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Fractional differential equations: Change of variables and nonlocal symmetries

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Abstract
The paper considers point changes of variables in integrals and fractional derivatives of various types. In the general case, such changes lead to the appearance of operators of fractional integro-differentiation of a function with respect to another function. The problem of extending the action of a group of point transformations to a given type of operators is solved, and the corresponding formulas for the continuation of the infinitesimal operator of the group are presented and proved. Using a simple example of an ordinary differential equation with a fractional derivative, we illustrate the application of continuation formulas to find some of its nonlocal symmetries and check them admitted by the equation.

Keywords: Fractional derivatives, continuation formulas, nonlocal symmetries

Introduction
The study of symmetric properties of differential equations containing fractional derivatives is currently an urgent problem in connection with the increasing use of such equations as mathematical models of various processes with anomalous kinetics [1, 2]. Moreover, in contrast to the classical derivative of integer order, there are many non-identical definitions of derivatives of fractional order [3, 4, 5, 6, 7], which leads to a variety of differential equations of fractional order that are close in form but significantly different in properties. The most frequently used in practice are the concepts of left-sided fractional derivatives of the Riemann-Liouville type [3].

$$\left({}_cD_x^\alpha y\right)(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_c^x\frac{y(t)}{(x-t)^{\alpha-n+1}}dt \tag{1}$$

and Caputo type [4]

$$\left({}_c^CD_x^\alpha y\right)(x)=\frac{1}{\Gamma(n-\alpha)}\int_c^x\frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt \tag{2}$$

(here $n=[\alpha]+1$, $\Gamma(x)$- the gamma function).

In the general case, the solution of the differential equation with the derivative (1) can contain an integrable singularity of order not higher than $1-\alpha$ at the point $x=c$, while the existence of the derivative (2) implies the boundedness of the solution at this point. It is known (See, for example, [4]) that if there is a finite limit $\lim\limits_{x\to c+}y(x)=y(c)$ then derivatives (1) and (2) are related by the relation

$$\left({}_cD_x^\alpha y\right)(x)=\left({}_c^CD_x^\alpha y\right)(x)+\frac{1}{\Gamma(1-\alpha)}\frac{y(c)}{(x-c)^\alpha} \tag{3}$$

In [8, 9, 10, 11], methods for constructing point groups of transformations admitted by differential equations were developed for equations containing fractional derivatives of the form (1) and (2). Formulas for the continuation of the infinitesimal operator of a group to integrals and derivatives of fractional order were constructed, algorithms for finding the admissible group for equations containing these derivatives were developed, and some problems of group classification of ordinary differential equations and partial differential equations of fractional order were solved. It turned out, however, that the class of changes of variables that preserve the form of fractional derivatives is very limited. For derivatives of the Riemann-Liouville type (1), the general form of such a point change is determined by the expression.

$$\left({}_cD_x^\alpha y\right)(x)=\left({}_c^CD_x^\alpha y\right)(x)+\frac{1}{\Gamma(1-\alpha)}\frac{y(c)}{(x-c)^\alpha} \quad (3)$$

In [8, 9, 10, 11], methods for constructing point groups of transformations admitted by differential equations were developed for equations containing fractional derivatives of the form (1) and (2). Formulas for the continuation of the infinitesimal operator of a group to integrals and derivatives of fractional order were constructed, algorithms for finding the admissible group for equations containing these derivatives were developed, and some problems of group classification of ordinary differential equations and partial differential equations of fractional order were solved. It turned out, however, that the class of changes of variables that preserve the form of fractional derivatives is very limited. For derivatives of the Riemann-Liouville type (1), the general form of such a point change is determined by the expression

$$\bar{x}=\frac{cc_1+(x-c)}{c_1+c_2(x-c)}, \qquad \bar{y}=\psi_0(x)+y\psi_1(x),$$

Where
c, c_1, c_2 - constants,
$\psi_0(x)$, $\psi_1(x)$ are some functions, the specific form of which is determined by the equation under study.

Nevertheless, derivatives of the form (1) and (2) are only partial, albeit the most frequently used, types of fractional derivatives. A more general case is the case of a fractional derivative of a function with respect to another function, which inevitably arises under a general change of variables in any derivative of a fractional order of a specific type. Differential equations with a fractional derivative of a function with respect to a function arise, in particular, in the construction of invariant solutions of partial differential equations of fractional order. For example, when constructing invariant solutions for the equations of anomalous transport with respect to the dilatation group, ordinary differential equations with fractional derivatives of the Erdelyi-Kober type arise [8, 12]. The existing methods for their solution are very complex and are suitable only for narrow classes of equations.
The use of fractional derivatives of a function with respect to a function allows, in the general case, to expand the class of possible changes of variables, considering them as a new form of equivalence transformations (a short discussion of this issue can be found in [13]). This approach opens up new possibilities for reducing the number of variables, and, in particular, for constructing invariant solutions.
In this regard, it seems relevant to extend the methods of group analysis to the class of fractional-order equations containing fractional derivatives of a function with respect to a function. The first step on this path is the construction of formulas for the continuation of the infinitesimal operator of the group to integrals and derivatives of fractional order of a function with respect to a function, which is the subject of the second section of this paper.
Since the operators of fractional derivatives are integro-differential operators, i.e. are nonlocal by definition, it seems natural that equations with such derivatives should have nonlocal symmetries. One of the ways to construct such symmetries is to use non-point (containing fractional derivatives or integrals) changes of variables. In this case, the action of the operators in the space extended to the corresponding non-local variables is determined.
Using the continuation formula to construct and test nonlocal symmetries is illustrated with a simple example. Note that when working with fractional derivatives, checking an admissible operator is often a nontrivial task, which is illustrated in the third section of this paper.

Fractional derivative of a function with respect to a function and a continuation formula
In general, an arbitrary change of variables $\bar{x}=\varphi(x,y)$, $\bar{y}=\psi(x,y)$ does not preserve the form of the fractional differentiation operator. In particular, with such a change, the fractional Riemann-Liouville derivative (1) of order $\alpha\in(0,1)$ transforms into the left-hand fractional derivative of function $\psi(x,y)$ with respect to function $\varphi(x,y)$:

$$\left({}_cD_{\varphi[x]}^\alpha\psi\right)[x]=\frac{1}{\Gamma(1-\alpha)}\frac{1}{D_x\varphi[x]}\frac{d}{dx}\int_{\bar{c}}^{x}\frac{\psi[t]D_t\varphi[t]}{(\varphi[x]-\varphi[t])^\alpha}dt,$$

Where
$\bar{c}:\varphi(x,y(x))\big|_{x=\bar{c}}=c$.

Here, to shorten the notation, we introduced the notation $f(x) \equiv f(x, y(x))$. For the definition and basic properties of derivatives of a function with respect to a function, see, for example, [3].
Let us give a number of examples of such changes of variables that transform the Riemann-Liouville operator into other well-known forms of fractional differentiation operators.

1) Transfer according to *x*

$$\bar{x} = x + a, \quad \bar{y} = y$$

Preserves the type of the operator and changes only the lower limit of integration:

$$\left({}_cD_x^{\alpha} y\right)(x) = \left({}_{\bar{c}}D_{\bar{x}}^{\alpha} \bar{y}\right)(\bar{x}), \quad \bar{c} = c + a .$$

2) The change of variables

$$\bar{x} = x^a, \quad \bar{y} = y$$

Leads to the replacement of the Riemann-Liouville operator with an operator of Erdelyi-Kober type [3]:

$$\left({}_cD_x^{\alpha} y\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\bar{x}^{b-1}} \frac{d}{d\,\bar{x}} \int_{\bar{c}}^{\bar{x}} \frac{\bar{y}(\bar{t})\bar{t}^{\,b-1}}{\left(\bar{x}^b - \bar{t}^b\right)^{\alpha}} d\bar{t}, \quad b = \frac{1}{a}, \; \bar{c} = c^a .$$

Such a replacement is often performed when searching for invariant solutions for the equations of anomalous transfer of fractional order on the dilation group [8].

3) Change of variables

$$\bar{x} = e^x, \quad \bar{y} = y$$

leads to the replacement of the Riemann-Liouville operator by the operator of the fractional derivative of the Hadamard type [3]:

$$\left({}_cD_x^{\alpha} y\right)(x) = \frac{\bar{x}}{\Gamma(1-\alpha)} \frac{d}{d\,\bar{x}} \int_{e^{\bar{c}}}^{\bar{x}} \frac{\bar{y}(\bar{t})}{\left(\ln \frac{\bar{x}}{\bar{t}}\right)^{\alpha}} \frac{d\,\bar{t}}{\bar{t}} .$$

Along with the notion of a fractional derivative of a function with respect to a function, the notion of a fractional-order integral $\beta > 0$ of a function with respect to a function is also used [3]:

$$\left({}_cI_{g(x)}^{\beta} y\right)(x) = \frac{1}{\Gamma(\beta)} \int_c^x \frac{y(t)g'(t)}{\left(g(x) - g(t)\right)^{1-\beta}} dt \tag{4}$$

It is assumed [3] that on the interval (c, d) the function $g(x) > 0$ and has a continuous derivative $g'(x)$ of constant sign ($g'(x) > 0$ or $g'(x) < 0$). The function $y(x)$ is considered Lebesgue integrable on the interval (c, d), that is, $L_1(c, d)$. In what follows, for simplicity of presentation, we will consider the left-sided fractional derivative of the order $\alpha \in (0, 1)$ of the function $y(x)$ with respect to the function $g(x)$:

$$\left({}_cD_{g(x)}^{\alpha} y\right)(x) \equiv \frac{1}{g'(x)} \frac{d}{dx} \left({}_cI_{g(x)}^{\alpha} y\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{g'(x)} \frac{d}{dx} \int_c^x \frac{y(t)g'(t)}{\left(g(x) - g(t)\right)^{\alpha}} dt \tag{5}$$

Fractional derivative (1) for $\alpha \in (0, 1)$ is a special case (5) for $g(x) = x$.
Fractional derivative (5) has two properties, which will be used in what follows to derive the continuation formula.

Property 1. The relation

$$ {}_cD^{\alpha}_{g(x)}\left(g(x)y(x)\right)=g(x)\,{}_cD^{\alpha}_{g(x)}y(x)+\alpha\,{}_cI^{1-\alpha}_{g(x)}y(x) \tag{6}$$

Proof. $ {}_cD^{\alpha}_{g(x)}\left(g(x)y(x)\right)\equiv\frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\int_c^x\frac{g(t)y(t)g'(t)}{[g(x)-g(t)]^{\alpha}}dt=$

$$=\frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\left[\int_c^x\frac{g(t)y(t)g'(t)}{[g(x)-g(t)]^{\alpha}}dt-\int_c^x[g(x)-g(t)]^{1-\alpha}y(t)g'(t)dt\right]=$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_c^x\frac{y(t)g'(t)}{[g(x)-g(t)]^{\alpha}}dt+\frac{1}{\Gamma(1-\alpha)}\frac{g(x)}{g'(x)}\frac{d}{dx}\int_c^x\frac{y(t)g'(t)}{[g(x)-g(t)]^{\alpha}}dt-$$

$$-\frac{1-\alpha}{\Gamma(1-\alpha)}\int_c^x\frac{y(t)g'(t)}{[g(x)-g(t)]^{\alpha}}dt\equiv\alpha\,{}_cI^{1-\alpha}_{g(x)}y(x)+g(x)\,{}_cD^{\alpha}_{g(x)}y(x).$$

Property 2. If $\lim\limits_{t\to c+}y(t)(g(t)-g(c))=0$, then

$$ {}_cD^{\alpha}_{g(x)}\left((g(x)-g(c))y(x)\right)=\frac{1}{\Gamma(1-\alpha)}\int_c^x\frac{D_t[y(t)(g(t)-g(c))]}{[g(x)-g(t)]^{\alpha}}dt \tag{7}$$

Proof. The proof is carried out by integrating by parts and then differentiating the resulting integral with a variable upper limit:

$$ {}_cD^{\alpha}_{g(x)}\left((g(x)-g(c))y(x)\right)=\frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\int_c^x\frac{y(t)(g(t)-g(c))}{[g(x)-g(t)]^{-\alpha}}dt=$$

$$=\frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\left[-y(t)[g(t)-g(c)]\frac{(g(x)-g(t))^{1-\alpha}}{1-\alpha}\Bigg|_c^x+\right.$$

$$\left.+\int_c^x D_t[y(t)(g(t)-g(c))]\frac{(g(x)-g(t))^{1-\alpha}}{1-\alpha}dt\right]=$$

$$=\frac{1}{(1-\alpha)\Gamma(1-\alpha)}\frac{1}{g'(x)}\int_c^x D_t[y(t)(g(t)-g(c))]\frac{d}{dx}(g(x)-g(t))^{1-\alpha}dt=$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_c^x\frac{D_t[y(t)(g(t)-g(c))]}{[g(x)-g(t)]^{\alpha}}dt.$$

Statement 1. Consider the one-parameter group of point transformations in infinitesimal form:

$$\bar{x}=x+a\xi[x]+o(a),\qquad \bar{y}(\bar{x})=y(x)+a\eta[x]+o(a).$$

Let the function $y(x)\in L_1(c,d)$ and have a continuous derivative $y'(x)$ for $x\in(c, d)$, the functions $\xi[x]=\xi(x,y(x))$ and $\eta[x]=\eta(x,y(x))$ are sufficiently smooth in each point (c, d), $g(x)$ is a monotone positive twice differentiable function. Then the infinitesimal transformation of the fractional integral (4) for $\beta=1-\alpha$ can be represented as

$$\left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(x)=\left({}_cI^{1-\alpha}_{g(x)}y\right)(x)+a\varsigma_{\alpha-1}[x]+o(a),$$

Where $\varsigma_{\alpha-1}$ is defined by the continuation formula

$$\varsigma_{\alpha-1}[x] = {}_c I_{g(x)}^{1-\alpha}(\eta - \xi y')(x) + \xi[x] g'(x)\left({}_c D_{g(x)}^{\alpha} y\right)(x) \tag{8}$$

Proof. We write the operator of fractional integration in new variables $\bar{x}, \bar{y}$:

$$\left({}_c I_{g(\bar{x})}^{1-\alpha} \bar{y}\right)(x) \equiv \frac{1}{\Gamma(1-\alpha)} \int_c^{\bar{x}} \frac{\bar{y}(\bar{\tau}) g'(\bar{\tau}) d\bar{\tau}}{[g(\bar{x}) - g(\bar{\tau})]^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_c^{x+a\xi[x]} \frac{\bar{y}(\bar{\tau}) g'(\bar{\tau}) d\bar{\tau}}{[g(x+a\xi[x]) - g(\bar{\tau})]^{\alpha}} + o(a) \tag{9}$$

To perform the replacement of the function $\bar{y}(\bar{\tau})$, some replacement of the integration variable $\bar{\tau}$ is necessary. The most natural form of the replacement $\bar{\tau} = \tau + a\xi[\tau]$ makes it easy to pass from $\bar{y}(\bar{\tau})$ to $y(\tau)(\bar{y}(\bar{\tau}) = y(\tau) + a\eta[\tau] + o(a))$. However, such a replacement leads to the appearance of the parameter a in the lower limit of integration, which significantly complicates further transformations and requires the imposition of additional restrictions on the form of the function $\xi[x]$ [9].

More optimal is the change of variables

$$\bar{\tau} = t + ah(x,t),$$

Where

$$h(x,t) = \xi[x] \frac{g'(x)}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{g'(t)} \tag{10}$$

Here t is the new variable of integration. Such a change preserves the form of the limits of integration, since $t=c$ goes to $\bar{\tau} = c$, and $t = x$ to $\bar{\tau} = x + a\xi[x]$.

Carrying out this change in (9), we obtain

$$\left({}_c I_{g(\bar{x})}^{1-\alpha} \bar{y}\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_c^{x} \left(\frac{\bar{y}(\bar{\tau}) g'(\bar{\tau})}{[g(x+a\xi[x]) - g(\bar{\tau})]^{\alpha}} \cdot \frac{d\bar{\tau}}{dt} \right)\Bigg|_{\bar{\tau}=t+ah(x,t)} dt + o(a) \tag{11}$$

Let us consider in detail the transformation of each of the integrand factors. To express $\bar{y}(\bar{\tau})$ in terms of t, it is necessary to substitute the argument τ in $y(\tau)+a\eta[\tau]+o(a)$, which transforms exactly into $\bar{\tau}$ when replaced. It is known that the inverse infinitesimal change has the form $\tau = \bar{\tau} - a\xi[\bar{\tau}] + o(a)$. Expressing $\bar{\tau}$ in terms of the previously defined t, we obtain

$\tau = t + ah(x, t) - a\xi[t] + o(a)$.

As a result, we have

$$\bar{y}(\bar{\tau})\big|_{\bar{\tau}=t+ah(x,t)} = \left(y(\tau) + a\eta[\tau] + o(a)\right)\big|_{\tau=t+ah(x,t)-a\xi[t]+o(a)} = y(t + ah(x,t) - a\xi[t]) + a\eta[t] + o(a) =$$

$$= y(t) + ay'(t)(h(x,t) - \xi[t]) + a\eta[t] + o(a) = y(t) + a\eta[t] - a\xi[t]y'(t) +$$

$$+ a\frac{\xi[x]g'(x)}{g(x) - g(c)} \cdot \frac{g(t) - g(c)}{g'(t)} y'(t) + o(a) \tag{12}$$

Further,

$$\left(g(x + a\xi[x]) - g(\bar{\tau})\right)\big|_{\bar{\tau}=t+ah(x,t)} = g(x) + a\xi[x]g'(x) - g(t) - ah(x,t)g'(t) + o(a) =$$

$$= g(x) - g(t) + a\left(\xi[x]g'(x) - \xi[x]\frac{g(t) - g(c)}{g(x) - g(c)} \frac{g'(x)}{g'(t)} g'(t) \right) + o(a) =$$

$$=[g(x)-g(t)]\left(1+a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right)+o(a),$$

Where

$$[g(x+a\xi[x])-g(\bar{\tau})]^{-\alpha}\Big|_{\bar{\tau}=t+ah(x,t)}=[g(x)-g(t)]^{-\alpha}\left(1-\alpha a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right)+o(a) \qquad (13)$$

Finally

$$\left(g'(\bar{\tau})\frac{d\bar{\tau}}{dt}\right)\Bigg|_{\bar{\tau}=t+ah(x,t)}=(g'(t)+ag''(t)h(x,t))(1+ah_t(x,t))+o(a)=$$

$$=g'(t)+a\frac{\xi[x]g'(x)}{g(x)-g(c)}\left(g''(t)\frac{g(t)-g(c)}{g'(t)}+g'(t)\frac{d}{dt}\frac{g(t)-g(c)}{g'(t)}\right)+o(a)=$$

$$=g'(t)\left(1+a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right)+o(a) \qquad (14)$$

Substituting (12), (13), (14) into (11), we obtain

$$\left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x})=\frac{1}{\Gamma(1-\alpha)}\int_c^x\left(\bar{y}(\bar{\tau})[g(x+a\xi[x])-g(\bar{\tau})]^{-\alpha}g'(\bar{\tau})\frac{d\bar{\tau}}{dt}\right)\Bigg|_{\bar{\tau}=t+ah(x,t)}dt+o(a)=$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_c^x\frac{y(t)+a\eta[t]-a\xi[t]y'(t)+a\dfrac{\xi[x]g'(x)}{g(x)-g(c)}\dfrac{g(t)-g(c)}{g'(t)}y'(t)}{(g(x)-g(t))^{\alpha}}\left(1-\alpha a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right)\times$$

$$\times g'(t)\left(1+a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right)dt+o(a)=\frac{1}{\Gamma(1-\alpha)}\int_c^x\frac{y(t)g'(t)dt}{(g(x)-g(t))^{\alpha}}+$$

$$+\frac{a}{\Gamma(1-\alpha)}\int_c^x\frac{g'(t)dt}{(g(x)-g(t))^{\alpha}}\left(\eta[t]-\xi[t]y'(t)+\frac{\xi[x]g'(x)}{g(x)-g(c)}\left[\frac{g(t)-g(c)}{g'(t)}y'(t)+(1-\alpha)y(t)\right]\right)+$$

$$+o(a)=\left({}_cI^{1-\alpha}_{g(x)}y\right)(x)+a\,{}_cI^{1-\alpha}_{g(x)}(\eta-\xi y')(x)+$$

$$+\frac{a\xi[x]g'(x)}{\Gamma(1-\alpha)(g(x)-g(c))}\int_c^x\frac{(g(t)-g(c))y'(t)+(1-\alpha)y(t)g'(t)}{(g(x)-g(t))^{\alpha}}dt+o(a).$$

Let's use properties 1 and 2 to transform the last integral:

$$\frac{1}{\Gamma(1-\alpha)}\int_c^x\ldots=\frac{1}{\Gamma(1-\alpha)}\int_c^x\frac{D_t[y(t)(g(t)-g(c))]-\alpha y(t)g'(t)}{(g(x)-g(t))^{\alpha}}dt=$$

$$={}_cD^{\alpha}_{g(x)}[y(x)(g(x)-g(c))]-\alpha\,{}_cI^{1-\alpha}_{g(x)}y(x)={}_cD^{\alpha}_{g(x)}(g(x)y(x))-g(c)\,{}_cD^{\alpha}_{g(x)}y(x)-\alpha\,{}_cI^{1-\alpha}_{g(x)}y(x)=$$

$$=g(x)\,{}_cD^{\alpha}_{g(x)}y(x)+\alpha\,{}_cI^{1-\alpha}_{g(x)}y(x)-g(c)\,{}_cD^{\alpha}_{g(x)}y(x)-\alpha\,{}_cI^{1-\alpha}_{g(x)}y(x)=$$

$$=\left(g(x)-g(c)\right)\left({}_cD^{\alpha}_{g(x)}y\right)(x).$$

As a result, we finally find

$\left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x})=\left({}_cI^{1-\alpha}_{g(x)}y\right)(x)+a\left({}_cI^{1-\alpha}_{g(x)}(\eta-\xi y')(x)+\xi[x]g'(x)\left({}_cD^{\alpha}_{g(x)}y\right)(x)\right)+o(a)$, which proves the statement.

Statement 2. Under the conditions of Statement 1, the infinitesimal transformation of the fractional derivative (5) of order $\alpha\in(0, 1)$ has the form

$$\left({}_cD^{\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x})=\left({}_cD^{\alpha}_{g(x)}y\right)(x)+a\varsigma_{\alpha}[x]+o(a),$$

Where

$$\varsigma_{\alpha}[x]={}_cD^{\alpha}_{g(x)}(\eta-\xi y')(x)+\xi[x]g'(x)\left({}_cD^{\alpha+1}_{g(x)}y\right)(x) \tag{15}$$

Proof. By definition

$$\left({}_cD^{\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x})\equiv\frac{1}{g'(\bar{x})}\frac{d}{d\,\bar{x}}\left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x})$$

Using infinitesimal expansions

$$\frac{d}{d\bar{x}}=\left(\frac{d\,\bar{x}}{dx}\right)^{-1}\frac{d}{dx}=\left(1-aD_x\xi[x]+o(a)\right)\frac{d}{dx},$$

$\dfrac{1}{g'(\bar{x})}=\dfrac{1}{g'(x)}\left(1-a\xi[x]\dfrac{g''(x)}{g'(x)}+o(a)\right)$, $\left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x})=\left({}_cI^{1-\alpha}_{g(x)}y\right)(x)+a\varsigma_{\alpha-1}[x]+o(a)$, we obtain

$$\left({}_cD^{\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x})=\frac{1}{g'}\left(1-a\xi\frac{g''}{g'}+o(a)\right)\left(1-aD_x\xi+o(a)\right)\frac{d}{dx}\left({}_cI^{1-\alpha}_{g}y+a\varsigma_{\alpha-1}+o(a)\right)=$$

$$=\frac{1}{g'}\left(1-a\frac{D_x(\xi g')}{g'}\right)\left(D_x\left({}_cI^{1-\alpha}_{g}y\right)+aD_x\varsigma_{\alpha-1}\right)+o(a)=$$

$$=\frac{1}{g'}D_x\left({}_cI^{1-\alpha}_{g}y\right)+\frac{a}{g'}\left(D_x\varsigma_{\alpha-1}-\frac{D_x(\xi g')}{g'}D_x\left({}_cI^{1-\alpha}_{g}y\right)\right)+o(a)=$$

$$={}_cD^{\alpha}_{g}y+\frac{a}{g'}\left[D_x\left({}_cI^{1-\alpha}_{g}(\eta-\xi y')\right)+D_x\left(\xi g'\,{}_cD^{\alpha}_{g}y\right)-D_x(\xi g')\,{}_cD^{\alpha}_{g}y\right]+o(a)=$$

$$={}_cD^{\alpha}_{g}y+a\left[\frac{1}{g'}D_x\left({}_cI^{1-\alpha}_{g}(\eta\xi y')\right)+\xi D_x\left({}_cD^{\alpha}_{g}y\right)\right]+o(a)=$$

$$={}_cD^{\alpha}_{g}y+a\left({}_cD^{\alpha}_{g}(\eta-\xi y')+\xi g'\,{}_cD^{1+\alpha}_{g}y\right)+o(a).$$

Here, for the sake of brevity, the argument is omitted for all functions.

Remark 1. For g(x)=x (15) turns into the extension formula obtained earlier [9] for the derivative of the Riemann-Liouville type, and for integer α it coincides with the well-known classical extension formulas for derivatives of integer orders [14].

Remark 2. In contrast to derivatives of integer order, it is generally impossible to expand the brackets on the right-hand side of (15), since the fractional derivative of the individual terms η and ξ*y*’ may not exist. An example of an operator with such coefficients is X_1 from Section 3.

Remark 3. It can be shown that formulas (8) and (15) are valid for fractional integrals and derivatives of arbitrary order, respectively.

Nonlocal symmetries

Nonlocal symmetries for differential equations with derivatives of integer order have been known for a long time [15] and allow, in a number of cases, constructing additional invariant solutions and conservation laws. It should be noted that there is no constructive algorithm for their construction. Several heuristic approaches are known that allow one to construct certain types of nonlocal symmetries. One of them is the introduction of non-local variables and the extension of the transformations to these variables. This approach can be successfully applied to equations with fractional derivatives. In this case, the continuation formulas (8), (15) constructed in the previous section can be used both for constructing nonlocal symmetries and for checking their admissibility by the equation.

Let's illustrate this with a simple example. Consider the equation

$$ {}_0D_x^{1+\alpha} y = 0, \quad \alpha \in (0, 1) \tag{16}$$

which has the well-known general solution $y = x^{\alpha-1}(c_1 x + c_2)$(c_1, c_2 are arbitrary constants). By definition of the fractional derivative, equation (16) can be written as

$$ D_x^2\left({}_0I_x^{1-\alpha} y\right) = 0\,, $$

Where

$$ \left({}_0I_x^{1-\alpha} y\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{y(t)dt}{(x-t)^\alpha} $$

a left-hand integral of fractional order 1–α. After the nonlocal replacement $z = {}_0I_x^{1-\alpha} y$ equation (16) can be written in the form

$$ z'' = 0 \tag{17}$$

which admits the well-known eight-parameter group [9] defined by the infinitesimal operators

$$ X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial z},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = z\frac{\partial}{\partial z},\ X_5 = x\frac{\partial}{\partial z},\ X_6 = z\frac{\partial}{\partial z}, $$
$$ X_7 = x^2\frac{\partial}{\partial x} + xz\frac{\partial}{\partial x}, \quad X_8 = xz\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial z}. $$

By virtue of the identity ${}_0D_x^{1-\alpha}\,{}_0I_x^{1-\alpha} y = y$, we can reverse the nonlocal change:

$$ y = {}_0D_x^{1-\alpha} z\,. $$

Using the continuation formula (15) with *g*(*x*)=*x*, we can construct an extension of the group of equation $z'' = 0$ to the fractional derivative ${}_0D_x^{1-\alpha} z$:

$$ \varsigma_{1-\alpha} = {}_0D_x^{1-\alpha}(\eta - \xi z') + \xi\,{}_0D_x^{2-\alpha} z \tag{18}$$

Further, for brevity, we omit the subscripts 0 and *x* for the operators of fractional differentiation and integration. The well-known relationship between the Riemann-Liouville and Caputo derivatives (3) in this case can be written as

$$ D^\beta f \equiv DI^{1-\beta} f = I^{1-\beta} f' + \frac{f(0)x^{-\beta}}{\Gamma(1-\beta)},\ \beta \in (0, 1) \tag{19}$$

After differentiating it, we have

$$D^{\beta+1} f = D^{\beta} f' + \frac{f(0)x^{-1-\beta}}{\Gamma(-\beta)} \tag{20}$$

Which allows for β=1–α to write

$$I^{\alpha} z' = D^{1-\alpha} z - \frac{z(0)x^{\alpha-1}}{\Gamma(\alpha)}, \quad D^{1-\alpha} z' = D^{2-\alpha} z - \frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)} \tag{21}$$

Note that since $z(0)$ exists, the fractional derivative ${}_0D_x^{1-\alpha} z'$ also exists. When constructing an extension, the Leibniz formula for fractional differentiation of the product of two functions also turns out to be useful (see [3]):

$$D^{\beta}(fg) = \sum_{k=0}^{\infty} \binom{\beta}{k} D^{\beta-k} f \, D^{k} g \tag{22}$$

Here $\binom{\beta}{k}$ are binomial coefficients $D^{\beta-k} f = I^{k-\beta} f$ for $k > \beta$. In particular

$$D^{\beta}(xf) = xD^{\beta}(f) + \beta D^{\beta-1}(f) \tag{23}$$

$$D^{\beta}(x^2 f) = x^2 D^{\beta}(f) + 2\beta x D^{\beta-1}(f) + \beta(\beta-1)D^{\beta-2}(f) \tag{24}$$

The fractional derivative of the power function has the form [3]

$$D^{\alpha} x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > -1, \ \alpha \in \mathrm{R} \tag{25}$$

Continuation of operator X_1: Here ξ=1, η=0 and

$$\varsigma_{1-\alpha} = -D^{1-\alpha}(z') + D^{2-\alpha}(z) = \frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)}.$$

Continuation of operator X_2: $\varsigma_{1-\alpha} = D^{1-\alpha}(1) = \dfrac{x^{\alpha-1}}{\Gamma(\alpha)}$

.

Continuation of operator X_5: $\varsigma_{1-\alpha} = D^{1-\alpha}(x) = \dfrac{x^{\alpha}}{\Gamma(\alpha+1)}$.

Continuation of operator X_6: $\varsigma_{1-\alpha} = D^{1-\alpha}(z)$.

Continuation of operator X_3: $\varsigma_{1-\alpha} = -D^{1-\alpha}(xz') + xD^{2-\alpha}(z)$.

It follows from the assumption of the existence of a finite value z(0) that $(xz)\big|_{x=0} = 0$. Then, by virtue of (19), we have $D^{1-\alpha}(xz)' = D^{2-\alpha}(xz)$ and, representing xz' as $(xz)'-z$, we have

$$\varsigma_{1-\alpha} = -D^{2-\alpha}(xz) + D^{1-\alpha} z + xD^{2-\alpha}(z) = (\alpha-1)D^{1-\alpha} z.$$

Continuation of operator X_4: $\varsigma_{1-\alpha} = -D^{1-\alpha}(zz') + zD^{2-\alpha}(z)$.

Using equation (17), Leibniz's rule (22), as well as representations (21), one can get rid of the nonlinearity under the fractional differentiation operator:

$$\varsigma_{1-\alpha}=-\sum_{n=0}^{\infty}\binom{1-\alpha}{n}D^{n}(z)D^{1-n-\alpha}z'+zD^{2-\alpha}(z)=-zD^{1-\alpha}z'-(1-\alpha)z'I^{\alpha}z'+zD^{2-\alpha}(z)=$$

$$=-(1-\alpha)z'\left(D^{1-\alpha}z-\frac{z(0)x^{\alpha 1}}{\Gamma(\alpha)}\right)+\frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)}=(\alpha-1)z'D^{1-\alpha}z-\frac{z'z(0)x^{\alpha-1}}{\Gamma(\alpha-1)}+\frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)}.$$

This form of the coefficient of the continued operator is not the only possible one. In particular, we can eliminate the variable z' using the representation of the fractional derivative $D^{1-\alpha}z$ - on equation (17):

$$D^{1-\alpha}z=\sum_{n=0}^{\infty}\binom{1-\alpha}{n}D^{n}(z)D^{1-n-\alpha}1=\frac{zx^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-\alpha)z'x^{\alpha}}{\Gamma(1+\alpha)},$$

Whence, in view of $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, we have

$$(1-\alpha)z'=\frac{\Gamma(1+\alpha)}{x^{\alpha}}D^{1-\alpha}z-\alpha\frac{z}{x}$$

As a result, we find

$$\varsigma_{1-\alpha}=-\left(\frac{\Gamma(1+\alpha)}{x^{\alpha}}D^{1-\alpha}z-\alpha\frac{z}{x}\right)\left(D^{1-\alpha}z-\frac{z(0)x^{\alpha-1}}{\Gamma(\alpha)}\right)+\frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)}=$$

$$=-\Gamma(1+\alpha)\frac{(D^{1-\alpha}z)^{2}}{x^{\alpha}}+\frac{\alpha zD^{1-\alpha}z}{x}+\frac{z(0)\Gamma(1+\alpha)D^{1-\alpha}z}{x\Gamma(\alpha)}+\left(\frac{1}{\Gamma(\alpha-1)}-\frac{\alpha}{\Gamma(\alpha)}\right)zz(0)x^{\alpha-2}=$$

$$=-\Gamma(1+\alpha)\frac{(D^{1-\alpha}z)^{2}}{x^{\alpha}}+\frac{\alpha(z+z(0))D^{1-\alpha}z}{x}-\frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha)}.$$

Continuation of operator X_7: $\varsigma_{1-\alpha}=-D^{1-\alpha}(xz'-x^{2}z')+x^{2}D^{2-\alpha}z$.

Proceeding similarly to the procedure of continuation of the operator X_3, we find

$$\varsigma_{1-\alpha}=D^{1-\alpha}(xz)-D^{1-\alpha}D(x^{2}z)+D^{1-\alpha}(2xz)+x^{2}D^{2-\alpha}z=3D^{1-\alpha}(xz)-D^{2-\alpha}(x^{2}z)+x^{2}D^{2-\alpha}z=$$

$$=3xD^{1-\alpha}z-D^{2-\alpha}(x^{2}z)+x^{2}D^{2-\alpha}z=3xD^{1-\alpha}z+(3-3\alpha)I^{\alpha}z-(4-2\alpha)xD^{1-\alpha}z-$$

$$-(2-\alpha)(1-\alpha)I^{\alpha}z=(2\alpha-1)xD^{1-\alpha}z+(1-\alpha^{2})D^{\alpha}z.$$

Continuation of operator X_8: $\varsigma_{1-\alpha}=-D^{1-\alpha}(z^{2}-xzz')+xzD^{2-\alpha}z$ (26)

Using Leibniz's rule (22), by virtue of equation (17), we find

$$D^{1-\alpha}(z^{2})=\sum_{n=0}^{\infty}\binom{1-\alpha}{n}D^{n}zD^{1-n-\alpha}z=zD^{1-\alpha}z+(1-\alpha)z'I^{\alpha}z \quad (27)$$

Similarly, applying the Leibniz rule for $xz\cdot z'$ and taking into account that, by virtue of equation (17) $D^{3}(xz)=0$, we have

$$-D^{1-\alpha}(xzz')=-xzD^{1-\alpha}(z')-(1-\alpha)(z+xz')I^{\alpha}z'-(1-\alpha)(-\alpha)z'I^{\alpha+1}z'=$$

$$=-xzD^{2-\alpha}z+xz\frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)}-(1-\alpha)(z+xz')\left[D^{1-\alpha}z-\frac{z(0)x^{\alpha-1}}{\Gamma(\alpha)}\right]+\alpha(1-\alpha)z'\left[I^{\alpha}z-\frac{z(0)x^{\alpha}}{\Gamma(\alpha+1)}\right]=$$

$$=-xzD^{2-\alpha}z+z\frac{z(0)x^{\alpha-1}}{\Gamma(\alpha-1)}-(1-\alpha)\left[zD^{1-\alpha}z-z\frac{z(0)x^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)}+xz'D^{1-\alpha}z-\frac{z'z(0)x^{\alpha}}{(\alpha-1)\Gamma(\alpha-1)}\right]+$$

$$+\alpha(1-\alpha)\left[z'I^{\alpha}z-\frac{z'z(0)x^{\alpha}}{\alpha(\alpha-1)\Gamma(\alpha-1)}\right]=$$

$$=-xzD^{2-\alpha}z-(1-\alpha)zD^{1-\alpha}z-(1-\alpha)xz'D^{1-\alpha}z+\alpha(1-\alpha)z'I^{\alpha}z \tag{28}$$

Substituting (27) and (28) into (26), we obtain

$\varsigma_{1-\alpha}=\alpha zD^{1-\alpha}z+(1-\alpha^2)z'I^{\alpha}z-(1-\alpha)xz'D^{1-\alpha}z$.

As a result, the continued operators take the form

$$\tilde{X}_1=\frac{\partial}{\partial x}+\frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\frac{\partial}{\partial z^{(1-\alpha)}},\qquad \tilde{X}_2=\frac{\partial}{\partial z}+\frac{x^{\alpha-1}}{\Gamma(\alpha)}\frac{\partial}{\partial z^{(1-\alpha)}},\qquad \tilde{X}_3=x\frac{\partial}{\partial x}+(\alpha-1)z^{(1-\alpha)}\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_4=z\frac{\partial}{\partial x}+\left((\alpha-1)z'z^{(1-\alpha)}-\frac{z'z(0)x^{\alpha-1}}{\Gamma(\alpha-1)}+\frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\right)\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_5=x\frac{\partial}{\partial z}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial}{\partial z^{(1-\alpha)}},\qquad \tilde{X}_6=z\frac{\partial}{\partial z}+z^{(1-\alpha)}\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_7=x^2\frac{\partial}{\partial x}+xz\frac{\partial}{\partial z}+\left[(2\alpha-1)xz^{(1-\alpha)}+(1-\alpha^2)z^{(-\alpha)}\right]\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_8=xz\frac{\partial}{\partial x}+z^2\frac{\partial}{\partial z}+\left[\alpha zz^{(1-\alpha)}-(1-\alpha)xz'z^{(1-\alpha)}+(1-\alpha^2)z'z^{(-\alpha)}\right]\frac{\partial}{\partial z^{(1-\alpha)}},$$

Where

$z^{(1-\alpha)}\equiv{}_0D_x^{1-\alpha}z$. Hence, after the inverse change of variables

$z={}_0I_x^{1-\alpha}y$, we find the symmetries of equation (16):

$$X_1=\frac{\partial}{\partial x}+\frac{y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\frac{\partial}{\partial y},\qquad X_2=x^{\alpha-1}\frac{\partial}{\partial y},\quad X_3=x\frac{\partial}{\partial x}+(\alpha-1)y\frac{\partial}{\partial y},$$

$$X_4=y^{(\alpha-1)}\frac{\partial}{\partial x}+\left((\alpha-1)yy^{(\alpha)}-\frac{y^{(\alpha)}y^{(\alpha-1)}(0)x^{\alpha-1}}{\Gamma(\alpha-1)}+\frac{y^{(\alpha-1)}y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\right)\frac{\partial}{\partial y},$$

$$X_5=x^{\alpha}\frac{\partial}{\partial y},\qquad X_6=y\frac{\partial}{\partial y},\qquad X_7=x^2\frac{\partial}{\partial x}+\left[(2\alpha-1)xy+(1-\alpha^2)Iy\right]\frac{\partial}{\partial y},$$

$$X_8=xy^{(\alpha-1)}\frac{\partial}{\partial x}+\left[\alpha yy^{(\alpha-1)}-(1-\alpha)xyy^{(\alpha)}+(1-\alpha^2)y^{(\alpha)}Iy\right]\frac{\partial}{\partial y}.$$

Here $y^{(\alpha-1)} \equiv {}_0 I_x^{1-\alpha} y,\ Iy \equiv {}_0 I_x y$.

Symmetries X_2, X_3, X_5, X_6 are local, the rest of the symmetries are nonlocal. Note that the initial value $y^{(\alpha-1)}(0)$ included in the operators X_1 and X_4 is a natural initial condition in the statement of the Cauchy problem for fractional differential equations. Let us show that the coefficients of the operators $X_1 \dots X_8$ satisfy the defining equation

$$\varsigma_{\alpha+1}\Big|_{D^{\alpha+1}y=0} = 0,$$

which for equation (16) takes the form

$$D^{\alpha+1}(\eta - \xi y')\Big|_{D^{\alpha+1}y=0} = 0.$$

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Operators X_2, X_5, X_6. The check is trivial. For X_6 we have $D^{\alpha+1}(y)\Big|_{D^{\alpha+1}y=0} = 0$. For X_2, we have $\eta - \xi y' = x^{\alpha}$. By virtue of (25), we obtain $D^{\alpha+1}x^{\alpha} = 0$, since the gamma function has poles of order 1 at the points $x = 0$, $x = -n$, $n \in \mathrm{R}$. Similarly for X_5: $D^{\alpha+1}x^{\alpha-1} = 0$.

Operator X_1. $\varsigma_{\alpha+1} = D^{\alpha+1}\left(\dfrac{y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)} - y'\right)$.

Note that $D^{\alpha+1}y'$ and $D^{\alpha+1}x^{\alpha-2}$ are not exist, so the operator $D^{\alpha+1}$ cannot be applied to individual terms in this case. Relation (19) allows us to write down the following representation y for $f = I^{1-\alpha}y$:

$$y = D^{1-\alpha} I^{1-\alpha} y = I^{\alpha} D I^{1-\alpha} y + \frac{\left(I^{1-\alpha} y\right)(0) \cdot x^{\alpha-1}}{\Gamma(\alpha)} = I^{\alpha} D^{\alpha} y + \frac{y^{(\alpha-1)}(0) \cdot x^{\alpha-1}}{\Gamma(\alpha)} \quad (30)$$

Where

$$y' = D^{1-\alpha} D^{\alpha} y + \frac{y^{(\alpha-1)}(0) \cdot x^{\alpha-2}}{\Gamma(\alpha-1)} \quad (31)$$

due to (α - 1) Γ (α - 1) = Γ (α). Then

$$\varsigma_{\alpha+1} = -D_x^{1+\alpha}\left(D_x^{1-\alpha} D_x^{1} y\right).$$

By virtue of relation (19) for and equation (16), we have

$$D^{1-\alpha} D^{\alpha} y = D I^{\alpha} D^{\alpha} y = I^{\alpha} D^{\alpha+1} y + \frac{\left(D^{\alpha} y\right)(0) \cdot x^{\alpha-1}}{\Gamma(1-\beta)} = \frac{\left(D^{\alpha} y\right)(0) \cdot x^{\alpha-1}}{\Gamma(1-\beta)} \quad (32)$$

(the existence $\left(D^{\alpha} y\right)(0)$ of follows from the statement of the Cauchy problem for the original equation or from the existence of $z'(0)$). The fractional derivative of order α + 1 of expression (32) is equal to zero by virtue of (25), whence

$$\left.\varsigma_{1+\alpha}\right|_{D^{\alpha+1} y=0} = 0.$$

Stretch group operator X_3: $\varsigma_{1+\alpha} = D^{\alpha+1}\left((\alpha-1) y - x y'\right)$. Using the representation $xy'=(xy)'-y$ and the relation $D^{\alpha+1}(xy)' = D^{\alpha+2}(xy)$ (true by virtue of $\left.(xy)\right|_{x=0} = 0$), we obtain

$$\varsigma_{1+\alpha} = D^{\alpha+1}\left(\alpha y - (xy)'\right) = \alpha D^{\alpha+1} y - D^{\alpha+2}(xy) = -x D^{\alpha+2}(y) - 2 D^{\alpha+1}(y),$$

and

$$\left.\varsigma_{1+\alpha}\right|_{D^{\alpha+1} y=0} = 0.$$

Operator X_4:

$$\varsigma_{1+\alpha} = D_x^{1+\alpha}\left((\alpha-1) y y^{(\alpha)} - \frac{y^{(\alpha)} y^{(\alpha-1)}(0) x^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{y^{(\alpha-1)} y^{(\alpha-1)}(0) x^{\alpha-2}}{\Gamma(\alpha-1)} - y^{(\alpha-1)} y'\right).$$

Using Leibniz's rule (22), it is easy to see that the fractional derivatives of the first and second terms by virtue of equation (16) are equal to zero:

$$\left.D_x^{1+\alpha}\left(y y^{(\alpha)}\right)\right|_{D^{1+\alpha} y=0} = \sum_{n=0}^{\infty} \binom{1+\alpha}{n} \left.D^{1+\alpha-n} y \cdot D^{n+\alpha} y\right|_{D^{1+\alpha} y=0} = 0,$$

$$\left.D_x^{1+\alpha}\left(x^{\alpha-1} y^{(\alpha)}\right)\right|_{D^{1+\alpha} y=0} = \sum_{n=0}^{\infty} \binom{1+\alpha}{n} \left.D^{1+\alpha-n} x^{\alpha-1} \cdot D^{n+\alpha} y\right|_{D^{1+\alpha} y=0} = 0.$$

To simplify the rest of the expression, we use the representation y' (31):

$$D_x^{1+\alpha}\left(\frac{y^{(\alpha-1)}(0) y^{(\alpha-1)} x^{\alpha-2}}{\Gamma(\alpha-1)} - y^{(\alpha-1)} y'\right) = -D_x^{1+\alpha}\left(I^{1-\alpha} y \cdot D^{1-\alpha} D^{\alpha} y\right) =$$

$$= \sum_{n=0}^{\infty} \binom{1+\alpha}{n} D^{1+\alpha-n}\left(D^{1-\alpha} D^{\alpha} y\right) \cdot D^{n}\left(I^{1-\alpha} y\right)$$

By virtue of the equation and its differential consequences $D^{n+\alpha}=0$, all terms with n> 1 vanish. The first two terms are also equal to zero due to the relation (32) fulfilled on the equation:

$$-D^{1+\alpha}\left(D^{1-\alpha}D^{\alpha}y\right)\cdot I^{1-\alpha}y-(1+\alpha)D^{\alpha}\left(D^{1-\alpha}D^{\alpha}y\right)\cdot D^{\alpha}y=0.$$

Remark. It is clear from the proof that an operator of a simpler form than X_4 is allowed:

$$\hat{X}_4=y^{(\alpha-1)}\frac{\partial}{\partial x}+\frac{y^{(\alpha-1)}y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\frac{\partial}{\partial y}.$$

Operator X_7: $\varsigma_{1+\alpha}=D^{1+\alpha}\left((2\alpha-1)xy+\left(1-\alpha^2\right)Iy-x^2y'\right)$. Equalities are true

$$D^{1+\alpha}Iy=D^2I^{1-\alpha}Iy=D^2I\left(I^{1-\alpha}y\right)=DI^{1-\alpha}y=D^{\alpha}y,$$

$$D^{\alpha}(xy')=D^{\alpha}(xy)'-D^{\alpha}y=D^{\alpha+1}(xy)-D^{\alpha}y=xD^{\alpha+1}y+\alpha D^{\alpha}y,$$

$$D^{\alpha+1}(xy')=DD^{\alpha}(xy')=(1+\alpha)D^{\alpha+1}y+xD^{\alpha+2}y.$$

Thus

$$D^{1+\alpha}Iy=D^{\alpha}y,\ D^{\alpha}(xy')\Big|_{D^{\alpha+1}y=0}=\alpha D^{\alpha}y,\ D^{\alpha+1}(xy')\Big|_{D^{\alpha+1}y=0}=0 \quad (33)$$

Then

$$\varsigma_{1+\alpha}=(2\alpha-1)D^{\alpha+1}(xy)+\left(1-\alpha^2\right)D^{1+\alpha}Iy-D^{1+\alpha}(x\cdot xy')=(2\alpha-1)xD^{1+\alpha}(y)+$$

$$+(2\alpha-1)(1+\alpha)D^{\alpha}y+\left(1-\alpha^2\right)D^{\alpha}y-xD^{1+\alpha}(xy')-(1+\alpha)D^{\alpha}(xy').$$

After substituting equation (16) and using relations (33), we obtain

$$\varsigma_{1+\alpha}\Big|_{D^{1+\alpha}y=0}=\left(2\alpha^2+\alpha-1+1-\alpha^2\right)D^{\alpha}y-0-\alpha(1+\alpha)D^{\alpha}y=0.$$

Operator X_8: $\varsigma_{1+\alpha}=D^{1+\alpha}\left(\alpha yy^{(\alpha-1)}-(1-\alpha)xyy^{(\alpha)}+\left(1-\alpha^2\right)y^{(\alpha)}Iy-xy^{(\alpha-1)}y'\right)$.

We use Leibniz's rule (22) to represent each of the terms, taking into account that, by virtue of (16), $D^{n+\alpha}y=0$ for n> 0.

$$D^{1+\alpha}\left(\alpha yI^{1-\alpha}\right)=\alpha\sum_{n=0}^{\infty}\binom{1+\alpha}{n}D^{1+\alpha-n}y\cdot D^{n+\alpha-1}y=\alpha(1+\alpha)\left(D^{\alpha}y\right)^2,$$

$$D^{1+\alpha}\left((\alpha-1)xyD^{\alpha}y\right)=(\alpha-1)\sum_{n=0}^{\infty}\binom{1+\alpha}{n}D^{1+\alpha-n}(xy)\cdot D^{n+\alpha}y=(\alpha-1)D^{1+\alpha}(xy)=$$

$$=(\alpha-1)xD^{1+\alpha}(y)+(\alpha-1)(\alpha+1)\left(D^{\alpha}y\right)^2=\left(\alpha^2-1\right)\left(D^{\alpha}y\right)^2.$$

$$D^{1+\alpha}\left(\left(1-\alpha^2\right)D^{\alpha}yIy\right)=\left(1-\alpha^2\right)\sum_{n=0}^{\infty}\binom{1+\alpha}{n}D^{1+\alpha-n}(Iy)\cdot D^{n+\alpha}y=$$

$$=\left(1-\alpha^2\right)D^{1+\alpha}(Iy)D^{\alpha}y=\left(\alpha^2-1\right)\left(D^{\alpha}y\right)^2,$$

$$D^{1+\alpha}\left(-xy' I^{1-\alpha} y\right) = -\sum_{n=0}^{\infty} \binom{1+\alpha}{n} D^{1+\alpha-n}(xy') D^{n+\alpha-1} y = -D^{1+\alpha}(xy') I^{1-\alpha} y -$$
$$-(1+\alpha) D^{\alpha}(xy') D^{\alpha} y = -\alpha(1+\alpha)\left(D^{\alpha} y\right)^2.$$

It is easy to see that the sum of the right-hand sides of all equalities gives 0, which is what was required to prove. Note that simpler operators are also allowed:

$$\hat{X}_8 = xy^{(\alpha-1)} \frac{\partial}{\partial x} + \alpha y y^{(\alpha-1)} \frac{\partial}{\partial y}, \quad \overline{X}_8 = \left((\alpha-1) x y y^{(\alpha)} + \left(1-\alpha^2\right) y^{(\alpha)} I y\right) \frac{\partial}{\partial y}$$

Remark. Earlier, in [9], from the invariance principle for equation (16), five local symmetries were obtained, including the projection operator

$$X_9 = x^2 \frac{\partial}{\partial x} + \alpha x y \frac{\partial}{\partial y}.$$

This operator cannot be obtained from X_1 , ..., X_8, but the closest to it is X_7, obtained from the projection operator for the equation $z'' = 0$. It is easy to verify that the nonlocal operator.

$$X_{10} \equiv X_7 - X_9 = \left[(\alpha-1) x y + \left(1-\alpha^2\right) I y\right] \frac{\partial}{\partial y}$$

Allowed by equation (16). Moreover, in the limiting case $\alpha = 1$, the operator X_{10} vanishes, that is, X_7 is the same as X_9.

Conclusion

The continuation formulas obtained in this work make it possible to study the symmetry properties of a new class of differential equations containing fractional derivatives of a function with respect to another function. In this case, the next important task to be solved is the development of a method for resolving the resulting governing equation. In this case, the main difficulty is the question of the rules for splitting the governing equation.

Another area of further research is the systematization of results on nonlocal symmetries of differential equations of fractional order and the development of new algorithms for their construction. It is also very important to develop rules for the classification of nonlocal symmetries of such equations.

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