Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
Impact Factor (RJIF): 5.94
JMPES 2025; 6(2): 222-226

www.mathematicaljournal.com
Received: 11-05-2025
Accepted: 14-06-2025

Ubong D Akpan
Department of Mathematics, Akwa Ibom State University, Mkpat Enin, Nigeria

Corresponding Author:
Ubong D Akpan
Department of Mathematics, Akwa Ibom State University, Mkpat Enin, Nigeria

Stability analysis of systems of three sub-fractional differential equations

Ubong D Akpan

DOI: https://www.doi.org/10.22271/math.2025.v6.i2b.234

Abstract
In this work, the stability analysis of systems of sub-fractional differential equations is carried out. Stability status of systems of fractional differential equations satisfying stability notions of fractional input stability, Mittag-Leffler input stability and global asymptotic stability have been determined. Stability theorems have been stated and examples have been given to illustrate the application of the theorems.

Keywords: Fractional differential equations, Fractional input stability, Mittag-Leffler input stability, global asymptotic stability

Introduction

Many scientific and engineering systems can be formulated as differential equations, especially those that require dynamical laws. There are basically two types of derivatives namely, fractional order derivative and integer order derivative. Fractional-order derivative deals with the whole time domain and space of a physical process, while the classical derivative is concerned with a particular time and local properties of a certain position. Fractional calculus, as a generalization of integer- order or classical calculus has been used as a valuable tool in the modelling of many physical phenomena and engineering systems. Research in fractional order systems is relatively new due to the absence of solution methods for fractional differential equations. Fractional order systems have more degrees of freedom in the model compared to integer order systems. Recently, fractional differential equations have become popular in modelling processes such as control theory, signal processing, bioengineering, circuit theory and viscoplasticity. Fractional differential equations also have great applications in other fields such as secure communication, data encryption, financial systems, chaos control and chaos synchronization.

Further studies on fractional calculus have given rise to the development of many fractional derivatives such as Liouville-Riemann derivative [1], Caputo-Fabrizio derivative [2], Conformable derivative [3], Caputo-Liouville derivative [1], Atangana-Baleanu derivative [4, 5, 6] and others. The importance of stability analysis of systems cannot be over-stated. Stability analysis is carried out on systems because of its time-serving and resources conservation gains. It determines the pattern and behavior of the solutions of differential equations. Before analyzing, many physical systems are expressed or modelled as differential equations. The solutions of these differential equations are analysed to obtain the stability status of the differential systems. Over the years, many works focusing on stability analysis of differential equations for both classical and fractional order systems have been done [7-18]. Many researchers have developed stability conditions either necessary or sufficient conditions which are used in determining the stability status of differential systems [19-21]. Also, researchers have developed new fractional derivatives and methods of establishing systems stability [22-24].

Stability analysis of fractional differential equations involving external inputs is relatively new and was introduced into the literature of fractional differential equations by Sontag [25]. This has given rise to the evolution of various stability notions involving external inputs. Studies have been carried out on fractional input stability and Mittag-Leffler input stability of systems demonstrating converging-input-converging-state and bounded-input bounded-state properties [26, 27]. Akpan [17] investigated the stability status of the system of fractional differential equations with two sub-fractional differential equations which are fractional input stable and globally asymptotically stable respectively.

This work examines a system of fractional differential equation with three sub-fractional differential equations which are fractional input stable, Mittag-Leffler input stable and globally asymptotically stable.

Definitions and Preliminary Analysis

In this section, some definitions and concepts that would be needed in the analysis are given.

Definition 2.1

The Mittag-Leffler function with two parameters is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where the parameters $\alpha > 0, \beta \in \mathbb{R}$ and $z \in \mathbb{C}$. For $\beta = 1, E_\alpha(z) = E_{\alpha,1}(z)$.

Definition 2.2

The set of all continuous map $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\alpha(0) = 0$, and $\alpha(t) > 0$ for all $t > 0$ represents the class PV function. A class K function is an increasing PV function. The class K_∞ represents the set of all unbounded K functions.

Definition 2.3

The Caputo-Liouville generalized fractional derivative denoted by $D_c^{\alpha,\rho}$ is defined by

$$\left(D_c^{\alpha,\rho} g\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\alpha} g'(s) ds,$$

for all $t > 0$, where the order $\alpha \in (0,1), \rho > 0$.

Definition 2.4

A fractional differential equation where a class KL function β exists such that for any initial condition y_0, the inequality

$$\|h(t, y_{0)}\| \leq \beta(\|y_0\|, t - t_0)$$

holds. Then, the equation is globally asymptotically stable.

Definition 2.5

The origin of the unforced equation defined by $D_c^\alpha x = f(x, 0)$ is said to be Mittag-Leffler stable, if for any initial condition x_0, its solution satisfies

$$\left\|x(t, x_{0)}\right\| \leq [d(\|x_0\|) E_\alpha(\lambda(t - t_0)^\alpha)]^{\frac{1}{q}}$$

where $q > 0$, and d is locally Lipschitz[27].

Definition 2.6

The equation defined by $D_\alpha^c x = f(h, x, v)$is said to be Mittag-Leffler input stable if, for any input $v \in \mathbb{R}^n$, there exists a K_∞ function γ such that for any initial condition x_0, its solution satisfies

$$\|x(h, x_0, v)\| \leq [\varphi(\|x_0\|) E_\alpha(\lambda(h - h_0)^\alpha)]^{\frac{1}{q}} + \gamma(\|v\|_\infty).$$

where φ and $q > 0$ are nonnegative constants [27].

Definition 2.7 [26]

The equation defined by $D_\alpha^c y = f(h, x, v)$is said to be fractional input stable if, for any input $v \in \mathbb{R}^n$, a class KL function β exists and a K_∞ function γ such that for any initial condition $x(h_0)$, its solution satisfies

$$\|x(h, x_0, v)\| \leq \beta(\|x_0\|, h - h_0) + \gamma(\|v\|_\infty)$$

Definition 2.8

The trivial solution to system $D_\alpha^c y = f(h, x, 0)$ is said to be stable if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that for any initial condition $\|x_0\| < \delta$, the solution $x(h)$ of the system $D_\alpha^c x = f(h, x, 0)$ satisfies the inequality $\|x_0\| < \epsilon$ for all $h > h_0$.
The system $D_\alpha^c y = f(h, x, v)$ which is stable and $\lim_{h \to \infty} y(h) = 0$. Then, the system asymptotically stable.

Main results

Consider the system of equations

$$\begin{cases} D_c^{\alpha,\rho} y = f(x_1, x_2, x_3) \\ D_c^{\alpha,\rho} g = f(x_2, x_3) \quad (3.1) \\ D_c^{\alpha,\rho} z = f(x_3) \end{cases}$$

consisting of three sub-equations. The following theorems are stated and proved as the main results.

Theorem 3.1

Consider the system of fractional differential equation (3.1) where $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$ is fractional input stable, $D_c^{\alpha,\rho} g = f(x_2, x_3)$ is Mittag-Leffler input stable and $D_c^{\alpha,\rho} z = f(x_3)$ is globally asymptotically stable. Then, the system (3.1) is globally asymptotically stable.

Proof

Consider the fractional input stability of the fractional differential equation $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$. This implies the input of the differential equation is converging. Therefore, the solution of the equation is converging. This property is known as the Converging –Input-Converging-State (CICS). Also, the input is bounded and the solution is also bounded. This is known as the Bounded-Input-Bounded-State (BIBS) property. The equation $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$ is therefore convergent. Mittag-Leffler input stability is a special case of FIS and therefore possesses the CICS and BIBS properties. Therefore, $D_c^{\alpha,\rho} g = f(x_2, x_3)$ is convergent. For global asymptotic stability of the equation defined by $D_c^{\alpha,\rho} z = f(x_3)$, it implies that $\lim_{t\to\infty} \|x_3(t)\| = 0$. Therefore, equation (3.1) is convergent.

Next, we use the stability notions to establish the stability of equation (3.1). If the equation $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$ is the fractional input stable, it implies that there exists a class KL function β_1 and a K_∞ function γ, such that for any initial condition ξ_1, its solution satisfies

$$\|\boldsymbol{x}(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})\| \le \beta_1(\|\xi_1\|, t - t_0) + \gamma(\|x_2\|_\infty)$$

$$\le \beta_1(\|\xi_1\|, 0) + \gamma(\|x_2\|_\infty)$$

$$\le \ \beta_1(\|\xi_1\|, 0) + \gamma(\beta_1(\|\xi_1\|, 0) \ (3.2)$$

From the Mittag-Leffler input stability of the fractional differential equation $D_c^{\alpha,\rho} x_2 = f(x_2, x_3)$, if for any input $x \in R^3$, there exists a class K_∞ function γ such that for any initial condition ξ_2, its solution satisfies

$$\|x(x_2, x_3)\| \le [\beta_2\|\xi_2\| E_\alpha(\lambda(t - t_0)^\alpha)]^{\frac{1}{q}} + \gamma(\|x_3\|_\infty)$$

$$\le \beta_2(\|\xi_2\|, t - t_0) + \gamma(\|x_3\|_\infty)$$

$$\le \ \beta_2(\|\xi_2\|, 0) + \gamma(\beta_2\|\xi_2\|, 0) \ (3.3)$$

From global asymptotic stability of fractional differential equation $D_c^{\alpha,\rho} z = f(x_3)$, there exists a class KL function β_3 such that for any initial condition ξ_3, we have

$\|\boldsymbol{x_3(t)}\| \le \beta_3(\|\xi_3\|, 0)$ (3.4)

From (3.2), (3.3) and (3.4), there exists ϵ such that $\|x(t)\| = \|x(x_1, x_2, x_3)\| \le \epsilon$.

Therefore, equation (3.1) is stable. Combining the convergence, boundedness and the stability of the component fractional differential equations, equation (3.1) is therefore Globally Asymptotically stable.

Theorem 3.2

Consider the system of fractional differential equation (3.1) where $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$, $D_c^{\alpha,\rho} g = f(x_2, x_3)$ are Mittag-Leffler input stable and $D_c^{\alpha,\rho} z = f(x_3)$ is globally asymptotically stable. Then, the system (3.1) is globally asymptotically stable.

Proof

From the Mittag-Leffler input stability of $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$ and $D_c^{\alpha,\rho} g = f(x_2, x_3)$, it follows that the fractional differential equations satisfy the CICS and BIBS properties. Therefore, the equations are convergent. From the global asymptotic stability of $D_c^{\alpha,\rho} z = f(x_3)$, it implies that $\lim_{t\to\infty} \|x_3(t)\| = 0$. Therefore, $D_c^{\alpha,\rho} x_3 = f(x_3)$ is convergent.

Also, from the Mittag-Leffler input stability of fractional differential equations $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$ and $D_c^{\alpha,\rho} g = f(x_2, x_3)$, the equations are stable(as in the proof of Theorem 3.1). Similarly, $D_c^{\alpha,\rho} z = f(x_3)$ is stable since it is asymptotically stable. Combining the convergence and stability of the individual fractional differential equations, the system (3.1) is Globally Asymptotically Stable.

Theorem 3.3

Consider the system of fractional differential equation (3.1) where $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$, $D_c^{\alpha,\rho} g = f(x_2, x_3)$ are Mittag-Leffler input stable and $D_c^{\alpha,\rho} z = f(x_3)$ is fractional input stable. Then, the system (3.1) is fractional input stable.

Proof

The Mittag-Leffler input stability of $D_c^{\alpha,\rho} y = f(x_1, x_2, x_3)$ and $D_c^{\alpha,\rho} g = f(x_2, x_3)$ implies the fractional differential equations are convergent and stable as in the proof of Theorem 3.1. Also, the fractional input stability of the equation $D_c^{\alpha,\rho} z = f(x_3)$ is convergent and stable as in the proof of Theorem 3.1. But Mittag-Leffler input stability is a special case of fractional input stability. Therefore, the system (3.1) is Fractional Input Stable.

Application and Analysis

Example 1

Consider the system of fractional differential equations consisting of sub-fractional differential equations

$$\begin{cases} D_c^{\alpha,\rho} y = -2x_1 + 2x_2 \\ D_c^{\alpha,\rho} g = 3x_1 - x_2 \\ D_c^{\alpha,\rho} z = -4x_3 \end{cases} \tag{4.1}$$

where $D_c^{\alpha,\rho} y = -2x_1 + 2x_2$ is fractional input stable, $D_c^{\alpha,\rho} g = 3x_1 - x_2$ is Mittag-Leffler input stable and $D_c^{\alpha,\rho} z = -4x_3$ is globally asymptotically stable. We show that the sub-fractional differential equation

$D_c^{\alpha,\rho} y = f(x_1, x_2, x_2) = -2x_1 + 2x_2$ (4.1a)

is fractional input stable. The solution of (4.1a) is given by the following [28].

$$y(t) = \xi E_\alpha\left(P\left(\frac{t^\rho}{\rho}\right)^\alpha\right) + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}(P\left(\frac{t^\rho}{\rho}\right)^\alpha x_2(s)\frac{ds}{s^{1-\rho}}$$

Further modifications result in the following

$$\|y(t)\| \le \mu_1(\|\xi_1\|, t^\rho) + G\|x_2\|$$

where $\mu_1(\|\xi_1\|, t^\rho) = \xi E_\alpha\left(P\left(\frac{t^\rho}{\rho}\right)^\alpha\right)$ and G is a constant. This shows that equation(4.1a) satisfies CICS as well as BIBS and $\lim_{t\to\infty}\|y(t)\| = 0$ and therefore fractional input stable.

The sub-fractional differential equation

$D_c^{\alpha,\rho} g = 3x_1 - x_2$ (4.1b)

exhibits the CICS and BIBS properties and therefore convergent. Also, $\lim_{t\to\infty}\|g(t)\| = 0$.

Therefore, equation (4.1b) is Mittag-Leffler input stable.

The sub-fractional differential equation

$D_c^{\alpha,\rho} z = -4x_3$ (4.1c)

is obviously globally asymptotically stable. Combining the convergence and the stability of the individual fractional differential equations as well as utilizing Theorem 3.1, we conclude that system (4.1) is globally asymptotically stable. This result agrees with the result obtained by using the Matignon condition on system (4.1).

Example 2

Consider the system

$$\begin{cases} D_c^{\alpha,\rho} y = -3x_1 + 3x_2 \\ D_c^{\alpha,\rho} g = -4x_1 + 5x_2 \\ D_c^{\alpha,\rho} z = -6x_3 \end{cases} \tag{4.2}$$

where $D_c^{\alpha,\rho} y = -3x_1 + 3x_2$ and $D_c^{\alpha,\rho} g = -4x_1 + 5x_2$ are Mittag-Leffler input stable. This implies that CICS and BIBS properties are satisfied. Also, $\lim_{t\to\infty}\|z(t)\| = 0$. Therefore, the system (4.2) is convergent.

The Mittag-Leffler input stability of these equations implies the stability of the sub-fractional differential equations. Also, the global asymptotic stability of $D_c^{\alpha,\rho} z = -6x_3$ implies the stability of the sub-fractional differential equation. Combining the convergence and stability as well as Theorem 3.2, we conclude that the system (4.2) is globally asymptotically stable. Applying the Matignon condition for establishing asymptotic stability, the result agrees with the result obtained by using Theorem 3.2. Similar analysis can be carried out to illustrate the utilization of Theorem 3.3.

Conclusion

Stability analysis of differential equations with external input is an emerging area of research with vast applications in sciences and engineering. The properties of the stability notions have been studied. Stability theorems for establishing the stability status of the fractional differential systems have been stated and proved. Examples have been given to illustrate the application of the theorems.

References

1. Fahd J, Abdeljawad T. A modified Laplace transform for certain generalized fractional operators. Res Nonl Anal. 2018;2:88-98.
2. Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Progr Fract Differ Appl. 2018;1(2):1-15.
3. Khalil R, Horani MA, Yousef A, Sababheh M. A new definition of fractional derivative. J Comput Appl Math. 2014;264:65-70.
4. Hristov J. On the Atangana-Baleanu derivative and its relation to the fading memory concept: the diffusion equation formulation. In: Gomez J, Torres L, Escobar R, editors. Fractional Derivatives with Mittag-Leffler Kernel, Studies in Systems, Decision and Control. Cham: Springer; 2019. p. 175-193.
5. Khan H, Jarad F, Abdeljawad T, Khan A. A singular ABC-fractional differential equation with p-Laplacian operator. Chaos Soli Fract. 2019;129:56-61.
6. Abdeljawad T, Baleanu D. On fractional derivatives with generalized Mittag-Leffler kernels. Adv Diff Equat. 2018;468.
7. Samuel OE, Ante JE, Otobi AO, Okeke SI, Akpan UD, Francis RE, Auta JT. Optimizing neural networks with convex hybrid activations for improved gradient flow. Adv J Sci Technol Eng. 2025;5(1):10-27. ISSN: 2997-5972.
8. Akpan UD, Oyesanya MO. On the stability analysis of MDGKN systems with control parameters. J Adv Math Comput Sci. 2017;25(5).
9. Akpan UD, Oyesanya MO. Stability analysis and response bounds of gyroscopic systems. Asian Res J Math. 2017;5(4). Available from: www.sciencedomain.org/issue/2848
10. Akpan UD. On the stability analysis of linear unperturbed non-integer differential systems. Asian Res J Math. 2021. Available from: www.journalarjom.com/index.php/ARJOM/article/view/30301
11. Akpan UD. The influence of circulatory forces on the stability of undamped gyroscopic systems. Asian Res J Math. 2021. Available from: www.journalarjom.com/index.php/ARJOM/article/view/30302
12. Akpan UD. Stability analysis of perturbed linear non-integer differential systems. J Adv Math Comput Sci. 2021. Available from: www.journaljamcs.com/index.php/JAMCS/article/view/30370
13. Akpan UD. On the analysis of damped gyroscopic systems using Lyapunov direct method. J Adv Math Comput Sci. 2021. Available from: www.journaljamcs.com/index.php/JAMCS/article/view/30372
14. Atsu JU, Ante JE, Inyang AB, Akpan UD. A survey on the vector Lyapunov functions and practical stability of nonlinear impulsive Caputo fractional differential equations via new modelled generalized Dini derivative. IOSR J Math. 2024;20(4):28-42.
15. Ante JE, Akpan UD, Igomah GO, Akpan CS, Ebere UE, Okoi PO. On the global existence of solution of the comparison system and vector Lyapunov asymptotic eventual stability for nonlinear impulsive differential systems. Br J Comput Netw Inf Technol. 2024;7(4):103-117.
16. Ineh MP, Akpan EP, Akpan UD, Maharaj A, Narain OK. On total stability analysis of captured fractional dynamic equations on time scale. Asia Pac J Math. 2025;12(39):1-14.
17. Akpan UD. Analysis of fractional input stability and global asymptotic stability of systems of fractional differential equations. Afr J Math Stat Stud. 2025;8(2).
18. Akpan UD. Mittag-Leffler stability analysis of differential equations with and without input using Caputo derivative. Int J Stat Appl Math. 2025.
19. Yao Z, Yang Z, Fu Y, Liu S. Stability analysis of fractional order differential equations with multiple delays. Chin J Phys. 2024;89:951-968.
20. Hai X, Yu Y, Xu C, Ren G. Stability analysis of fractional differential equations with the short-term memory property. Fract Calc Appl Anal. 2022;25(2).
21. Ristiani LD, Angelina S, Trirahma T. Stability analysis of fractional order differential equations with time delays. Int J Appl Math Comput. 2024;1(2):27-35.
22. Zhao K. Stability of nonlinear Langevin system of MK-type fractional derivative affected by time-varying delays and differential feedback control. Fractal Fract. 2022;6(12):725.
23. Swashankar M, Sabarinathan S, Govindan V, Fernandez-Gamiz U, Noeiaghdam S. Stability analysis of COVID-19 outbreak using Caputo-Fabrizio fractional derivative. AIMS Math. 2023;8(8):2720-2735.
24. Hattaf K. Stability of fractional differential equations with new generalized Hattaf fractional derivative. Math Probl Eng. 2021;2021(1):Article ID 8608447, 7 pages.
25. Sontag ED. Smooth stabilization implies coprime factorization. Syst Contr Let. 1989;34:435-443.
26. Sene N. Fractional input stability for electrical circuits described by the Riemann-Liouville and the Caputo fractional derivatives. AIMS Math. 2019;4(1):147-165.
27. Sene N. Mittag-Leffler input stability of fractional differential equations and its applications. Discrete Contin Dyn Syst Ser S. 2020;13(3).
28. Sene N. Stability analysis of the generalized fractional differential equations with and without exogenous inputs. J Non-linear Sci Appl. 2019;12:562-572.