop. 2.5] and from [5].

Lemma 1.2: Let M be a module. Then
1. For submodules N, K, L 0f M with $K \leq N$, we have
a) If $N \ll_e M$, then $K \ll_e M$ and $N/K \ll_e M/K$.
b) $N + L \ll_e M$ if and only if $N \ll_e M$ and $L \ll_e M$.
2. If $K \ll_e M$ and $f: M \to N$ is a homomorphism, then $f(K) \ll_e N$. In particular, if $K \ll_e M \leq N$, then $K \ll_e N$.
3. Let N, K, L, and T be submodules of M. If $K \ll_e L$ and $N \ll_e T$, then $K + N \ll_e L + T$.
4. Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_g M_1 \oplus M_2$ if and **0**nly if $K_1 \ll_e M_1$ besides $K_2 \ll_e M_2$.

Definition 1.3: [14] Let M be a module. Define
$\mathrm{Rad}_e(M) = \cap \{N \trianglelefteq M \mid N \text{ is maximal in } M\}$.
If M have no maximal essential submodules, then we denote $\mathrm{Rad}_e(M) = M$.
Clearly, $\mathrm{Rad}(M) \leq \mathrm{Rad}_e(M)$ and $\mathrm{Soc}(M) \leq \mathrm{Rad}_e(M)$. For an arbitrary ring R, let $\mathrm{Rad}_e(R) = \mathrm{Rad}_e({}_RR)$. For more information about $\mathrm{Rad}_e(M)$ see for example [3-5]. In the next we use g-small submodules to characterize $\mathrm{Rad}_e(M)$.

Theorem 1.4: Let M be an R-module. Then $\mathrm{Rad}_e(M) = \sum_{N \ll_e M} N$.

Proof: By [14, Theorem 2.10]. □

Lemma 1.5: Let M and N be modules. Then
(1) If $f: M \to N$ is an R-homomorphism, then $f(\mathrm{Rad}_e(M)) \leq \mathrm{Rad}_e(N)$.
(2) If every proper essential subm**o**dule of M is contained in a maximal submodule of M, then $\mathrm{Rad}_e(M)$ is the unique largest g-small submodule of M.

Proof: See [14, Corollary 2.11]. □

Remark 1.6: t is clear that, in general, $\mathrm{Rad}_e(M)$ need not be e-small in M. But if M is a coatomic module, i.e. every proper submodule of M is contained in a maximal submodule in M, then $\mathrm{Rad}_e(M)$ is e-small in M using Lemma 1.5(2).

Lemma 1.7: [5] If $M = \oplus_{i \in I} M_i$ then $\mathrm{Rad}_e(M) = \oplus_{i \in I} \mathrm{Rad}_e(M_i)$.

2. On Small Multiplication Module

Over the past decades, multiplication modules besides their diverse generalizations have received considerable attention, with their properties receiving thorough investigation. Given the central role that small submodules play in module theory, and the significance of multiplication modules in this context, this study seeks to propose a novel generalization of multiplication modules through the framework of small submodules.

Definition 2.1: [9] Let M be an R-module. M is called small multiplication module if, for every $N \ll M$, there exists ideal I in R such that $N = IM$.

Example 2.2 [9]
1. Suppose M is a module with $\mathrm{Rad}(M) = 0$. Then M is clearly a small multiplication module. Consequently, every semisimple module satisfies the small multiplication condition. In particular, all modules over a V-ring are small multiplication modules.
2. Consider $M = \mathbb{Q}$ as a $\mathbb{Z}$-module and N a finitely generated submodule in M, with $N \ll M$ it is established that $N \neq M$. Assume there exists an ideal $n\mathbb{Z}$ to $\mathbb{Z}$ such that $N = n\mathbb{Q}$. This leads to $N = M$, contradicting the initial condition. Hence, M is not a small multiplication module. In general, an injective R-module over a principal ideal domain with non-zero radical, $\mathrm{Rad}(M) \neq 0$ cannot be small multiplication as each $x \in \mathbb{R}$, $xM = M$.
3. Let the $\mathbb{Z}$-module $M = \mathbb{Z}_n$ where n is not a square-free natural number. Now all$N \leq M$, there is an integer $x \in \mathbb{Z}$ in which $N = x\mathbb{Z}M$. Therefore M is small multiplication.
4. A module M is consider hollow if, each proper submodule in M is small of M. For hollow modules, the notions of multiplication module and small multiplication are equivalent. However, this equivalence does not extend to all modules, For

instance, some injective $\mathbb{Z}$-module may fail to meet the criteria of being either multiplication or small multiplication modules (as demonstrated earlier in 2). The following example highlights a broader category of modules that fulfill the small multiplication condition but do not satisfy the requirements of a full multiplication module.

Example 2.3: [9] Every semi simple module M over local ring R can be considered a small multiplication even though M is not a multiplication module. Since M is semisimple, it can be written as $M \cong \bigoplus_{i\in I} R/J(R)$ for some index set I. Now, assume there exists a nontrivial submodule $N \subseteq M$. when M is multiplication, then there would be some ideal $I \subseteq R$ where $K = I(\bigoplus_{i\in I} R/J(R)) \cong N$. Then $K = \bigoplus_{i\in I} I(R/J(R)) = \bigoplus_{i\in I} (IR + J(R))/J(R) = \bigoplus_{i\in I} J(R)/J(R) = 0$.
Which is a contradiction, because R is a local and $I \subseteq J(R)$. As a result, we observe that M qualifies as a small multiplication module, as illustrated in Ex 2.2.

Lemma 2.4: [9] Presume M is a Noetherian module in which for each cyclic submodule Rx an ideal I of R exists in which $Rx = IM$. Now M is small multiplication. Consequently, all finitely generated $\mathbb{Z}$-module is small multiplication.

Proposition 2.5: [9] The next conditions are equivalent for a module M,
- M is small multiplication;
- For each $N \ll M$ we have $N \subseteq (N:M)M$;
- For each $N \ll M$ we have $N = (N:M)M = \mathrm{Ann}_R(M/N) \cdot M$.

Proposition 2.6: [9] The next conditions are equivalent for a left module M over a ring R**,**
1. M is small multiplication;
2. For every ideal I of R in which $I \subseteq \mathrm{Ann}_R(M)$, the R/I-module M is small multiplication;
3. There is an ideal I in R in which $I \subseteq \mathrm{Ann}_R(M)$ besides M is a small multiplication R/I-module.

Let R be a ring and M be an R-module. Recall that $L \leq M$ is called fully invariant provided $\varphi(L) \subseteq L$ for each endomorphism φ in M. Clearly 0 and M are fully invariant submodules in M. Note that if R is a commutative ring, then all submodule of an R-module R is fully invariant.

Proposition 2.7. [9] Assume M is a small multiplication module and $N \ll M$. Then M/N is small multiplication.

Proposition 2.8: [9] Assume M is a small multiplication module. Then,
1. Each small submodule to M is a fully invariant submodule of M,
2. Assume $N \leq M$ in which $N \cap IM = IN$ for each ideal I of R. Now, N is a small multiplication module.

Suppose M is a small multiplication which does not possess any nontrivial fully invariant submodule. Then as stated in Proposition 2.8, the zero submodule is the only small submodule of M. As a result, we have $\mathrm{Rad}(M) = 0$.

Corollary 2.9: [9] Each direct summand of small multiplication module, inherits the property.

Proposition 2.10. [9] Assume M is a small multiplication R-module and P is an ideal to R in which $M \neq PM$. If every cyclic $Y \leq M$ we have $\mathrm{Rad}(Y) \neq 0$, then there is a cyclic $X \leq M$ such that $P \nsubseteq \mathrm{Ann}_R(M/X)$.

Proposition 2.11: [9] For a module M, the following conditions are equivalent;
1. M is small multiplication;
2. For each cyclic small $X \leq M$,there is an ideal B of R in which $X = BM$;
3. For each small $X \leq M$ there is set $\{X_i\}_{i\in I} \leq X$ and a set $\{B_i\}_{i\in I}$ of ideals of R such that $X = \sum_{i\in I} X_i$ and $X_i = B_i M$ for each $i \in I$.

The next presents an easy characterization of small multiplication modules.

Lemma 2.12: [9] Presume M is an R-module. ThenM is small multiplication if and only if for each small $N \leq M$ and each $m \in M$, there is an ideal I of R such that $N \cap Rm = IM$.

Remark 2.13: [9] Presume R is ring with just one non-trivial ideal I. Presume M is small multiplication module besides presume N is small submodule to M. Choose any nonzero element $x \in N$, then the submodule Rx is small in M. Given that R has only one nontrivial ideal and M is small multiplication module, we have $Rx = IM = N$. This implies that N must be a simple submodule of M . Therefore, over a ring like $R = \mathbb{Z}_{P^2}$, every small submodule of a small multiplication is simple.

Proposition 2.14. [9] Let M be an Artinian small multiplication module, and suppose J is the Jacobson radical of R. Then the following assertions hold.
1. If $JM \ll M$, then the module M/JM is a semisimple small multiplication module.
2. If $JM \ll M$ and M/MJ is cyclic, then M is a cyclic module.
3. If M is a finitely generated module, then M is a cyclic module.

3. e-Small Multiplication Modules

In this section, we aim to investigate the connection between multiplication modules besides g-small submodules. Our focus is on e-small submodules as a generalization, rather than relying on classical small submodules. Inspired by this perspective, we propose a new approach to generalizing multiplication modules through the concept of e-small submodules.

Definition 3.1: A module M is called e-small multiplication module if for every e-small submodule N of M, there exists an ideal I of R such that $N = IM$.
It is obvious that every e-small multiplication module is small multiplication. Besides, by [9], every δ-small multiplication module is small multiplication. Hence, we have the following hierarchies:

Multiplication module ⇒ e-Small multiplication ⇒ δ-Small multiplication ⇒Small multiplication

Example 3.2

- Let M be a module with $\mathrm{Rad}_e(M) = 0$. Then the only e-small submodule of M is zero submodule and we have $0 = 0M$. Hence M is e-small multiplication.
- Let M be an e-hollow module, i.e. every proper submodule of M is e-small in M. Then M is an e-small multiplication module if and only if M is multiplication. In particular, for a hollow module (and e-hollow module), the three concepts e-small multiplication, small multiplication and multiplication coincide.
- Let M be projective and singular module, we claim that M is an e-small multiplication. This equivalence arises from two facts: first, in projective modules, e-small submodules are necessarily δ-small (as established in [14]). Second, the singularity of M ensures that the distinction between e-small and δ-small submodules vanishes. Consequently, verifying the e-small multiplication condition request to checking the small multiplication condition in this context [9].
- Consider an injective (divisible) R-module M, where R is a principal ideal domain and $\mathrm{Rad}_e(M) \neq 0$. In this scenario, M does not satisfy the e-small multiplication property. Since for every element $x \in R$, we have $xM = M$.

  An example is given of a small multiplication module that fails to be g-small multiplication, presentation that the collection to small multiplicati0n modules strictly includes all e-small multiplication modules.

Example 3.3: Suppose that F is a field. Set $M = \bigoplus_{i=1}^{\infty} F$. Then M is a semisimple projective F-module and$\mathrm{Rad}(M) = 0$. Therefore, M is a small multiplication F-module. Now, take $N = F \oplus F \leq M$. It is straightforward to confirm that N is an e-small submodule of M. However, there exists no ideal I of F such that $N = IM$, then $N = 0$ or $N = M$ that both of which are contradictions. Thus, we conclude that M is not e-small multiplication.

Lemma 3.4: Let M be an R-module. Then M is e-small multiplication if and only if for every e-small submodule N of M and each $m \in M$, there is an ideal I of R such that $N \cap Rm = IM$.

Proof: Assume M is e-small multiplication, $N \ll_e M$ and$m \in M$. Then $N \cap Rm$ is e-small submodule of both M and$N$. Consequently, there exists an ideal I of R such that $N \cap Rm = IM$. Conversely, assume N is an e-small submodule of M. Now for every $y \in N$, There exists an ideal I tof R such that $Ry = I_y M$. Hence, we can write $N = \sum_{y \in N} Ry = IM$ where I is an ideal of R which is the sum of all ideals I_y. □

Proposition 3.5: For a module M, the following conditions are equivalent:

1. M is e-small multiplication;
2. For each e-small submodule N of M, we have $N \subseteq (N: M)M$;
3. For each e-small submodule N of M, we have $N = (N: M)M = \mathrm{Ann}_R(M/N)M$.

Proof: (1) ⇒ (2) Since M is e-small multiplication then for every e-small submodule N of M there exist an ideal I of R such that $N = IM$. It follows that $I \subseteq (N: M)$. Therefore, $N = IM \subseteq (N: M)M$.
(2) ⇒ (3) Let N is e-small submodule of M and $x = \sum_{i=1}^{n} r_i x_i \in (N: M)M$ such that $x_i \in M$ and $r_i \in (N: M)$. As $r_i \in (N: M)$, we conclude that $r_i x_i \in N$. It follows that x belongs to N. Thus $N = (N: M)M$. Note also that $(N: M) = \mathrm{Ann}_R(M/N)$.

(3) ⇒ (1) Take $(N: M)$ as the ideal related to N. □

Let M be a module $N \leq K \leq M$. If K/N is an e-small submodule in M/N and $N \leq_e M$, then it can be easily shown that $K \leq_e M$.

Proposition 3.6: Suppose M is a e-small multiplication module and N is an e-small submodule in M. Then M/N is e-small multiplication.

Proof: Assume M is a g-small multiplication besides N g-small submo0dule to M. Consider K/N as an e-small submodule of M/N where $K \leq M$ and $N \leq K$. Since $N \ll_e M$, this means $K \ll_e M$. As well. Because. M an e-small multiplication module, there must exist an ideal I to R in which $K = IM$. One can easily verify that
$K/N = I(M/N)$. As a result M/N is e-small multiplication. □

Proposition 3.7: Suppose that M is an e-small multiplication module. Then the following hold.

1. Every e-small submodule of M is a fully invariant submodule of M.
2. Let N be a submodule of M such that $N \cap IM = IN$ for every ideal I of R. Then N is an e-small multiplication module. For example, every direct summand of an e-small multiplication module, inherits the property.

Proof

1. Suppose $N \leq_e M$ and φ be an endomorphism of M. Since M satisfies the e-small multiplication property, there exists an ideal I in R in which $N = IM$. It follows that $\varphi(N) = \varphi(IM) = I\varphi(M) \subseteq IM = N$. This shows that N is fully invariant .
2. Suppose $N \leq M$ in which $N \cap IM = IN$ for every ideal I of R. Let K be e-small submodule to N. Note that K is an e-small submodule in M, too. Given that M has the e-small multiplication, there is an ideal I to R in which $K = IM$. Consequently, $K = IM = K \cap IM \subseteq N \cap IM = IN \subseteq IM = K$. In fact, $K = IN$, which completes the argument. For the latter, set $M = N \oplus N'$ where $N' \leq M$. Now for every ideal I of R, $IM = IN \oplus IN'$. It follows that $N \cap IM = N \cap (IN \oplus IN')$, which implies $N \cap IM = IN$. Thus, N is e-small multiplication. □

Let M be a module with no nontrivial fully invariant submodule. If M is an e-small multiplication module, then either $\mathrm{Rad}_e(M) = 0$ or M is a semisimple projective module. For, if N is an e-small submodule of M, then $N = 0$ or $N = M$. coincide with the entire module, $N = M$. Observe that the condition $M \ll_e M$ leads to M being semisimple projective. In particular, the $\mathbb{Z}$-module $\mathbb{Q}$ can not be e-small multiplication note that $\mathbb{Q}$ is neither semisimple projective nor $\mathrm{Rad}_e(\mathbb{Q}) = 0$. It is clear that $\mathrm{Rad}_e(\mathbb{Q}) = \mathrm{Rad}(\mathbb{Q}) = \mathbb{Q}$.

Proposition 3.8: For an R-module M, the following conditions are equivalent;

1. M is e-small multiplication;
2. For every cyclic e-small submodule X of M, there exists an ideal B of R such that $X = BM$;
3. For every e-small submodule X of M, there exists a set $\{X_i\}_{i\in I}$ of submodules of X and a set $\{B_i\}_{i\in I}$ of ideals of R such that $X = \sum_{i\in I} X_i$ and $X_i = B_i M$ for each $i \in I$.

Proof: The implication (1)⟹(2) is clear.

(2) ⟹(3) Presume X is e-small submodule of M. Then each cyclic submodule of X is e-small submodule of M. Note that $X = \sum_{y\in x} Ry$. By (2), for each $y \in X$, there exists an ideal B_y of R such that $Ry = B_y M$. This completes the proof.

(3) ⟹(1) Think through an arbitrary e-small submodule X of M and a set $\{B_i\}_{i\in I}$ of ideals of R such that $X = \sum_{i\in I} X_i$ and $X_i = B_i M$ for each $i \in I$. We denote by B the sum of all ideals B_i of R. Then $X = \sum_{i\in I} X_i = \sum_{i\in I} B_i M = (\sum_{i\in I} B_i)M = BM$. Thus, M is e-small multiplication ϒ.

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