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On the existence of soft fixed points in soft s-metric spaces via altering distance functions

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Abstract
In this article, we introduce a novel type of soft contractive condition and establish several soft fixed point theorems by employing soft altering distance functions in soft S-metric spaces. The proposed results not only generalize existing findings but also provide useful consequences, thereby enriching the theory of soft fixed point analysis.

Keywords: Soft fixed point, soft contractive conditions, soft s-metric space, soft altering distance function

1. Introduction

The concept of soft set theory, introduced by Molodtsov [20] (1999), has emerged as a powerful mathematical tool to handle uncertainties that classical mathematical structures often fail to address. Building on this foundation, researchers have developed various extensions, including soft metric spaces, soft S-metric spaces, and soft topological spaces, to study problems involving vagueness and imprecision.

In recent years, the study of fixed point theorems in soft structures has gained considerable attention. A soft fixed point theorem investigates the existence and uniqueness of fixed points for mappings defined in soft metric or soft S-metric spaces. These theorems generalize the classical fixed point results of Banach [2], Kannan [12], and Chatterjea [3] into the framework of soft mathematics, thereby providing a broader applicability in areas where uncertainty is inherent. Researchers have also introduced soft contractive conditions, soft altering distance functions, and hybrid contraction principles to establish fixed point results in soft settings. Such generalizations enrich the theory and allow connections with decision-making, optimization, engineering models, and computational intelligence (see for detailed survey [8], [9], [11], [13], [15], [16], [17], [19]). Overall, soft fixed point theorems not only extend classical results to a more flexible environment but also open new directions for applications in applied sciences, particularly where data or conditions are not precisely defined.

On the basis of the theory of soft elements of soft metric spaces, Abbas *et al.* [1] proposed the idea of soft contraction mapping. The concept of soft mapping was presented by Wardowski [21], who also determined its fixed point in the context of soft topological spaces. A soft Banach contraction principle was one of the findings they derived from their study of fixed points of soft contraction mappings. The concept of soft S-metric spaces was formally introduced by Cigdem Gunduz Aras *et al.* [4] in 2018. In that study, they defined soft S-metric spaces, investigated their foundational properties, and used them to prove fixed point theorems under soft contractive mappings. This marks the earliest known formal introduction and treatment of the concept in the soft set framework.

2. Prilimeries

2.1 Definition [14] "A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following property is satisfied:

- (Θ_1) $\psi(0) = 0$,
- (Θ_2) ψ is monotonically non-decreasing function,
- (Θ_3) ψ is a continuous function,

By Ψ we denote the set of all altering distance functions."

M. S. Khan *et al.* [14] proved the following Banach Contraction Principal by using altering distance function

2.2 Theorem [14] "Let (M, d) be a complete metric space. Let $\psi \in \mathrm{K}$ and $T: M \to M$ be a mapping which satisfies the following inequality:

$$\psi(d(Tu, Tv)) \leq \mathrm{k}\, \psi\left(d(u, v)\right),$$

for some $0 < \mathrm{k} < 1$ and all $u, v \in M$. Then $w \in M$ is a unique fixed point of T such that for each $u \in M$, $\lim_{n\to\infty} T^n u = w$."

2.3 Definition [20]**:** "A pair (F, E) is called a soft set over a given universal set X, if and only if F is a mapping from a set of parameters E (each parameter could be a word or a sentence) into the power set of X denoted by $P(X)$. That is, $F: E \to P(X)$. Clearly, a soft set over X is a parameterized family of subsets of the given universe X."

2.4 Definition [18]**:** "A soft set (F, E) over X is said to be a null soft set denoted by $\widetilde{\Phi}$, if for all $e \in E, F(e) =$ null set ϕ."

2.5 Definition [18]**:** "A soft set (F, E) over X is said to be an absolute soft set denoted by $\tilde{X}$ if for all $e \in E, F(e) = X$."
Das and Samanta ([6]-[7]) presented the concepts of soft real numbers and soft real sets and discussed about their distinctive features. Based on these notions, they introduced in the concept of soft metric.

2.6 Definition [6]**:** "Let $\mathbb{R}$ be the set of real numbers and $\mathcal{B}(\mathbb{R})$ the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F: E \to \mathcal{B}(\mathbb{R})$ is called a soft real set. If a real soft set is a singleton soft set, it will be called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0$, $\tilde{1}(e) = 1$, for all $e \in E$ respectively."
In 2018, Aras *et al.* [4] introduced the concept of soft S-metric spaces and also discussed its important properties which are as follows:
"Let $\widetilde{\mathcal{X}}$ be an absolute soft set, E be a non-empty set of parameters and $SP(\widetilde{\mathcal{X}})$ be the collection of all soft points of $\widetilde{\mathcal{X}}$. Let $\mathbb{R}(E)^*$ denotes the set of all non-negative soft real numbers."

2.7 Definition [4] "A soft S-metric on $\widetilde{\mathcal{X}}$ is a mapping $\mathcal{S} : SP(\widetilde{\mathcal{X}}) \times SP(\widetilde{\mathcal{X}}) \times SP(\widetilde{\mathcal{X}}) \to \mathbb{R}(E)^*$ which satisfies the following conditions:
$(\overline{\mathcal{S}_1})$ $\mathcal{S}(\hat{u}_a, \hat{v}_b, \widehat{w}_c) \geq \tilde{0}$;

$(\overline{\mathcal{S}_2})$ $\mathcal{S}(\hat{u}_a, \hat{v}_b, \widehat{w}_c) = \tilde{0}$, if and only if $\hat{u}_a = \hat{v}_b = \widehat{w}_c$;

$(\overline{\mathcal{S}_3})$ $\mathcal{S}(\hat{u}_a, \hat{v}_b, \widehat{w}_c) \leq \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{t}_d) + \mathcal{S}(\hat{v}_b, \hat{v}_b, \hat{t}_d) + \mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{t}_d)$.

For all $\hat{u}_a, \hat{v}_b, \widehat{w}_c, \hat{t}_d \in SP\left(\widetilde{\mathcal{X}}\right)$, then the soft set $\tilde{X}$ with a soft S-metric is called soft S-metric space and denoted by $\left(\widetilde{\mathcal{X}}, S, E\right)$."

2.8 Lemma [5] "Let $\left(\widetilde{\mathcal{X}}, S, E\right)$ is a soft S-metric space. Then we have

$\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b) = \mathcal{S}(\hat{v}_b, \hat{v}_b, \hat{u}_a)$."

2.9 Definition [5] "A soft sequence $\{\hat{u}^n_{a_n}\}$ in $\left(\widetilde{\mathcal{X}}, S, E\right)$ converges to $\hat{v}_b$ if and only if $\mathcal{S}\left(\hat{u}^n_{a_n}, \hat{u}^n_{a_n}, \hat{v}_b\right) \to \tilde{0}\ as\ n \to \infty$ and we denote this by $\lim_{n\to\infty} \hat{u}^n_{a_n} = \hat{v}_b$."

2.10 Definition [5] "A soft sequence $\{\hat{u}^n_{a_n}\}$ in $\left(\widetilde{\mathcal{X}}, S, E\right)$ is called a Cauchy sequence if for $\tilde{\varepsilon} > \tilde{0}$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{S}\left(\hat{u}^n_{a_n}, \hat{u}^n_{a_n}, \hat{u}^m_{a_m}\right) < \tilde{\varepsilon}$ for each $m, n \geq n_0$."

2.11 Definition [25] "A soft S-metric space $\left(\widetilde{\mathcal{X}}, S, E\right)$ is said to be complete if every Cauchy sequence is convergent."

2.12 Definition [5] "Let $\left(\widetilde{\mathcal{X}}, S, E\right)$ and $\left(\widetilde{\Upsilon}, S', E'\right)$ be two soft S-metric spaces. The mapping $f_\varphi: \left(\widetilde{\mathcal{X}}, S, E\right) \to \left(\widetilde{\Upsilon}, S', E'\right)$ is a soft mapping, where $f: \widetilde{\mathcal{X}} \to \widetilde{\Upsilon}$ and $\varphi :\ E \to\ E'$ are two mappings."

2.13 Definition [5] "Let $f_\varphi: \left(\widetilde{\mathcal{X}}, S, E\right) \to \left(\widetilde{\Upsilon}, S', E'\right)$ be a soft mapping from soft S-metric space $\left(\widetilde{\mathcal{X}}, S, E\right)$ to a soft S-metric space $\left(\widetilde{\Upsilon}, S', E'\right)$. Then f_φ is soft continuous at a soft point $\hat{u}_a \in SP(\widetilde{\mathcal{X}})$ if and only if $f_\varphi\left(\{\hat{u}^n_{a_n}\}\right) \to f_\varphi(\hat{u}_a)$."

2.14 Definition [5] "Let $\left(\widetilde{\mathcal{X}}, S, E\right)$ be a soft S-metric space. A map $f_\varphi: \left(\widetilde{\mathcal{X}}, S, E\right) \to \left(\widetilde{\mathcal{X}}, S, E\right)$ is said to be a soft contraction mapping if there exists a soft real number $\tilde{k} \in \mathbb{R}(E), \tilde{0} \leq \tilde{k} < \tilde{1}$ (where $\mathbb{R}(E)$ denotes the soft real number set) such that

$$\mathcal{S}\left(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b)\right) \leq \tilde{k}\ \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b),$$

for all $\hat{u}_a, \hat{v}_b \in SP\left(\widetilde{\mathcal{X}}\right)$."
In 2018, Elif G. *et al.* [10] establish the following definition of soft altering distance function in soft metric space.

2.15 Definition [10] "A soft function $\psi : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ is called a soft altering distance function if ψ satisfies the following property:

- (Θ_1) $\psi(\bar{0}) = \bar{0}$,
- (Θ_2) ψ is monotonically non-decreasing function,
- (Θ_3) ψ is a sequentially continuous function i.e., $\hat{u}_{a_n}^n \to \hat{u}_a$, then $\psi(\hat{u}_{a_n}^n) \to \psi(\hat{u}_a)$."

2.16 Theorem [10] "Let $(\tilde{X}, d)$ be a complete metric space. Let $\psi : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be a soft altering distance function and $T: \tilde{X} \to \tilde{X}$ be a soft mapping which satisfies the following inequality:

$$\psi\big(d(T(\hat{u}_{a_n}^n), T(\hat{v}_{b_n}^n)\big) \leq \bar{c}\, \psi\big(d(\hat{u}_a, \hat{v}_b)\big),$$

for some $\bar{0} < \bar{c} < \bar{1}$ and $\hat{u}_a, \hat{v}_b \epsilon\, SP(\tilde{X})$. Then T has a unique soft fixed point."

3. Soft fixed point theorems using altering distance function

3.1 Theorem: Let $(\tilde{\mathcal{X}}, S, E)$ be a complete soft S-metric space. Let $\psi : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be a soft altering distance function and f_φ be soft self mapping on $(\tilde{\mathcal{X}}, S, E)$ which satisfies the following inequality:

$$\psi\big(\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b))\big) \leq \bar{c}\, \psi\big(\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)\big), \quad (3.1)$$

for some $\bar{0} < \bar{c} < \bar{1}$ and $\hat{u}_a, \hat{v}_b \,\epsilon\, SP(\tilde{X})$. Then f_φ has a unique soft fixed point.

Proof: Let $\hat{u}_{a_0}^0 \epsilon\, SP(\tilde{X})$ be an arbitrary point and let $\{\hat{u}_{a_n}^n\}$ be a soft sequence defined as follows

$$\hat{u}_{a_{n+1}}^{n+1} = f_\varphi(\hat{u}_{a_n}^n) = f_\varphi^{\;n+1}(\hat{u}_a),\ \hat{t}_n = \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

We first prove that f_φ has a soft fixed point in $(\tilde{\mathcal{X}}, S, E)$. We may assume that $\hat{t}_n > \bar{0}$ for each $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.1), we obtain

$$\psi\big(\mathcal{S}(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_{n+1}}^{n+1}))\big) \leq \bar{c}\, \psi\left(\mathcal{S}\big(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}\big)\right)$$

$$\psi\big(\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+2}}^{n+2})\big) \leq \bar{c}\, \psi\left(\mathcal{S}\big(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}\big)\right)$$

$$\psi(\hat{t}_{n+1}) \leq \bar{c}\, \psi(\hat{t}_n) \leq \psi(\hat{t}_n).$$

Since ψ is non-decreasing function, $\{\hat{t}_n\}$ is a decreasing sequence of soft real numbers. Hence $\{\hat{t}_n\}$ has a limit point. We put $\lim_{n\to\infty} \hat{t}_n = \tilde{t}$ and suppose $\tilde{t} > \bar{0}$. Hence $\hat{t}_n \geq \tilde{t}$ implies that $\psi(\tilde{t}\,) \leq \bar{c}\, \psi\,(\tilde{t}\,) < \psi\,(\tilde{t}\,)$ which is a contradiction. So $\tilde{t} = \bar{0}$. Therefore, $\{\hat{t}_n\}$ converges to $\bar{0}$.

Now, we will prove that $\{\hat{u}_{a_n}^n\}$ is a Cauchy sequence in $(\tilde{\mathcal{X}}, S, E)$. Suppose that $\{\hat{u}_{a_n}^n\}$ is not a Cauchy sequence which means that there is a constant $\bar{\in} > \bar{0}$ and two subsequence $\left\{\hat{u}_{a_{n_k}}^{n_k}\right\}$ and $\left\{\hat{u}_{a_{m_k}}^{m_k}\right\}$ of $\{\hat{u}_{a_n}^n\}$ such that for every $n \in \mathbb{N} \cup \{0\}$, we find that $n_k > m_k > n$, $\mathcal{S}\left(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}\right) \geq \bar{\in}$ and $\mathcal{S}\left(\hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{m_k}}^{m_k}\right) < \bar{\in}$. For each $n > 0$, we put

$\widetilde{B_n} = \mathcal{S}\left(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}\right)$. Then we have

$$\bar{\in} \leq \mathcal{S}\left(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}\right) \leq 2\, \mathcal{S}\left(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k-1}}^{n_k-1}\right) + \mathcal{S}\left(\hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{m_k}}^{m_k}\right) \leq 2\, \hat{t}_{n-1} + \bar{\in}$$

Since $\{\hat{t}_n\} \to \bar{0}$, we obtain $\{\widetilde{B_n}\}$ converges to $\bar{\in}$.

Similarly, we can show that $\mathcal{S}\left(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1}\right)$ converges to $\bar{\in}$.

From the hypothesis, we deduce

$$\psi\left(\mathcal{S}(f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{m_k}}^{m_k}))\right) \leq \bar{c}\, \psi\left(\mathcal{S}\left(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}\right)\right)$$

$$\psi\left(\mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1})\right) \leq \bar{c}\, \psi\left(\mathcal{S}\left(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}\right)\right)$$

Letting $k \to \infty$, we obtain that $\psi(\bar{\in}) \leq \bar{c}\, \psi\,(\bar{\in}) \leq \psi\,(\bar{\in})$, which is contradiction. Hence, $\{\hat{u}_{a_n}^n\}$ is a Cauchy sequence. By

completeness of $(\widetilde{\mathcal{X}}, S, E)$, $\{\hat{u}^{n}_{a_n}\}$ converges to some soft point $\widehat{w}_c$.
Now, we show that $\widehat{w}_c$ is a fixed soft point of f_φ. If we substitute $\hat{u}_a = \hat{u}^{n-1}_{a_{n-1}}$ and $\hat{v}_b = \widehat{w}_c$ in (3.1), we obtain

$$\psi\big(\mathcal{S}(f_\varphi(\hat{u}^{n-1}_{a_{n-1}}), f_\varphi(\hat{u}^{n-1}_{a_{n-1}}), f_\varphi(\widehat{w}_c)\big) \leq \bar{c}\, \psi\left(\mathcal{S}\big(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c\big)\right)$$

$$\psi\big(\mathcal{S}(\hat{u}^{n}_{a_n}, \hat{u}^{n}_{a_n}, f_\varphi(\widehat{w}_c)\big) \leq \bar{c}\, \psi\left(\mathcal{S}\big(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c\big)\right).$$

Taking limit as $n \to \infty$ and using the continuity of ψ and f_φ, we get

$$\psi\big(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, f_\varphi(\widehat{w}_c)\big) \leq \bar{c}\, \psi\big(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c)\big) = \bar{c}.\psi(\bar{0}) = \bar{0},$$

which implies $\psi\big(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, f_\varphi(\widehat{w}_c)\big) = \bar{0}$ that is $f_\varphi(\widehat{w}_c) = \widehat{w}_c$.

To prove the uniqueness, we assume that $\widehat{w}_c$ and $\hat{r}_d$ be two different fixed soft point of f_φ. Then from (3.1), we obtain that

$$\psi\big(\mathcal{S}(f_\varphi(\widehat{w}_c), f_\varphi(\widehat{w}_c), f_\varphi(\hat{r}_d)\big) \leq \bar{c}\, \psi\big(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\big)$$

$$\psi(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d) \leq \bar{c}\, \psi\big(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\big) < \psi\big(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\big),$$

which is contradiction. Hence f_φ has a soft unique fixed point.
Here completes the proof.

Note: If we consider $\psi(\bar{t}) = \bar{t}$. Then the above theorem reduces to contraction condition $\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b) \leq \bar{c}\, \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)$, for some $\bar{0} < \bar{c} < \bar{1}$, which is given by Aras [5].

3.2 Theorem: Let $(\widetilde{\mathcal{X}}, S, E)$ be a complete soft S-metric space. Let $\psi : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be a soft altering distance function with $\psi(\bar{x}) \neq \bar{0}$ for all $\bar{x} \neq \bar{0}$ and f_φ be soft self mapping on $(\widetilde{\mathcal{X}}, S, E)$ which satisfies the following inequality:

$$\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b)) \leq \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b) - \psi\big(\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)\big), \quad (3.2)$$

for all $\hat{u}_a, \hat{v}_b\, \epsilon\, SP(\widetilde{X})$. Then f_φ has a unique soft fixed point in $\widetilde{\mathcal{X}}$.

Proof: Let $\hat{u}^{0}_{a_0} \epsilon\, SP(\tilde{X})$ be an arbitrary point and let $\{\hat{u}^{n}_{a_n}\}$ be a soft sequence defined as follows

$$\hat{u}^{n+1}_{a_{n+1}} = f_\varphi(\hat{u}^{n}_{a_n}) = f_\varphi^{\,n+1}(\hat{u}_a),\ \hat{t}_n = \mathcal{S}(\hat{u}^{n}_{a_n}, \hat{u}^{n}_{a_n}, \hat{u}^{n+1}_{a_{n+1}}) \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

We first prove that f_φ has a soft fixed point in $(\widetilde{\mathcal{X}}, S, E)$. We may assume that $\hat{t}_n > \bar{0}$ for each $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.2), we obtain

$$\mathcal{S}\left(f_\varphi(\hat{u}^{n}_{a_n}), f_\varphi(\hat{u}^{n}_{a_n}), f_\varphi(\hat{u}^{n+1}_{a_{n+1}})\right) \leq \mathcal{S}(\hat{u}^{n}_{a_n}, \hat{u}^{n}_{a_n}, \hat{u}^{n+1}_{a_{n+1}}) - \psi\left(\mathcal{S}(\hat{u}^{n}_{a_n}, \hat{u}^{n}_{a_n}, \hat{u}^{n+1}_{a_{n+1}})\right)$$

$$\mathcal{S}(\hat{u}^{n+1}_{a_{n+1}}, \hat{u}^{n+1}_{a_{n+1}}, \hat{u}^{n+2}_{a_{n+2}}) \leq \mathcal{S}(\hat{u}^{n}_{a_n}, \hat{u}^{n}_{a_n}, \hat{u}^{n+1}_{a_{n+1}}) - \psi\left(\mathcal{S}(\hat{u}^{n}_{a_n}, \hat{u}^{n}_{a_n}, \hat{u}^{n+1}_{a_{n+1}})\right)$$

$$\hat{t}_{n+1} \leq \hat{t}_n - \psi(\hat{t}_n) \leq \hat{t}_n. \quad (3.3)$$

Since $\{\hat{t}_n\}$ is a decreasing sequence of soft real numbers. Hence $\{\hat{t}_n\}$ has a limit point. We put $\lim_{n\to\infty} \hat{t}_n = \tilde{t}$ and suppose $\tilde{t} > \bar{0}$. Since ψ is non decreasing, $\hat{t}_n \geq \tilde{t}$ implies that $\psi(\hat{t}_n) \geq \psi(\tilde{t}) > \bar{0}$. By (3.3) we have $\hat{t}_{n+1} \leq \hat{t}_n - \psi(\tilde{t})$.
Thus, $\hat{t}_{n+M} \leq \hat{t}_n - \bar{M}\, \psi(\tilde{t})$ is a contradiction for M large enough. So $\tilde{t} = \bar{0}$. Therefore, $\{\hat{t}_n\}$ converges to $\bar{0}$. As in above theorem it is easy to show that $\{\hat{u}^{n}_{a_n}\}$ is a Cauchy sequence in $(\widetilde{\mathcal{X}}, S, E)$. By completeness of $(\widetilde{\mathcal{X}}, S, E)$, $\{\hat{u}^{n}_{a_n}\}$ converges to some soft point $\widehat{w}_c$.
Now, we show that $\widehat{w}_c$ is a fixed soft point of f_φ. If we substitute $\hat{u}_a = \hat{u}^{n-1}_{a_{n-1}}$ and $\hat{v}_b = \widehat{w}_c$ in (3.2), we obtain

$$\mathcal{S}(f_\varphi(\hat{u}^{n-1}_{a_{n-1}}), f_\varphi(\hat{u}^{n-1}_{a_{n-1}}), f_\varphi(\widehat{w}_c)) \leq \mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c) - \psi\left(\mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c)\right)$$

$$\mathcal{S}(\hat{u}^{n}_{a_n}, \hat{u}^{n}_{a_n}, f_\varphi(\widehat{w}_c)) \leq \mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c) - \psi\left(\mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c)\right).$$

Taking limit as $n \to \infty$ and using the continuity of ψ and f_φ, we get

$\mathcal{S}\left(\widehat{w}_c, \widehat{w}_c, f_\varphi(\widehat{w}_c)\right) \leq \mathcal{S}(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c) - \psi(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c)) = \psi(\bar{0}) = \bar{0},$

which implies $\mathcal{S}(\widehat{w}_c, \widehat{w}_c, f_\varphi(\widehat{w}_c)) = \bar{0}$ that is $f_\varphi(\widehat{w}_c) = \widehat{w}_c$.

To prove the uniqueness, we assume that $\widehat{w}_c$ and $\hat{r}_d$ be two different fixed soft point of f_φ. Then from (3.2), we obtain that
$\mathcal{S}\left(f_\varphi(\widehat{w}_c), f_\varphi(\widehat{w}_c), f_\varphi(\hat{r}_d)\right) \leq \mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d) - \psi(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d))$

$\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d) \leq \mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d) - \psi(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)).$

Which implies $\psi(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)) \leq \bar{0}.$

Thus, $\psi(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)) = \bar{0}$ and hence we get $\widehat{w}_c = \hat{r}_d$.
Therefore, f_φ has a soft unique fixed point.
Here completes the proof.

Note: If we consider $\psi(\bar{t}) = \bar{k}.\bar{t}$ where $\bar{0} < \bar{k} \leq \bar{1}$, then the above theorem reduces to contraction condition $\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b)) \leq \bar{c}\, \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)$, for some $\bar{0} < \bar{c} < \bar{1}$, which is given by Aras [5].

3.3 Theorem: Let $(\widetilde{\mathcal{X}}, S, E)$ be a complete soft S-metric space. Let $\psi, \phi : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be the two soft altering distance function with $\psi(\tilde{x}) \neq \bar{0}$ and $\phi(\tilde{x}) \neq \bar{0}$ for all $\tilde{x} \neq \bar{0}$ and f_φ be soft self mapping on $(\widetilde{\mathcal{X}}, S, E)$ which satisfies the following inequality:

$\psi(\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b))) \leq \psi(\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)) - \phi(\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b))$, (3.4) for all $\hat{u}_a, \hat{v}_b \,\epsilon\, SP(\tilde{X})$. Then f_φ has a unique soft fixed point in $\widetilde{\mathcal{X}}$.

Proof: Let $\hat{u}^0_{a_0} \epsilon\, SP(\tilde{X})$ and let $\{\hat{u}^n_{a_n}\}$ be a soft sequence defined as follows

$\hat{u}^{n+1}_{a_{n+1}} = f_\varphi(\hat{u}^n_{a_n}) = f_\varphi{}^{n+1}(\hat{u}_a)$, $\hat{t}_n = \mathcal{S}(\hat{u}^n_{a_n}, \hat{u}^n_{a_n}, \hat{u}^{n+1}_{a_{n+1}})$ for each $n \in \mathbb{N} \cup \{0\}$.

We first prove that f_φ has a soft fixed point in $(\widetilde{\mathcal{X}}, S, E)$. We may assume that $\hat{t}_n > \bar{0}$ for each $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.4), we obtain

$$\psi\left(\mathcal{S}\left(f_\varphi(\hat{u}^n_{a_n}), f_\varphi(\hat{u}^n_{a_n}), f_\varphi(\hat{u}^{n+1}_{a_{n+1}})\right)\right) \leq \psi\left(\mathcal{S}(\hat{u}^n_{a_n}, \hat{u}^n_{a_n}, \hat{u}^{n+1}_{a_{n+1}})\right) - \phi\left(\mathcal{S}(\hat{u}^n_{a_n}, \hat{u}^n_{a_n}, \hat{u}^{n+1}_{a_{n+1}})\right)$$

$$\psi\left(\mathcal{S}(\hat{u}^{n+1}_{a_{n+1}}, \hat{u}^{n+1}_{a_{n+1}}, \hat{u}^{n+2}_{a_{n+2}})\right) \leq \psi\left(\mathcal{S}(\hat{u}^n_{a_n}, \hat{u}^n_{a_n}, \hat{u}^{n+1}_{a_{n+1}})\right) - \phi\left(\mathcal{S}(\hat{u}^n_{a_n}, \hat{u}^n_{a_n}, \hat{u}^{n+1}_{a_{n+1}})\right)$$

$\psi(\hat{t}_{n+1}) \leq \psi(\hat{t}_n) - \phi(\hat{t}_n) \leq \psi(\hat{t}_n)$. (3.5)

Since ψ is non-decreasing function, $\{\hat{t}_n\}$ is a decreasing sequence of soft real numbers. Hence $\{\hat{t}_n\}$ has a limit point. We put $\lim_{n\to\infty} \hat{t}_n = \tilde{t}$ and suppose $\tilde{t} > \bar{0}$. Letting $n \to \infty$ in (3.5) and using continuity of ψ, we obtain $\psi(\hat{t}_n) \leq \psi(\tilde{t}) - \phi(\tilde{t}) < \psi(\tilde{t})$ which is a contradiction. So $\tilde{t} = \bar{0}$.

Therefore, $\{\hat{t}_n\}$ converges to $\bar{0}$.

Now, we will prove that $\{\hat{u}^n_{a_n}\}$ is a Cauchy sequence in $(\widetilde{\mathcal{X}}, S, E)$. Suppose that $\{\hat{u}^n_{a_n}\}$ is not a Cauchy sequence which means that there is a constant $\bar{\in} > \bar{0}$ and two subsequence $\left\{\hat{u}^{n_k}_{a_{n_k}}\right\}$ and $\left\{\hat{u}^{m_k}_{a_{m_k}}\right\}$ of $\{\hat{u}^n_{a_n}\}$ such that for every $n \in \mathbb{N} \cup \{0\}$, we find that $n_k > m_k > n$, $\mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{m_k}_{a_{m_k}}\right) \geq \bar{\in}$ and $\mathcal{S}\left(\hat{u}^{n_k-1}_{a_{n_k-1}}, \hat{u}^{n_k-1}_{a_{n_k-1}}, \hat{u}^{m_k}_{a_{m_k}}\right) < \bar{\in}$. For each $n > 0$, we put

$\widetilde{B_n} = \mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{m_k}_{a_{m_k}}\right)$. Then we have

$$\bar{\in} \leq \mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{m_k}_{a_{m_k}}\right) \leq 2\, \mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k-1}_{a_{n_k-1}}\right) + \mathcal{S}\left(\hat{u}^{n_k-1}_{a_{n_k-1}}, \hat{u}^{n_k-1}_{a_{n_k-1}}, \hat{u}^{m_k}_{a_{m_k}}\right) \leq 2\, \hat{t}_{n-1} + \bar{\in}$$

Since $\{\hat{t}_n\}$ converges to $\bar{0}$, we obtain $\{\widetilde{B_n}\}$ converges to $\bar{\in}$.

Similarly, we can show that $\mathcal{S}\left(\hat{u}^{n_k+1}_{a_{n_k+1}}, \hat{u}^{n_k+1}_{a_{n_k+1}}, \hat{u}^{m_k+1}_{a_{m_k+1}}\right)$ converges to $\bar{\in}$.

From the hypothesis, we deduce

$$\psi\left(\mathcal{S}(f_{\varphi}(\hat{u}^{n_k}_{a_{n_k}}), f_{\varphi}(\hat{u}^{n_k}_{a_{n_k}}), f_{\varphi}(\hat{u}^{m_k}_{a_{m_k}})\right) \leq \psi\left(\mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{m_k}_{a_{m_k}}\right)\right) - \phi\left(\mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{m_k}_{a_{m_k}}\right)\right)$$

$$\psi\left(\mathcal{S}(\hat{u}^{n_k+1}_{a_{n_k+1}}, \hat{u}^{n_k+1}_{a_{n_k+1}}, \hat{u}^{m_k+1}_{a_{m_k+1}})\right) \leq \psi\left(\mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{m_k}_{a_{m_k}}\right)\right) - \phi\left(\mathcal{S}\left(\hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{n_k}_{a_{n_k}}, \hat{u}^{m_k}_{a_{m_k}}\right)\right)$$

Letting $k \to \infty$, we obtain that $\psi(\overline{\epsilon}) \leq \psi(\overline{\epsilon}) - \phi(\overline{\epsilon}) \leq \psi(\overline{\epsilon})$, which is contradiction. Hence, $\{\hat{u}^n_{a_n}\}$ is a Cauchy sequence. By completeness of $(\widetilde{\mathcal{X}}, \mathcal{S}, E)$, $\{\hat{u}^n_{a_n}\}$ converges to some soft point $\widehat{w}_c$.
Now, we show that $\widehat{w}_c$ is a fixed soft point of f_φ. If we substitute $\hat{u}_a = \hat{u}^{n-1}_{a_{n-1}}$ and $\hat{v}_b = \widehat{w}_c$ in (3.4), we obtain

$$\psi\left(\mathcal{S}(f_\varphi(\hat{u}^{n-1}_{a_{n-1}}), f_\varphi(\hat{u}^{n-1}_{a_{n-1}}), f_\varphi(\widehat{w}_c))\right) \leq \psi\left(\mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c)\right) - \phi\left(\mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c)\right)$$

$$\psi\left(\mathcal{S}(\hat{u}^{n}_{a_{n}}, \hat{u}^{n}_{a_{n}}, f_\varphi(\widehat{w}_c))\right) \leq \psi\left(\mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c)\right) - \phi\left(\mathcal{S}(\hat{u}^{n-1}_{a_{n-1}}, \hat{u}^{n-1}_{a_{n-1}}, \widehat{w}_c)\right).$$

Taking limit as $n \to \infty$ and using the continuity of ψ and f_φ, we get

$$\psi\left(\mathcal{S}\left(\widehat{w}_c, \widehat{w}_c, f_\varphi(\widehat{w}_c)\right)\right) \leq \psi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c)\right) - \phi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \widehat{w}_c)\right) = \overline{0},$$

which implies $\mathcal{S}\left(\widehat{w}_c, \widehat{w}_c, f_\varphi(\widehat{w}_c)\right) = \overline{0}$ that is $f_\varphi(\widehat{w}_c) = \widehat{w}_c$.

To prove the uniqueness, we assume that $\widehat{w}_c$ and $\hat{r}_d$ be two different fixed soft point of f_φ. Then from (3.4), we obtain that

$$\psi\left(\mathcal{S}\left(f_\varphi(\widehat{w}_c), f_\varphi(\widehat{w}_c), f_\varphi(\hat{r}_d)\right)\right) \leq \psi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\right) - \phi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\right)$$

$$\psi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\right) \leq \psi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\right) - \phi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\right).$$

Which implies $\phi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\right) \leq \overline{0}$. Thus, $\phi\left(\mathcal{S}(\widehat{w}_c, \widehat{w}_c, \hat{r}_d)\right) = \overline{0}$ and hence we get $\widehat{w}_c = \hat{r}_d$.

Therefore, f_φ has a soft unique fixed point.

Note: In Theorem 3.3 If we particularly take $\phi(\bar{t}) = (\bar{1} - \bar{k})\ \psi(\bar{t})$, for all $\bar{t} > \bar{0}$ where $\bar{0} < \bar{k} < \bar{1}$ then we obtain the result of Theorem 3.1. Again, by taking $\psi(\bar{t}) = \bar{t}$ for all $\bar{t} > \bar{0}$, in Theorem 3.3, we obtain the result of Theorem 3.2..

4. Conclusion

The study of soft S-metric spaces provides a natural extension of classical metric and S-metric spaces within the framework of soft set theory. By incorporating parameterization, these spaces successfully handle uncertainty, vagueness, and imprecise data, making them more suitable for real-life applications. The fundamental results on convergence, Cauchy sequences, and completeness form the basis for further theoretical developments. In particular, the establishment of **soft fixed point theorems** under various contractive conditions not only generalizes existing results but also enhances their applicability in decision-making, optimization, engineering, and computational intelligence. Thus, soft S-metric spaces open new directions for both theoretical research and practical applications.

References

1. Abbas M, Murtaza G, Romaguera S. Soft contraction theorem. J Nonlinear Convex Anal. 2015;16(3):423-435.
2. Banach S. Sur les operations dans les ensembles abstraits et leur application aux equations intégrales. Fundam Math. 1922;9(3):133-181.
3. Chatterjea SK. Fixed point theorems. C R Acad Bulg Sci. 1972;25(6):727-730.
4. Cigdem GA, Sadi B, Vefa C. A study on soft S-metric space. Commun Math Appl. 2018;9(4):713-723.
5. Cigdem GA, Sadi B, Vefa C. Fixed point theorems on soft S-metric space. Commun Math Appl. 2018;9(4):725-733.
6. Das S, Samanta SK. Soft real sets, soft real numbers and their properties. J Fuzzy Math. 2012;20(3):551-576.
7. Das S, Samanta SK. Soft metric. Ann Fuzzy Math Inform. 2013;6:77-94.
8. Donchev T. Lower semi-continuous differential inclusions: one-sided Lipschitz approach. Colloq Math. 1997;74(2):177-184.
9. Edelstein M. An extension of Banach's contraction principle. Proc Am Math Soc. 1961;12:7-10.
10. Elif G, Vildan C, Halis A. Soft fixed point theorems in terms of soft altering distance function. Sigma J Eng Nat Sci. 2018;9(3):285-293.
11. Fisher B. Common fixed point in Banach space. Chung Yuan J. 1982;9:12-15.
12. Kannan R. Some results on fixed point II. Am Math Mon. 1969;76:405-408.
13. Khan MS, Fisher B. Some fixed point theorem for commuting mappings. Math Nachr. 1982;106:323-326.
14. Khan MS, Swalech M, Sessa S. Fixed point theorems by altering distances between the points. Bull Austral Math Soc. 1984;30:1-9.
15. Kharal A, Ahmad B. Mappings on fuzzy soft classes. Adv Fuzzy Syst. 2009;2009:1-6.

16. Majumdar P, Samanta SK. On soft mappings. Comput Math Appl. 2010;60:2666-2672.
17. Maji PK, Roy AR, Biswas R. An application of soft sets in a decision making problem. Comput Math Appl. 2002;44(8-9):1077-1083.
18. Maji PK, Biswas R, Roy AR. Soft set theory. Comput Math Appl. 2003;45:555-562.
19. Meir A, Keeler E. A theorem on contraction mappings. J Math Anal Appl. 1969;28:326-329.
20. Molodtsov D. Soft set theory: first results. Comput Math Appl. 1999;37(4-5):19-31.
21. Wardowski D. On a soft mapping and its fixed points. Fixed Point Theory Appl. 2013;2013:1-11.