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Proof of Beal's conjecture

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Abstract
The following article is an edited version of the article previously published at https://doi.org/10.22271/math.2024.v5.i1b.137. It has been edited for clarity and completeness.
This article presents a straight-forward proof to Beal's Conjecture. It begins by proving that there exists a factor common to all three numbers, A^x, B^y, and C^z, comprising Beal's Conjecture. Then, using another number theory relationship, GCF(a,b) * LCM(a,b) = a*b, proves that the common factor must be greater than 1. Finally, the proof shows that, pursuant to the Fundamental Theorem of Arithmetic, all numbers greater than 1 are either prime or the product of prime factors.

Keywords: Beal's conjecture, Fermat's last theorem, greatest common factor (GCF), least common multiple (lcm), least common denominator (LCD), Euclid's algorithm, Mihailescu's theorem, Catalan's conjecture, fundamental theorem of arithmetic

Introduction
In December 1997, The American Mathematical Society (AMS) published a challenge to the mathematics community [1]. For that challenge, D. Andrew Beal, a banker and mathematics enthusiast, conjectured a generalization to Fermat's Last Theorem that for any solution of $A^x + B^y = C^z$, where A, B, C, x, y, and z are natural numbers and x, y, and z > 2, then A, B, and C must have a common prime factor. His challenge, Beal's Conjecture, was published in an article authored by R. Daniel Mauldin, titled "A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem." *Notices of the AMS*, Vol. 44, No. 11 (pp. 1436-37).
This article proposes a number-theoretical proof of Beal's Conjecture.

Materials and Methods
This article is strictly a mathematical proof involving no data sets nor methodology.

Discussion and Proof
Since the call of the conjecture looks for a common factor, the logical starting point is the Greatest Common Factor (GCF) function and Euclid's Algorithm. The Greatest Common Factor returns a number greater than or equal to 1 when comparing any two positive integers. Euclid's original Algorithm allows us to replace the greater of two numbers by the difference of the smaller number from the larger number [2]. Euclid's Algorithm traces its originally-published roots to Euclid's Elements, Book VII, Propositions 1 and 2 [3].
To prove Beal's Conjecture, we will show that A^x, B^y, and C^z have a greatest common factor and then we will show that the greatest common factor is greater than 1. Then, pursuant to the Fundamental Theorem of Arithmetic, show that common factor must itself either be prime or the product of prime factors.

There exists a Greatest Common Factor
To prove the existence of a greatest factor common to all three numbers, we start by comparing two of the terms at a time.

Case 1: Find the greatest common factor for A^x and C^z
To find the common factor between A^x and C^z, we attempt to find their greatest common factor:
GCF(A^x, C^z).

Applying the original Euclidean algorithm, which provides that we can substitute the greater number with the difference between the greater and the lesser number without changing the GCF, we get:

GCF(A^x, C^z) = GCF(A^x, C^z-A^x).

By substituting C^z - A^x = B^y, our equation becomes

GCF(A^x, C^z) = GCF(A^x, B^y)

Case 2: Find the greatest common factor for B^y and C^z
To find the common factor between B^y and C^z, we attempt to find their greatest common factor:

GCF(B^y, C^z).

Again, applying the Euclidean algorithm, we get:
GCF(B^y, C^z) = GCF(B^y, C^z - B^y)

And by substituting C^z - B^y = A^x, our equation becomes
GCF (B^y, C^z) = GCF(B^y, A^x)

Because the solutions to both cases are equal, we combine the equations to get
GCF(A^x, C^z) = GCF(A^x, B^y) = GCF(B^y, C^z)

Because the greatest common factor between A^x and C^z is equal to (1) the greatest common factor between B^y and C^z and (2) the greatest common factor between A^x and B^y, we have proved that there exists a greatest common factor for A^x, B^y, and C^z. However, at this point, we have not proven that the greatest common factor is greater than 1.

The Greatest Common Factor is Greater than 1
Now that we have proven that there exists a greatest common factor between for A^x, B^y, and C^z, we need to prove that the common factor is greater than 1 (because if the GCF = 1, then A^x, B^y, and C^z would be coprime).

We begin by assuming that A^x, B^y, and C^z are coprime, that is, GCF=1. We know that
GCF(a, b) * LCM(a,b) = a*b for any a,b >0, and by substituting our terms we have

GCF(A^x, B^y) * LCM(A^x, B^y) = A^x * B^y

Since we have assumed that A^x, B^y, and C^z are coprime, we can substitute GCF(A^x, B^y) = 1 to get
1 * LCM(A^x, B^y) = A^x * B^y

We know that the Least Common Multiple is the same as the Least Common Denominator when we are considering a fraction. It is permissible in algebra to divide both sides of an equation by any number except zero. So, we carefully select a number to divide both sides of the equation. Since A^x + B^y = C^z, and each of those numbers is greater than zero, we can choose either A^x + B^y or C^z [4].

Looking back at our original equation
A^x + B^y = C^z, we see that by diving both sides by A^x + B^y, we get

(A^x + B^y) / (A^x + B^y)= C^z / (A^x + B^y) = 1, and separating the terms becomes

A^x/(A^x + B^y) + B^y/(A^x + B^y) = 1. The fractions created for A^x and B^y have a common denominator, A^x + B^y.
It is at this point that we have a mathematical contradiction, that is, there exists a denominator that is less than the least common denominator, LCD, that is A^x + B^y < A^x*B^y, the Least Common Multiple/Least Common Denominator created by our assumption that GFC(A^x, B^y)=1. Here's how we arrive at that conclusion.
We examine to see if (A^x + B^y) is actually less than (A^x * B^y).
While Beal's original conjecture allows A, B, and C to be equal to 1, in actuality they cannot be based upon Mihailescu's Theorem [5] (the proof of Catalan's Conjecture) [6]. Mihailescu's Theorem proves that there on only two consecutive power numbers, 8 = (2^3) and 9 = (3^2). Beal's Conjecture requires that all exponents must be greater than 2 - thus eliminating 9 (or 3^2) as a possible value in Beal's Conjecture. Beal's original equation is A^x + B^y = C^z. It is the equivalent of B^y = C^z - A^x and A^x = C^z - B^y. A^x and B^y are themselves the difference of two power numbers and, pursuant to Mihailescu's Theorem that there are no applicable consecutive powers numbers (because 3^2 cannot be considered) then neither A^x nor B^y can be equal to 1.
Examining the least permissible value that A^x and B^y can be based upon Beal's defined domain, a base of 2 and the power of 3, we get (2^3+2^3) < (2^3*2^3)=16/64, which is true. We see that for all other numbers whose base is greater than 1 (due to Mihailescu's Theorem) and whose power is greater than 2 (due to Beal's defined domain) that (A^x + B^y) < (A^x * B^y) will always be true under Beal's constraints. Since there is always a common denominator, (A^x + B^y), which is always less than the Least Common Denominator (A^x * B^y), then GFC(A^x, B^y) cannot be equal to 1 and, since the GCF function returns values greater or equal to 1, the greatest common factor must always be greater than 1.

The Fundamental Theorem of Arithmetic
The Fundamental Theorem of Arithmetic [7] provides that "every integer greater than 1 is prime or can be represented uniquely as a product of prime numbers, up to the order of the factors." Since, in Section 1 we proved that there exists a unique number which is the greatest common factor of three numbers, A, B, and C, and in Section 2 we proved that that unique number is greater than 1, then the Fundamental Theorem of Arithmetic applied to our proof assures us that, whatever that number is, it is either prime or is a product of prime factors.

Conclusion
Having proven that A^x, B^y, and C^z have a common factor and that the common factor greater than 1, and that all numbers greater than 1 are either prime or consist of prime factors, we have thus proven Beal's Conjecture, QED.

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References
1. Beal Prize: 21 July 2025, https://www.ams.org/prizes-awards/paview.cgi?parent_id=41
2. The modern version of Euclid's Algorithm allows us to replace the greater number with the remainder of the larger number divided by the smaller number. See

https://en.wikipedia.org/wiki/Euclidean_algorithm.

3. Euclid's Elements is found at 10 June 2024 http://aleph0.clarku.edu/~djoyce/java/elements/elements.html
4. It makes no difference which we pick: (1) if we pick A^x + By, then we can work directly with the equation; if we pick C^z, then we can immediately substitute $A^x + B^y$ for C^z and continue working from there.
5. Preda Mihailescu, "Reflection, Bernoulli numbers and the proof of Catalan's conjecture", European Congress of Mathematics, 325-340, Eur. Math. Soc., Zurich, 2005.
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7. https://en.wikipedia.org/wiki/Fundamental_theorem_of_arithmetic