E-ISSN: 2709-9407
P-ISSN: 2709-9393
Impact Factor (RJIF): 5.94
JMPES 2025; 6(2): 516-523

www.mathematicaljournal.com
Received: 13-07-2025
Accepted: 16-08-2025

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Delay differential equations of fourth order with middle term: Oscillation properties with sublinear neutral term

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DOI: https://www.doi.org/10.22271/math.2025.v6.i2c.252

Abstract
The aim of this paper is to derive the oscillation properties of fourth order delay differential equation with middle term of the form $(a(t)((m(t)x(t)+p(t)x^{\alpha}(\tau(t)))''')^{\gamma})' + r(t)((m(t)x(t)+p(t)x^{\alpha}(\tau(t)))''')^{\gamma} + q(t)x^{\beta}(\sigma(t)) = 0$ (1) is considered. By applying the Riccati Transformation technique and new comparison principles, we establish sufficient conditions for the oscillation of the equation is obtained under the condition that $R(t) = \int_{t_0}^{\infty} \frac{1}{a(s)} \exp(-\int_{t_0}^{s} \frac{r(u)}{a(u)} du) ds = \infty$ as $t \to \infty$ Also, the results are an extension and simplification as well as improvement of the previous results.

Keywords: Oscillation, fourth order, neutral differential equation, sublinear neutral term

Introduction

In this paper we study the delay differential equations of fourth order with middle term of the type$(a(t)((m(t)x(t)+p(t)x^{\alpha}(\tau(t)))''')^{\gamma})' + r(t)((m(t)x(t)+p(t)x^{\alpha}(\tau(t)))''')^{\gamma} + q(t)x^{\beta}(\sigma(t)) = 0.$

In the last few decades the use of fourth order delay differential equations has been considered to describe many Real-life applications, such as models related to Biological, Chemical and Physical phenomena see (Osama Moaaz, et.al [7] and Waed Muhsin et.al. [16]).

Many Researchers have focused on understanding the oscillatory behavior of various differential equations of different orders and have provided advanced technologies for obtaining oscillation criteria for fourth and higher order differential equations. We can refer to (Li T. et.al [4], Tunc C. et.al [15], Zhang C. et.al. [18]) where the Authors extended the understanding of the oscillation of a class of nonlinear delay equations.

Oscillatory behavior of second order differential equations is extensively studied by (El-Gabera A.A [2], Meraa Arab [6]) and also Oscillatory behavior of Third order differential equations is extensively studied by (Zuhur Alqahtani et.al. [17], Sai Kumar P.V.H.S et.al. [12]), and for the Oscillatory behavior of Fourth order we refer to (Clemente Cesarano et.al. [1], Mohamed Mazen et.al [5], Omar Bazighifan [8], Omar Bazighifan [9], Sai Kumar P.V.H.S [10]) and for the Oscillation of fifth order we refer to (Sai Kumar P.V.H.S [11].).

E. M. Elabbasy, O. Moaaz [3] studied Asymptotic behavior of third order nonlinear functional differential equations with middle term of the form

$$(r(l)x''(l))' + p(l)x''(l) + \sum_{i=1}^{n} q_i(l)f(x(g_i(l))) = 0$$

and

$$(r(l)x''(l))' + \emptyset(t, x'(l)) + \sum_{i=1}^{n} q_i(l)f(x(g_i(l))) = 0$$

under the condition that

$$\int_{l_0}^{\infty} \frac{1}{r(l)} dl = \infty.$$

M. Sathish Kumar et.al. [13] studied the Qualitative behavior of third order nonlinear differential equations with several delays described by

$$(r_1(t)((r_2(t)x'(t))')^{\gamma})' + p(t)((m_2(t)x'(t))')^{\gamma} + \sum_{i=1}^{n} q_i(t)f(x(g_i(t))) = 0$$

Using the integral average and generalized Riccati techniques, new necessary criteria for the oscillation of equation solutions are established.
Grace [14] investigated Oscillation criteria for third order nonlinear delay differential equations with damping of the form

$$\left(m_2(t)(m_1(t)\left(x'(t)\right)'\right)' + p(t)x'(t) + q(t)f(x(g(t))) = 0$$

under the assumption

$$\int_{t_1}^{t} \frac{1}{m_i(s)} ds = \infty \text{ for } i = 1,2.$$

Zhang et.al. [18] examined the oscillatory properties of the higher order differential equation of equation

$$(\beta(t)(x^{n-1}(t))^k)' + \alpha(t)x^k(\delta(t)) = 0$$

under the condition that

$$\int_{t_0}^{\infty} \frac{1}{\beta(t)} dt < \infty.$$

By a solution of equation (1) we mean a function $x(t) \in C([T_x,\infty))$, $T_x \geq t_0$ which has the properties $z(t) \in C'\left([T_x,\infty)\right), a(t)\left(z'''(t)\right)^{\gamma} \in C'\left(([T_x,\infty)\right)$ and satisfies equation (1) on $([T_x,\infty))$.We consider only those solutions x of equation (1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$, and assume that the equation (1) possesses such solutions. As usual, a solution of equation (1) is called oscillatory if it has a zero on $[T,\infty)$ for all $T \geq T_x$; otherwise it is called nonoscillatory. If all solutions of a differential equations are oscillatory, then the equation itself is called oscillatory

2. Method

In this paper we use few Lemmas and Inequality (16) which are helpful to prove our results by applying the Riccati Transformation technique.

3. Main Results and Discussion

We need the following in our discussion
(H_1): $0 < \alpha \leq 1, \beta$ and γ are ratios of odd natural numbers, $m(t)$ is a real valued continuous function.

(H_2): $a \in C'[t_0,\infty),(0,\infty)), a'(t) \geq 0, p,q,r \in C[t_0,\infty),(0,\infty)), \lim_{t\to\infty} p(t) = 0$ and

$q(t) > 0, \tau(t) \leq t,\ \tau'(t) \geq 0$ and $\sigma(t) \leq t, \sigma'(t) > 0, \lim_{t\to\infty} \tau(t) = \infty.$

(H_3): We define $R(t) = \int_{t_0}^{\infty} \frac{1}{a(s)} \exp(-\int_{t_0}^{s} \frac{r(u)}{a(u)} du) ds = \infty\ as\ t \to \infty$. (2)

Lemma 3.1[1]. Suppose that $\sigma \in C^n([v_0,\mathbb{R}^+)$, where $\rho^{(n)}(v)$ has a constant sign and is non-zero on $[v_0,\infty)$. Additionally, suppose that there is $v_1 \geq v_0$ such that $\rho^{n-1)}(v)\rho^n(v) \leq 0$ for every $v \geq v_1$. If $\lim_{v\to\infty} \rho(v) \neq 0$, then for any $\delta \in (0,1)$, there is $v_\epsilon \in [v_1,\infty)$ such that $\rho(v) \geq \frac{\epsilon}{(n-1)!} v^{n-1}|\rho^{(n-1)}(v)|$, for $\rho \in [v_\epsilon,\infty)$.

Lemma 3.2 [1] Let $\rho \in C^n\left([v_0,\infty),(0,\infty)\right), \rho^{(i)}(v) > 0$ for $i = 1,2,\ldots..n,$ and $\rho^{(n+1)}(v) \leq 0$, eventually. Then, eventually, $\frac{\rho(v)}{\rho'(v)} \geq \frac{\epsilon v}{n}$ for every $\epsilon \in (0,1)$.

Lemma 3.3. Assume that $x(t)$ is an eventually positive solution of (1). Then, $x(t)$ eventually satisfies the following cases

C_1: $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) > 0, (a(t)(z'''(t))^{\gamma})' < 0,$

C_2: $z(t) > 0, z'(t) > 0, z''(t) < 0, z'''(t) > 0, (a(t)(z'''(t))^\gamma)' < 0,$

Theorem 3.1: Assume that (2) holds. If $\beta \geq \gamma$ and there is a nondecreasing function $\rho \in C'([t_0,\infty),(0,\infty))$ such that

$$\lim_{t\to\infty} sup \int_{t_0}^{t} \left(\begin{array}{c} \rho(s)\left(q(s)\frac{1}{m^\beta(\sigma(s))}\left(1-\frac{p(\sigma(s))}{{c_1}^{1-\alpha}}\right)^\beta\right) \\ -\frac{2^\gamma \rho(s) a(\sigma(s))}{(\gamma+1)^{\gamma+1}{c_2}^{\beta-\gamma}(\epsilon\sigma^2(s)\sigma'(s))^\gamma}\left[\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right]^{\gamma+1} \end{array} \right) ds = \infty \quad (3)$$

holds for every $c_1, c_2 > 0$, then (1) is oscillatory.
Proof: We assume for contradiction that (1) has an eventually positive solution of $x(t)$. Set

$$z(t) = m(t)x(t) + p(t)x^\alpha(\tau(t)) \quad (4)$$

Then $z(t) \geq x(t)$. By (1) and (2), we obtain that for $t_1 \geq t_0$

$$z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) > 0, (a(t)(z'''(t))^\gamma)' \leq 0, t \geq t_1. \quad (5)$$

Since $\sigma(t) \leq t$, then we have from (5) that

$$a(t)(z'''(t))^\gamma \leq a(\sigma(t))\left(z'''(\sigma(t))\right)^\gamma, t \geq t_1 \quad (6)$$

Since that $z'(t) > 0$. Hence there exists a constant $c_1 > 0$ such that $z(t) \geq c_1$ for all t large enough.
By (5), one gets

$$m(t)x(t) \geq z(t) - p(t)z^\alpha(\tau(t))$$

$$x(t) \geq \frac{1}{m(t)}\left(1-\frac{p(t)}{{c_1}^{1-\alpha}}\right)z(t) \ (7)$$

Then from equation (1), we have

$$(a(t)((m(t)x(t)+p(t)x^\alpha(\tau(t)))''')^\gamma)' + r(t)((m(t)x(t)+p(t)x^\alpha(\tau(t)))''')^\gamma = -q(t)x^\beta(\sigma(t)) \quad (8)$$

Define

$$\omega(t) = \rho(t)\frac{a(t)(z'''(t))^\gamma}{z^\beta(\sigma(t))}, t \geq t_1 \ (9)$$

$$\omega'(t) = \rho'(t)\frac{a(t)(z'''(t))^\gamma}{z^\beta(\sigma(t))} + \rho(t)\frac{\left(a(t)(z'''(t))^\gamma\right)'}{z^\beta(\sigma(t))} - \beta\rho(t)\frac{a(t)(z'''(t))^\gamma z'(\sigma(t))\sigma'(t)}{z^{\beta+1}(\sigma(t))}$$

$$\omega'(t) = \rho'(t)\frac{a(t)(z'''(t))^\gamma}{z^\beta(\sigma(t))} + \rho(t)\left[\frac{-r(t)(z'''(t))^\gamma - q(t)x^\beta(\sigma(t))}{z^\beta(\sigma(t))}\right] - \beta\rho(t)\frac{a(t)(z'''(t))^\gamma z'(\sigma(t))\sigma'(t)}{z^{\beta+1}(\sigma(t))}$$

$$\omega'(t) = \rho'(t)\frac{a(t)(z'''(t))^\gamma}{z^\beta(\sigma(t))} - \rho(t)\frac{r(t)(z'''(t))^\gamma}{z^\beta(\sigma(t))} - \rho(t)\frac{q(t)x^\beta(\sigma(t))}{z^\beta(\sigma(t))} - \beta\rho(t)\frac{a(t)(z'''(t))^\gamma z'(\sigma(t))\sigma'(t)}{z^{\beta+1}(\sigma(t))}$$

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{r(t)}{a(t)}\omega(t) - \rho(t)\frac{q(t)x^\beta(\sigma(t))}{z^\beta(\sigma(t))} - \beta\sigma'(t)\omega(t)\frac{z'(\sigma(t))}{z(\sigma(t))} (10)$$

We see from (7), (8), (9) and (10) we obtain

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{r(t)}{a(t)}\omega(t) - \rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^\beta\right) - \beta\sigma'(t)\omega(t)\frac{z'(\sigma(t))}{z(\sigma(t))}. (11)$$

Since $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) > 0$, and, $(a(t)(z'''(t))^\gamma)' < 0$ according to Lemma 3.1[1], we can deduce that $z'(t) \geq \frac{\epsilon}{2}t^2 z'''(t)$

and

$z'(\sigma(t)) \geq \frac{\epsilon}{2}\sigma^2(t)z'''(\sigma(t))$ (12)

for all $\in \epsilon(0,1)$ and every sufficiently large t. Substituting (12) into (11), we obtain

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{r(t)}{a(t)}\omega(t) - \rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1 - \frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^\beta\right) - \frac{\epsilon}{2}\beta\sigma^2(t)\sigma'(t)\frac{z'''(\sigma(t))}{z(\sigma(t))}\omega(t)$$

$$\omega'(t) \leq -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1 - \frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^\beta\right) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) \;-\frac{\epsilon}{2}\beta\frac{\sigma^2(t)\sigma'(t)}{a^{\frac{1}{\gamma}}(\sigma(t))}\frac{a^{\frac{1}{\gamma}}(\sigma(t))z'''(\sigma(t))}{z(\sigma(t))}\omega(t).$$

Since $(a(t)(z''')^\gamma(t))' < 0$, we conclude that

$$a^{\frac{1}{\gamma}}(t)z'''(t) \leq a^{\frac{1}{\gamma}}(\sigma(t))z'''(\sigma(t)).$$

Then,

$$\omega'(t) \leq -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1 - \frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^\beta\right) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) \;-\frac{\epsilon\beta}{2}\frac{\sigma^2(t)\sigma'(t)}{a^{\frac{1}{\gamma}}(\sigma(t))}\omega(t)\frac{a^{\frac{1}{\gamma}}(t)z'''(t)}{z(\sigma(t)}.$$

$$\omega'(t) \leq -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1 - \frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^\beta\right) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) \;-\frac{\epsilon\beta}{2}\frac{\sigma^2(t)\sigma'(t)}{(\rho(t)a(\sigma(t))^{\frac{1}{\gamma}}}\left(z(\sigma(t)\right)^{\frac{\beta-\gamma}{\gamma}}\omega^{\frac{\gamma+1}{\gamma}}(t). \quad (13)$$

Because $z'(t) > 0$ and $\beta \geq \gamma$, there exists constants $c_2 > 0$ and $t_2 \geq t_1$ such that

$z(\sigma(t)) \geq c_2$

$z^{\frac{\beta-\gamma}{\gamma}}(\sigma(t)) \geq {c_2}^{\frac{\beta-\gamma}{\gamma}}, t \geq t_2$. (14)

Substituting the inequality (14) in (13) gives

$$\omega'(t) \leq -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1 - \frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^\beta\right) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)\omega(t) \;-\frac{\epsilon\gamma {c_2}^{\frac{\beta-\gamma}{\gamma}}}{2}\frac{\sigma^2(t)\sigma'(t)}{(\rho(t)a(\sigma(t))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t). \quad (15)$$

Using the following inequality in (15),

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma}, \quad (16)$$

where $A > 0, B \geq 0, \gamma > 0$ with

$$B = \frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}, A = \frac{\epsilon\gamma {c_2}^{\frac{\beta-\gamma}{\gamma}}\sigma^2(t)\sigma'(t)}{2(\rho(t)a(\sigma(t))^{\frac{1}{\gamma}}} \text{ and } u(t) = \omega(t)$$

We get

$$\omega'(t) \leq -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1 - \frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^\beta\right) + \frac{2^\gamma\rho(t)a(\sigma(t))}{(\gamma+1)^{\gamma+1}{c_2}^{\beta-\gamma}(\epsilon\sigma^2(t)\sigma'(t))^\gamma}\left[\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right]^{\gamma+1}$$
(17)

Integrating (17) from $t_3 \geq t_2$ to t, we obtain

$$\int_{t_3}^{t}\left(\rho(s)\left(q(s)\frac{1}{m^\beta(\sigma(s))}\left(1 - \frac{p(\sigma(s))}{{c_1}^{1-\alpha}}\right)^\beta\right) - \frac{2^\gamma\rho(s)a(\sigma(s))}{(\gamma+1)^{\gamma+1}{c_2}^{\beta-\gamma}(\epsilon\sigma^2(s)\sigma'(s))^\gamma}\left[\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right]^{\gamma+1}\right) ds \leq \omega(t_3)$$

This is a contradiction to (3) as $t \to \infty$. Thus the proof is completed.

Theorem 3.2: Assume that (2) holds. If $\beta \geq \gamma$ and there is a nondecreasing function $\rho_1 \in C'([t_0,\infty),(0,\infty))$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\begin{array}{c} M_1^{\beta-\gamma}\rho_1(s)\left(q(s)\dfrac{1}{m^{\beta}(\sigma(s))}\left(1-\dfrac{p(\sigma(s))}{{c_1}^{1-\alpha}}\right)^{\beta}\right)\left(\dfrac{\sigma^3(s)}{s}\right)^{\frac{3\beta}{\epsilon}} \\ -\dfrac{2^{\gamma}}{(\gamma+1)^{\gamma+1}}\dfrac{(a(t)\rho_1(t))}{(\epsilon t^2)^{\gamma}}\left[\dfrac{\rho_1'(t)}{\rho_1(t)}-\dfrac{r(t)}{a(t)}\right]^{\gamma+1} \end{array} \right) ds = \infty.$$

(18)

holds for every $c_1, c_2 > 0$, then (1) is oscillatory.

Proof. We suppose for contradiction that (1) has an eventually positive solution. Now we define a function

$$\omega_1(t) = \rho_1(t)\frac{a(t)(z'''(t))^{\gamma}}{z^{\gamma}(t)} \quad (19)$$

which yields $\omega_1(t) > 0$, and

$$\omega_1'(t) = \rho_1'(t)\frac{a(t)(z'''(t))^{\gamma}}{z^{\gamma}(t)} + \rho_1(t)\frac{\left(a(t)(z'''(t))^{\gamma}\right)'}{z^{\gamma}(t)} - \gamma\rho_1(t)\frac{a(t)(z'''(t))^{\gamma}z'(t)}{z^{\gamma+1}(t)}$$

$$\omega_1'(t) = \rho_1'(t)\frac{a(t)(z'''(t))^{\gamma}}{z^{\gamma}(t)} + \rho_1(t)\left[\frac{-r(t)(z'''(t))^{\gamma} - q(t)x^{\beta}(\sigma(t))}{z^{\gamma}(t)}\right] - \gamma\rho_1(t)\frac{a(t)(z'''(t))^{\gamma}z'(t)}{z^{\gamma+1}(t)}$$

$$\omega_1'(t) = \rho_1'(t)\frac{a(t)(z'''(t))^{\gamma}}{z^{\gamma}(t)} - \rho_1(t)\frac{r(t)(z'''(t))^{\gamma}}{z^{\gamma}(t)} - \rho_1(t)\frac{q(t)x^{\beta}(\sigma(t))}{z^{\gamma}(t)} - \gamma\rho_1(t)\frac{a(t)(z'''(t))^{\gamma}z'(t)}{z^{\gamma+1}(t)}$$

$$\omega_1'(t) = \frac{\rho_1'(t)}{\rho_1(t)}\omega_1(t) - \frac{r(t)}{a(t)}\omega_1(t) - \rho_1(t)\frac{q(t)x^{\beta}(\sigma(t))}{z^{\gamma}(t)} - \gamma\omega_1(t)\frac{z'(t)}{z(t)}$$

(20)

From (7), (8), (19) and (20) that

$$\omega_1'(t) \leq -\rho_1(t)\left(q(t)\frac{1}{m^{\beta}(\sigma(t))}\left(1-\frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^{\beta}\right)\frac{z^{\beta}(\sigma(t))}{z^{\gamma}(t)} + \left[\frac{\rho_1'(t)}{\rho_1(t)} - \frac{r(t)}{a(t)}\right]\omega_1(t) - \gamma\frac{z'(t)}{z(t)}\omega_1(t).$$

(21)

We deduce from **Lemma 3.2** [1], that

$$z(t) \geq \frac{\epsilon}{3}tz'(t),$$

and hence,

$$\frac{z(\sigma(t))}{z(t)} \geq \left(\frac{\sigma^3(t)}{t}\right)^{\frac{3}{\epsilon}} \quad (22)$$

From Lemma 3.1[1], we conclude that

$$z'(t) \geq \frac{\epsilon}{2}t^2 z'''(t) \quad (23)$$

for all $\epsilon \in (0,1)$. Thus, by (21), (22) and (23), we have

$$\omega_1'(t) \leq -\rho_1(t)\left(q(t)\frac{1}{m^{\beta}(\sigma(t))}\left(1-\frac{p(\sigma(t))}{{c_1}^{1-\alpha}}\right)^{\beta}\right)z^{\beta-\gamma}(t)\frac{z^{\beta}(\sigma(t))}{z^{\beta}(t)} + \left[\frac{\rho_1'(t)}{\rho_1(t)} - \frac{r(t)}{a(t)}\right]\omega_1(t) - \frac{\epsilon\gamma}{2}t^2\frac{z'''(t)}{z(t)}\omega_1(t).$$

$$\omega_1'(t) = -\rho_1(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)z^{\beta-\gamma}(t)\left(\frac{\sigma^3(t)}{t}\right)^{\frac{3\beta}{\epsilon}} + \left[\frac{\rho_1'(t)}{\rho_1(t)}-\frac{r(t)}{a(t)}\right]\omega_1(t) - \frac{\epsilon\gamma}{2}\frac{t^2}{(a(t)\rho_1(t))^{\frac{1}{\gamma}}}\,\omega_1{}^{\frac{1+\gamma}{\gamma}}(t).$$

(24)

Since $z'(t) > 0$, there exists a $t_3 \geq t_2$ and a constant $M_1 > 0$ such that

$z(t) > M_1$.

Since $\beta \geq \gamma$, then

$z^{\beta-\gamma}(t) > M_1^{\beta-\gamma}$ (25).

Thus inequality (24) gives

$$\omega_1'(t) = -M_1^{\beta-\gamma}\rho_1(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)\left(\frac{\sigma^3(t)}{t}\right)^{\frac{3\beta}{\epsilon}} + \left[\frac{\rho_1'(t)}{\rho_1(t)}-\frac{r(t)}{a(t)}\right]\omega_1(t) - \frac{\epsilon\gamma t^2}{2(a(t)\rho_1(t))^{\frac{1}{\gamma}}}\omega_1{}^{\frac{1+\gamma}{\gamma}}(t).$$

(26)

Using the inequality (16) with

$B = \left[\frac{\rho_1'(t)}{\rho_1(t)}-\frac{r(t)}{a(t)}\right], A = \frac{\epsilon\gamma t^2}{2(a(t)\rho_1(t))^{\frac{1}{\gamma}}}$ and $u(t) = \omega_1(t)$

we can derive the following inequality

$$\omega_1'(t) = -M_1^{\beta-\gamma}\rho_1(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)\left(\frac{\sigma^3(t)}{t}\right)^{\frac{3\beta}{\epsilon}} + \frac{2^\gamma}{(\gamma+1)^{\gamma+1}}\frac{(a(t)\rho_1(t))}{(\epsilon t^2)^\gamma}\left[\frac{\rho_1'(t)}{\rho_1(t)}-\frac{r(t)}{a(t)}\right]^{\gamma+1}$$

(27)

On integrating (27) from $t_4 \geq t_3$ to t, we get

$$\int_{t_4}^{t}\left(\begin{array}{c} M_1^{\beta-\gamma}\rho_1(s)\left(q(s)\frac{1}{m^\beta(\sigma(s))}\left(1-\frac{p(\sigma(s))}{c_1{}^{1-\alpha}}\right)^\beta\right)\left(\frac{\sigma^3(s)}{s}\right)^{\frac{3\beta}{\epsilon}} \\ -\frac{2^\gamma}{(\gamma+1)^{\gamma+1}}\frac{(a(t)\rho_1(t))}{(\epsilon t^2)^\gamma}\left[\frac{\rho_1'(t)}{\rho_1(t)}-\frac{r(t)}{a(t)}\right]^{\gamma+1}\end{array}\right)ds \leq \omega_1(t_2),$$

which contradicts (18) as $t \to \infty$. Thus the proof is completed.

Theorem 3.3: Suppose that (2) holds. If $0 < \beta < \gamma$ and there is a nondecreasing function $\rho \in C'([t_0,\infty),(0,\infty))$ such that

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\left(\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right) - \frac{2^\beta a(\sigma(t))\rho(t)}{(\beta+1)^{\beta+1}c_3{}^{\beta-\gamma}(\epsilon\sigma^2(t)\sigma'(t))^\beta}\left[\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right]^{\beta+1}\right)ds = \infty.$$

(28)

holds for every $c_1, c_3 > 0$, then (1) is oscillatory.

Proof: We suppose for contradiction that (1) has an eventually positive solution $x(t)$. As in the proof of Theorem 3.1**,** the function $\omega(t)$ is defined as in (9) and then (10) holds

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{r(t)}{a(t)}\omega(t) - \rho(t)\frac{q(t)x^\beta(\sigma(t))}{z^\beta(\sigma(t))} - \beta\sigma'(t)\omega(t)\frac{z'(\sigma(t))}{z(\sigma(t))}$$

By (1), and (7) - (10), we conclude that

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{r(t)}{a(t)}\omega(t) - \rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right) - \beta\sigma'(t)\omega(t)\frac{z'(\sigma(t))}{z(\sigma(t))}.$$

By (12) we observe that

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)+\left(\frac{\rho'(t)}{\rho(t)}-\frac{r(t)}{a(t)}\right)\omega(t) -\frac{\epsilon\beta}{2}\sigma^2(t)\sigma'(t)\left(z'''(t)\right)^{\frac{\beta-\gamma}{\beta}}\frac{\left(z'''(t)\right)^{\frac{\gamma}{\beta}}}{z(\sigma(t))}\omega(t).$$

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)+\left(\frac{\rho'(t)}{\rho(t)}-\frac{r(t)}{a(t)}\right)\omega(t) -\frac{\epsilon\beta}{2}\frac{\sigma^2(t)\sigma'(t)}{(\rho(t)a(t))^{\frac{1}{\beta}}}\left(z'''(t)\right)^{\frac{\beta-\gamma}{\beta}}\omega^{\frac{\beta+1}{\beta}}(t).$$

(29)

Given that $0<\beta<\gamma$ and (C_1) hold and since $a'(t)\ge 0$, it follows that $z''''(t)\le 0$. This implies that $z'''(t)$ is nonincreasing. Then there exist constants $c_3>0$ and $t_3\ge t_2$ such that $z'''(t)\le c_3$.

$(z'''(t))^{\frac{\beta-\gamma}{\beta}} \ge c_3{}^{\frac{\beta-\gamma}{\beta}}, t\ge t_3.$ (30)

From (29) and (30) it follows that

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)+\left(\frac{\rho'(t)}{\rho(t)}-\frac{r(t)}{a(t)}\right)\omega(t) -\frac{\epsilon\beta c_3{}^{\frac{\beta-\gamma}{\beta}}}{2}\frac{\sigma^2(t)\sigma'(t)}{(\rho(t)a(t))^{\frac{1}{\beta}}}\omega^{\frac{\beta+1}{\beta}}(t).$$

(31)

Using the inequality (16) with

$B=\frac{\rho'(t)}{\rho(t)}-\frac{r(t)}{a(t)}, A=\frac{\epsilon\beta c_3{}^{\frac{\beta-\gamma}{\beta}}\sigma^2(t)\sigma'(t)}{2(\rho(t)a(t)^{\frac{1}{\beta}}}$ and $u(t)=\omega(t)$

It can be deduced that from (31) that

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)+\frac{2^\beta a(t)\rho(t)}{(\beta+1)^{\beta+1}c_3{}^{\beta-\gamma}(\epsilon\sigma^2(t)\sigma'(t))^\beta}\left[\frac{\rho'(t)}{\rho(t)}-\frac{r(t)}{a(t)}\right]^{\beta+1}$$

(32)

On integrating (32) over the interval t_5 to t, one arrives at

$$\int_{t_5}^{t}\left(\begin{array}{c}\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta\right)\\ -\frac{2^\beta a(\sigma(t))\rho(t)}{(\beta+1)^{\beta+1}c_3{}^{\beta-\gamma}(\epsilon\sigma^2(t)\sigma'(t))^\beta}\left[\frac{\rho'(s)}{\rho(s)}-\frac{r(s)}{a(s)}\right]^{\beta+1}\end{array}\right)ds \le \omega(t_5)$$

This is a contradiction to (28) as $t\to\infty$. Thus the proof is completed.

4. Conclusion

The goal of this paper is to study the ‘Oscillatory behavior of fourth order differential equations with Middle Term” of equation (1) by using Riccati Transformation technique. Further extension of these results can be used to study a class of system of higher order Neutral differential equations as well as Fractional order equations. Some research in this area is in progress.

The results of this study complement many of previously published findings in the literature. To our knowledge, this equation has not been studied by many researchers, so it would be a good idea to apply these results to nonlinear higher order NDE’s in future.

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