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Further modified forms of extended Bata Function and their Properties

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Abstract
The main object of this paper is to introduce modified forms a new extension of extended beta function and modified forms for the extension of hypergeometric and confluent hypergeometric functions. Some functional relations, integral representations and Mellin transform are obtained for these extended functions, some known and new relations for various forms of extended Beta functions are obtained as special cases of the main results.

Keywords: Beta function, Hypergeometric function, Confluent hypergeometric function, summation formulas, Integral representations, Mellin transform

1. Introduction
The classical Gamma and Beta functions are defined respectively as follows (Srivastava and Manocha, 1984):

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \ (Re(x) > 0). \tag{1}$$

And

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \ (Re(x) > 0, Re(y) > 0\,). \tag{2}$$

For each $x, y \in (0, +\infty)$, The Gamma and Beta functions have the following relation:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3}$$

Many authors have introduced certain extensions for Beta function (Chaudhry *et al*., 1997; Chaudhry *et al*., 2004; Kulib *et al*. 2020; Ozergin *et al*., 2011) [1, 2, 4, 5]. (Chaudhry *et al*., 1997) [1] introduced the following extended Beta function:

$$B(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left(\frac{-p}{t(1-t)}\right) dt \tag{4}$$

$$(Re(x) > 0, Re(y) > 0, Re(p) \geq 0)$$

(Choi *et al*., 2014) [3] presented the following extended Beta function:

$$B(x,y;p;q) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left(\frac{-p}{t} - \frac{q}{(1-t)}\right) dt \tag{5}$$

$$(Re(x) > 0, Re(y) > 0, Re(p) \geq 0), \;\; Re(q) \geq 0)$$

Note that, for $p = q$. (5) reduce to (4) and for $p = q = 0$. (5) reduce to (2). Also, the extended hyper geometric and confluent hyper geometric functions are introduced in (Choi *et al*., 2014) [3] as follows:

$$F_p^q(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_p^q(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!}, \tag{6}$$

Where

$$Re(p) \geq 0, Re(q) \geq 0 \; and \; Re(c) > Re(b) > 0, |z| < 1.$$

$$\phi_p^q(b; c; z) = \sum_{n=0}^{\infty} \frac{B_p^q(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{7}$$

Where

$$Re(p) \geq 0, Re(q) \geq 0 \; and \; Re(c) > Re(b) > 0$$

It is clear that when $p = q = 0$, the equations (6) and (7) reduce to the well-known classical hyper geometric and confluent hyper geometric functions respectively.
Recently, (Saif *et al*., 2020) [7] presented modified Laplace transform as follows:

$$L_a\{f(t)\} = \int_0^{\infty} a^{-st} f(t) dt, \qquad (Re(s) > 0, \;\; a \in (0, \infty)/\{1\}) \tag{8}$$

Which for $a = e$ reduces to the known Laplace transform given as:

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt, \qquad (Re(s) > 0), \tag{9}$$

Very recently, (Kulib *et al*., 2020) [4] presented a new modified forms of the extended Beta function as follows:

$$B_p(x, y; a) = \int_0^1 t^{x-1} (1-t)^{y-1} \; a^{\left(\frac{-p}{t(1-t)}\right)} dt. \tag{10}$$

$$(Re(x) > 0, Re(y) > 0 \,, Re(p) \geq 0, a \in (0, \infty) \setminus \{1\})$$

Motivated by the idea given in (Saif *et al*., 2020) [7], in this paper. We introduce a new modified forms of the extended Beta, hyper geometric and confluent hyper geometric functions defined in equations (6) – (7) respectively.

2. Modified Beta function

In this section, we introduce a new modified forms of extended Beta functions in the following form:

$$B_p^q(x, y; a) = \int_0^1 t^{x-1} (1-t)^{y-1} \; a^{\left(\frac{-p}{t} - \frac{q}{(1-t)}\right)} dt \tag{11}$$

$$(Re(x) > 0, Re(y) > 0 \,, Re(p) \geq 0, Re(p) \geq 0, a \in (0, \infty) \setminus \{1\})$$

Remark 2.1.

1. For $p = q$, definition (11) reduces to a new modified extended Beta function (10).
2. For $p = q = 0$, definition (11) reduces to classical Beta function (2).
3. For $a = e$ definition (11) reduces to extended Beta function (5).
4. Now, we introduce certain properties of the modified forms of the extended Beta function in the form of the following theorem:

Theorem 2.1. The following summation formula for $B_p^q(x, y; a)$ holds true:

$$B_p^q(x, y; a) = \sum_{n.m=0}^{\infty} \frac{p^n\, q^m\, (-loga)^{n+m}}{n!\, m!} \quad B(x - n. y - m) \tag{12}$$

Proof. To prove (12), expanding the R.H.S. of equation (11) by using the following relation:

$$a^x = \sum_{n=0}^{\infty} \frac{x^n\, (loga)^n}{n!}$$

we get

$$B_p^q(x, y; a) = \int_0^{\infty} t^{x-n-1}(1-t)^{y-m-1} \sum_{m.n=0}^{\infty} \frac{p^n\, q^m\, (-loga)^{n+m}}{n!\, m!} \; dt \tag{13}$$

Which on interchanging the order of summation and integration and then using relation (2) yields the desired result (12).

Remark 2.2. Using relations (3) and (13) in the R.H.S. of relation (12) and then applying some properties of Pochhammer symbol, we get the following hypergeometric representation for $B_p^q(x, y; a)$:

$$B_p^q(x, y; a) = B(x, y)F_2(1 - x - y, -, -; 1 - x, 1 - y; -4p\; loga\,, -\, 4q\; loga) \tag{14}$$

Where $F_2\,[.]$ denotes one of the Appell series $F_j(j = 1, 2, 3, 4)$ (Ozergin *et al*., 2011; Rainville, 1971) [5, 6].

Theorem 2.2. The following summation formulas for $B_p(x, y; a)$ hold true:

$$B_p^q(x, 1 - y; a) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^q(x + n, 1; a) \tag{15}$$

$$B_p^q(x, y; a) = \sum_{n=0}^{\infty} B_p^q(x + n, y + 1; a) \tag{16}$$

$$B_p^q(x, y; a) = \sum_{k=0}^{n} \binom{n}{k} B_p^q(x + k, y + n - k; a)\,, \qquad n \in \mathbb{N}_0 \tag{17}$$

Proof. To prove result (15), from (12), we have

$$B_p^q(x, 1-y; a) = \int_0^1 t^{x-1}(1-t)^{-y} \, a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)} \, dt$$

Using the generalized binomial theorem

$$(1-t)^{-y} = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^n, \quad |t| < 1 \tag{18}$$

we obtain

$$B_p^q(x, 1-y; a) = \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^{x+n-1} \, a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)} dt$$

Now interchanging the order of summation and integration in the above equation and using (11), we get the required result (15). The proof of result (16) is similar to that of (15).
To prove of (17), we use the mathematical induction on $(n \in \mathbb{N}_0)$ as follows:
Clearly, For $n = 0$ the equation (17) holds.
For $n = 1$, we have

$$B_p^q(x+1, y; a) + B_p^q(x, y+1; a)$$

$$= \int_0^1 \{t^x(1-t)^{y-1} + t^{x-1}(1-t)^y\} \, a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)} dt$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1}\{t + (1-t)\} \, a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)} dt$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1} \, a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)} dt,$$

$$= B_p^q(x, y; a).$$

Therefore, the equation (17) holds for $n = 1$.
Continuing this process for all $(n \in \mathbb{N}_0)$, we finally obtain the desired relation (17) and thus the proof of Theorem 2.2 is completed.

Theorem 2.3. The following integral formula for $B_p(x, y; a)$ holds true:

$$\int_0^\infty \int_0^\infty p^{r-1} q^{s-1} B_p^q(x, y; a) \, dp \, dq = \Gamma(r; a)\Gamma(s; a) \, B(x+r-1, y+s-1) \tag{19}$$

$$(Re(s) > 0, Re(r) > 0, Re(x+r) > 0, Re(y+s) > 0)$$

Proof. Multiplying each side of (11) by $p^{s-1} \, q^{r-1}$ and integrating the resulting identity with respect to p and q $(0 \le p.q < \infty)$, we obtain

$$\int_0^\infty \int_0^\infty p^{r-1} q^{s-1} B_p^q(x,y;a)\, dp\, dq$$

$$= \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left(\int_0^1 t^{x-1}(1-t)^{y-1} \; a^{\left(\frac{-p}{t} - \frac{q}{(1-t)}\right)} dt \right) dpdq \quad (20)$$

The uniform convergence of the integral in (19) guarantees that the order of the triple integrals in (20) can be interchanged. We therefore, have

$$\int_0^\infty \int_0^\infty p^{r-1} q^{s-1} B_p^q(x,y;a)\, dp\, dq =$$

$$\int_0^1 t^{x-1}\,(1-t)^{y-1} \left(\int_0^\infty p^{r-1}\; a^{\left(\frac{-p}{t}\right)} dp \cdot \int_0^\infty q^{s-1}\, a^{-\frac{q}{(1-t)}}\, dq \right) dt \quad (21)$$

Moreover, the integrals in (21) can be simplified in terms of the Gamma function to prove Theorem 1:

$$\int_0^\infty \int_0^\infty p^{r-1} q^{s-1} B_p^q(x,y;a)\, dp\, dq = \Gamma(r;a)\Gamma(s;a)\, B(x+r, y+s)$$

Remark 2.3. Putting $r = s = 1$ in relation (19), we get the following results:

Corollary 2.1. The following integral formula for $B_p^q\,(x,y;a)$ holds true:

$$\int_0^\infty B_p^q(x,y;a) dp\, dq = \frac{1}{(loga)^2} B(1+x, 1+y) \quad (22)$$

Theorem 2.4. The following integral representations hold true:

$$B_p^q(x,y;a) = 2\int_0^{\frac{\pi}{2}} cos^{2x-1}\theta sin^{2y-1}\,\theta\; a^{\left(\frac{-p}{cos^2\theta} - \frac{q}{sin^2\theta}\right)} d\theta \quad (23)$$

$$B_p^q(x,y;a) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, a^{\left(\frac{-(1+u)(p+qu)}{u}\right)} du \quad (24)$$

$$B_p^q(x,y;a) = (c-a)^{1-x-y}$$

$$\int_a^c (u-a)^{x-1}(c-u)^{y-1} a^{\left\{\frac{c-a}{(u-a)(c-u)}([(p-q)u+(qa-pc)])\right\}}\, du \quad (25)$$

$$B_p^q(x,y;a) = 2^{1-x-y} \int_{-1}^1 (1+u)^{x-1}(1-u)^{y-1} a^{\left\{\frac{2(p+q)+2(q-p)u}{(1-u)^2}\right\}}\, du \quad (26)$$

Proof. To prove result (23), putting $\mathrm{t} = \cos^2\theta$ in (11), we have

$$B_p^q(x,y;a) = 2\int_0^{\frac{\pi}{2}} cos^{2x-2}\theta\; sin^{2y-2}\theta\;\; a^{\left(\frac{-p}{cos^2\theta} - \frac{q}{sin^2\theta}\right)} cos\theta sin\theta d\theta$$

Which yields the desired result (23). Similarly, results (24), (25) and (26) can be proved by taking the transformation $=\frac{u}{1+u}$, $t=\frac{u-a}{c-a}$ and $t=\frac{1+u}{2}$ in (11) respectively. Thus the proof of Theorem 2.4 is completed.

Remark 2.4.

A) For $a=e$ and using relation $B_p^q(x,y;e)=B_p^q(x,y)$, all the above relations related to $B_p^q(x,y;a)$ reduce to the known results due to (Choi *et al*., 2014) [3].

B) For $=q$, all the above relations related to $B_p^q(x,y;a)$ reduce to the known results due to (Kulib *et al*., 2020) [4].

3. Modified hyper geometric functions

In this section, we introduce modified forms for the extension of hyper geometric and confluent hyper geometric functions given in (6) and (17) by using the modified Beta function defined in (12).

Definition 3.1. The modified forms of the extension of hyper geometric and confluent hyper geometric functions are defined as:

$$F_p^q(\alpha,\beta;\gamma;z;a)=\sum_{n=0}^{\infty}(\alpha)_n\frac{B_p^q(\beta+n,\gamma-\beta;a)}{B(\beta,\gamma-\beta)}\frac{z^n}{n!}, \tag{27}$$

$$(Re(\gamma)>Re(\beta)>0, Re(p)\geq 0, Re(q)\geq 0, a\in(0,\infty)\setminus\{1\}, |z|<1)$$

$$\phi_p^q(\beta;\gamma;z;a)=\sum_{n=0}^{\infty}\frac{B_p^q(\beta+n,\gamma-\beta;a)}{B(\beta,\gamma-\beta)}\frac{z^n}{n!} \tag{28}$$

$$(Re(\gamma)>Re(\beta)>0, Re(p)\geq 0, Re(q)\geq 0, a\in(0,\infty)\setminus\{1\}, |z|<1)$$

Remark 3.1. It is clear that, for $a=e$, (27) and (28) reduce to the extended hyper geometric and confluent hyper geometric functions (6) and (7) respectively.

Now, we prove some properties of the modified hyper geometric and confluent hyper geometric functions $F_p^q(\alpha,\beta;\gamma;z;a)$ and $\phi_p^q(\beta;\gamma;z;a)$ in the form of the following theorems:

Theorem 3.1. The modified hyper geometric and confluent hyper geometric functions $F_p^q(\alpha,\beta;\gamma;z;a)$ and $\phi_p^q(\beta;\gamma;z;a)$ have the following integral representations:

$$F_p^q(\alpha,\beta;\gamma;z;a)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\, a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)}\,dt \tag{29}$$

$$(Re(\gamma)>Re(\beta)>0, a\in(0,\infty)\setminus\{1\}, Re(p)\geq 0, Re(q)\geq 0, |arg(1-z)|<\pi)$$

$$\phi_p^q(\beta;\gamma;z;a)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}exp(zt)\; a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)}dt \tag{30}$$

$$(Re(\gamma)>Re(\beta)>0, a\in(0,\infty)\setminus\{1\}, Re(p)\geq 0, Re(q)\geq 0)$$

Proof. Using relation (11) in equation (27), we have

$$F_p^q(\alpha,\beta;\gamma;z;a)$$

$$= \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\, a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)}\sum_{n=0}^{\infty}(\alpha)_n\frac{(zt)^n}{n!}dt \tag{31}$$

Which on using relation (18) yields the desired result (29).
Similarly, result (30) can be obtained by using relation (11) in equation (28) and then using the definition of the exponential function. thus the proof of Theorem 3.1 is completed.

Remark 3.2. Substituting $t=\frac{u}{1+u}$, $t=sin^2\theta$ and $t=tanh^2\theta$ in relation (29) respectively, we get the following results:

Corollary 3.1. The modified hypergeometric function $F_p^q(\alpha,\beta;\gamma;z;a)$ has the following integral representations:

$$F_p^q(\alpha,\beta;\gamma;z;a) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 u^{\beta-1}(1+u)^{\alpha-\gamma}\big(1-u(z-1)\big)^{-\alpha} a^{\left[(1+u)\left(\frac{-p}{u}-q\right)\right]}du \tag{32}$$

$$F_p^q(\alpha,\beta;\gamma;z;a) = \frac{2}{B(\beta,\gamma-\beta)}\int_0^{\frac{\pi}{2}}\frac{sin^{2b-1}\theta cos^{2c-2b-1}\theta}{(1-zsin^2\theta)^a}\; a^{(-p\,csc^2\theta-q\,sec^2\theta)}d\theta \tag{33}$$

$$F_p^q(\alpha,\beta;\gamma;z;a) = \frac{2}{B(\beta,\gamma-\beta)}\int_0^{\infty}\frac{sinh^{2b-1}\theta cosh^{2a-2c+1}\theta}{(\cosh^2\theta-zsinh^2\theta)^a}\; a^{(-p\,cosh^2\theta\,coth^2\theta-(p-q)\,cosh^2\theta)}d\theta \tag{34}$$

Remark 3.3. Substituting $t=1-t$ in relation (3.4), we get the following result:

Corollary 3.2. The modified confluent hypergeometric function $\phi_p^q(\beta;\gamma;z;a)$ has the following integral representation:

$$\phi_p^q(\beta;\gamma;z;a) = \frac{exp(z)}{B(\beta,\gamma-\beta)}\int_0^1 t^{\gamma-\beta-1}(1-t)^{\beta-1}exp(-zt)\; a^{\left(\frac{-p}{t}-\frac{q}{(1-t)}\right)}dt \tag{35}$$

Remark 3.4. Taking $z=1$ in relation (29) and using relation (12), we get the following result:

$$F_p^q(\alpha,\beta;\gamma;1;a) = \frac{B_p^q(\beta,\gamma-\alpha-\beta;a)}{B(\beta,\gamma-\beta)} \tag{36}$$

Theorem 3.2. The following differentiation formulas for $F_p^q(\alpha,\beta;\gamma;z;a)$ and
$\phi_p^q(\beta;\gamma;z;a)$ hold true:

$$\frac{d^k}{dz^k}\{F_p^q(\alpha,\beta;\gamma;z;a)\} = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}F_p^q(\alpha+k,\beta+k;\gamma+k;z;a) \tag{37}$$

$$\frac{d^k}{dz^k}\{\phi_p^q(\beta;\gamma;z;a)\} = \frac{(\beta)_k}{(\gamma)_k}\phi_p^q(\beta+k;\gamma+k;z;a) \tag{38}$$

Proof. Differentiating (27) with respect to z, we have

$$\frac{d}{dz}\{F_p^q(\alpha,\beta;\gamma;z;a)\} = \frac{d}{dz}\sum_{n=0}^{\infty}(\alpha)_n\frac{B_p^q(\beta+n,\gamma-\beta;a)}{B(\beta,\gamma-\beta)}\frac{z^n}{n!},$$

$$= \sum_{n=1}^{\infty} (\alpha)_n \frac{B_p^q(\beta+n, \gamma-\beta; a)}{B(\beta, \gamma-\beta)} \frac{z^{n-1}}{(n-1)!} \tag{39}$$

Putting $n = n + 1$ in (39), we have

$$\frac{d}{dz}\{F_p^q(\alpha, \beta; \gamma; z; a)\} = \sum_{n=0}^{\infty} (\alpha)_{n+1} \frac{B_p^q(\beta+n+1, \gamma-\beta; a)}{B(\beta, \gamma-\beta)} \frac{z^n}{n!} \tag{40}$$

which on using the following relation

$$B(b, c-b) = \frac{c}{b} B(b+1, c-b) \tag{41}$$

In the R.H.S. gives

$$\frac{d}{dz}\{F_p^q(\alpha, \beta; \gamma; z; a)\} = \frac{\alpha\beta}{\gamma} F_p^q(\alpha+1, \beta+1; \gamma+1; z; a) \tag{42}$$

Again differentiating (42) with respect to z, we obtain

$$\frac{d^2}{dz^2}\{F_p^q(\alpha, \beta; \gamma; z; a)\} = \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} F_p^q(\alpha+2, \beta+2; \gamma+2; z; a) \tag{43}$$

Continuing up to k times, we get the required result (37).
Applying the similar procedure used in to prove result (37), we get the desired result (38), thus the proof of Theorem 3.2 is completed.

Theorem 3.3. The modified hypergeometric and confluent hypergeometric functions $F_p^q(\alpha, \beta; \gamma; z; a)$ and $\phi_p^q(\beta; \gamma; z; a)$ have the following Mellin transforms:

$$\mathcal{M}\{F_p^q(\alpha, \beta; \gamma; z; a); p \to s\} = \frac{\Gamma_p(r; a)\,\Gamma_q(s; a)}{B(\beta, \gamma-\beta)} B(\beta+r, \gamma-\beta+s)\ {}_2F_1(\alpha, \beta+s; \gamma+r+s; z) \tag{44}$$

$$\mathcal{M}\{\phi_p^q(b; c; z; a); p \to s\} = \frac{\Gamma_p(r; a)\,\Gamma_q(s; a)}{B(\beta, \gamma-\beta)} B(\beta+r, \gamma+s-\beta)\phi(\beta+r; \gamma+r+s; z) \tag{45}$$

Proof. Applying Mellin transform on both sides of (29), we have

$$\mathcal{M}\{F_p^q(\alpha, \beta; \gamma; z; a); p \to s\} = \frac{1}{B(\beta, \gamma-\beta)} \int_0^{\infty} p^{s-1} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\ a^{\left(\frac{-p}{t(1-t)}\right)} dt dp$$

Interchanging the order of integrations in above equation, we have

$$\mathcal{M}\{F_p^q(\alpha, \beta; \gamma; z; a); p \to s\} = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}$$

$$\left(\int_0^{\infty} p^{r-1}\, a^{\left(\frac{-p}{t}\right)} dp \cdot \int_0^{\infty} q^{s-1}\, a^{-\frac{q}{(1-t)}} dq\right) dt \tag{46}$$

Using the following relations:

$$\int_0^{\infty} p^{r-1}\, a^{\left(\frac{-p}{t}\right)} dp \ = t^r (1-t)^r\, \Gamma_p(r;a) \tag{47}$$

and (24) in the RH.S. of (46), we get:

$$\mathcal{M}\{F_p^q(\alpha,\beta;\gamma;z;a);p\to s\}= \frac{\Gamma_p(r;a)\,\Gamma_q(s;a)}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!}\int_0^1 t^{\beta+r+n-1}(1-t)^{\gamma-\beta+s-1}dt$$

$$=\frac{\Gamma_p(r;a)\,\Gamma_q(s;a)}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}\frac{\Gamma(\beta+r+n)\Gamma(\gamma-\beta+s)}{\Gamma(\gamma+r+s+n)}$$

$$=\frac{\Gamma_p(r;a)\,\Gamma_q(s;a)}{B(\beta,\gamma-\beta)}B(\beta+r,\gamma-\beta+s)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta+r)_n}{(\gamma+r+s)_n}\frac{z^n}{n!}$$

which yields the desired result (44).
In similar way, we can prove the result (45), thus the proof of Theorem 3.3 is completed.

Remark 3.5. Taking the inverse Mellin transform of both sides of equations (44) and (45), we get the following results:

$$F_p^q(\alpha,\beta;\gamma;z;a)=\frac{1}{2\pi i\ B(\beta,\gamma-\beta)}\int_{\gamma_1 i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\Gamma_p(r;a)\,\Gamma_q(s;a)B(\beta+r,\gamma-\beta+s)$$

$$\times\ {}_2F_1(\alpha,\beta+r;\gamma+r+s;z)\,p^{-s}\,q^{-s}\,dr\,ds \tag{48}$$

And

$$\phi_p^q(b;c;z;a)=\frac{1}{2\pi i\ B(\beta,\gamma-\beta)}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty}\Gamma_p(r;a)\,\Gamma_q(s;a)B(\beta+r,\gamma-\beta+s)$$

$$\times\,\phi(\beta+r;\gamma+r+s;z)\,p^{-r}\,q^{-s}\,dr\,ds. \tag{49}$$

Respectively.

Theorem 3.4. The modified hypergeometric and confluent hypergeometric functions $F_p^q(\alpha,\beta;\gamma;z;a)$ and $\phi_p^q(\beta;\gamma;z;a)$ have the following transformations formulas:

$$F_p^q(\alpha,\beta;\gamma;z;a)=(1-z)^{-\delta}F_p^q\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1};a\right) \tag{50}$$

$$\phi_p^q(\beta;\gamma;z;a)=exp(z)\phi_p^q(\gamma-\beta;\gamma;-z;a) \tag{51}$$

where $|arg(1-z)|<\pi$.

Proof. Putting $t=1-t$ in (29) and then using

$$\left(1-z(1-t)\right)^{-\delta}=(1-z)^{-\delta}\left(1+\frac{z}{1-z}t\right)^{-\delta} \tag{52}$$

In the resultant equation. we obtain

$$F_p^q(\alpha,\beta;\gamma;z;a) = \frac{(1-z)^{-\delta}}{B(\beta,\gamma-\beta)} \int_0^1 t^{\gamma-\beta-1}(1-t)^{\beta-1} \left(1+\frac{z}{1-z}t\right)^{-\delta} a^{\left(\frac{-p}{t(1-t)}\right)} dt,$$

Which in view of definition (27) yields the required result (50).
From (30) and (35), we can easily establish the required result (51), thus the proof of Theorem 3.4 is completed.

Remark 3.6. For $a = e,$ all the above relations related to reduce to the known results due to (Chaudhry *et al*., 2004) [2].

4. References

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