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Role of numerical range on some operator equations in Hilbert spaces

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Abstract
In operator theory equations occur in various situations. Some authors who have used the numerical range in solutions of such equations have mainly put some conditions on the closure of the numerical range rather than its interior. In this paper our task is to try and relax the conditions on the closure to mere interior of the numerical range.
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Introduction

In the sequel, H denotes a complex Hilbert space and B(H) denotes the Banach algebra of bounded linear operators on H. The elements of B(H) will be denoted by capital letters e.g. $A, B \in$ B(H), sometimes we may also use the letter H for an element of B(H). For $T \in$ B(H) the numerical range is given as $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$.

Several authors over the past few decades have used the numerical range for desired solutions in operator equations, by requiring that either 0 does not belong to the interior of the numerical range of the intertwining operator or does not belong to the closure of the numerical range of such operator. For example, M.R. Embry [2] in considering the equality of H and K in the equation $AH = KA$ imposed the conditions that $0 \notin W(A)$ implies $H = K$ provided H and K are commuting normal operators. Thus, means that the condition $0 \notin W(A)$ is not sufficient in itself. I.H. Sheth and J.M Khalagai [3] considered this same equation and removed the commutativity condition on H and K by showing that if $AH = KA$ and $AK = HA$, H and K normal with $0 \notin W(A)$ then $H = K$.

J.P Williams [5] considered the equation $ST = T^*S$ in which the desire was to look for conditions under which $T = T^*$. Thus T is self adjoint. More specifically he showed that if $ST = T^*S$ with $0 \notin \overline{W(S)}$ (closure of $W(S)$) and T is hyponormal then $T = T^*$. We note that in this case the condition that $0 \notin \overline{W(S)}$ is more stringent than $0 \notin W(S)$. I.H. Sheth and J.M. Khalagai [4] also considered the equation $TST^* = S$ for unitary solutions where they first proved the following results. If $TST^* = S$ with T left invertible then T is invertible provided $0 \notin \overline{W(S)}$ or $0 \notin \overline{W(ReS)}$ or $0 \notin \overline{W(ImS)}$.

In this paper we continue with this study of looking into operator equation with the aim of trying to relax the condition $0 \notin \overline{W(S)}$ to $0 \notin W(S)$.

Notations, definitions and terminologies

We note that as indicated in the abstract H denotes a complex Hilbert space and B(H) denotes Banach algebra of all bounded linear operators on H. For $A \in$ B(H) the numerical range $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$ its closure is denoted by $\overline{W(A)}$ while the spectrum of As defined and denoted as $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \ \mathit{is\ not\ invertible}\}$

where $\mathbb{C}$ denotes the complex number field. The point spectrum of A is given by $\sigma_p(A) = \{\lambda \in \mathbb{C}: Ax = \lambda x\}$. Thus $\sigma_p(A) \subseteq \sigma(A) \subseteq \overline{W(A)}$.

The commutator of two operators A and B is given by $[A, B] = AB - BA$.

An operator $A \in$ B(H) is said to be;

- Self adjoint if $A = A^*$ where A^* denotes the adjoint of A.
- Isometric if $A^*A = I$
- Co-Isometric if $AA^* = I$
- Unitary if $A^*A = AA^* = I$
- Normal operator if $[A, A^*]$
- Compact if for any bounded sequence (x_n) in H the sequence (Ax_n) contains a convergent subsequence.
- Hyponormal if $A^*A \geq AA^*$
- M-hyponormal if $(A - \lambda)(A - \lambda)^* \leq M(A - \lambda)^*(A - \lambda)$ for $\lambda \in \mathbb{C}$ and M positive number
- W-hyponormal if the inequality $|\tilde{A}| \geq |A| \geq |\tilde{A}^*|$ holds
- P-hyponormal if inequality $(A^*A)^p \geq (AA^*)^p$ holds for $1 \geq p > 0$.
- Log-hyponormal if A is invertible and $log(A^*A) \geq log(AA^*)$.

We have the following set inclusion of some classes of operators.

1. $\{Unitary\} \subseteq \{Isometry\}$
2. $\{Unitary\} \subseteq \{Co - sometry\}$
3. $\{Self - adjoint\} \subseteq \{Normal\} \subseteq \{Hyponormal\} \subseteq \{M - hyponormal\}$
4. $\{Hyponormal\} \subseteq \{P - hyponormal\} \subseteq \{w - hyponormal\}$
5. $\{Hyponormal\} \subseteq \{log - hyponormal\} \subseteq \{w - hyponormal\}$

Main results

Theorem 3.1

Let $T, S \in$ B(H) satisfy the equation $TST^* = S - (1)$ where $0 \notin W(S)$ and T left invertible. Then T is invertible.

Proof

Let T_l be the left inverse of T. Then $TST^* = S$ implies $T_lTST^* = T_lS$ i.e. $ST^* = T_lT$ using (1) we have $TT_lS = S$ i.e. $(TT_l - I)S = 0$. Since $0 \notin W(S)$ it follows that S has dense range thus we have that $TT_l - I = 0$ i.e. $TT_l = I = T_lT$

Hence T is invertible.

Remark 3.2

1. It follows easily that in equation (1) above if T^* is right invertible then T^* is invertible and hence T is invertible
2. The following corollary is immediate. Thus, we have

Corollary 3.3

Let $T, S \in$ B(H) satisfy equation (1) above with $0 \notin W(S)$. Then we have that.

1. T is unitary if it is isometric
2. T is unitary if T^* is a coisometry

Proof

1. T is isometric implying it is left invertible. Hence result follows from theorem 3.1 above.
2. T^* is coisometry implies it is right invertible from remark above and hence it is invertible. Thus, the result follows from theorem 3.1 above.

Corollary 3.4

Let $T, S \in$ B(H) satisfy equation (1) where T is left invertible. Then T is invertible under each of the following conditions

1. $0 \notin W(ReS)$
2. $0 \notin W(ImS)$

Proof

Given $TST^* = S$ taking adjoints gives, $TS^*T^* = S^*$ i.e. $TST^* + TS^*T^* = S + S^*$ i.e. $T(S + S^*)T^* = S + S^*$ i.e. $T(ReS)T^* = ReS$ Similarly $TST^* - TS^*T^* = S - S^*$ i.e. $T(S - S^*)T^* = S - S^*$ i.e. $T(ImS)T^* = ImS$

The proof of theorem 3.1 can now be traced to give results in both parts (i) and (ii) of corollary 3.4 above

Corollary 3.5

Let T and S be operators satisfying equation (1). If T is an isometry then T is unitary under each of the following conditions

1. $0 \notin W(ReS)$
2. $0 \notin W(ImS)$

Proof

We note that T is an isometry implies T is left invertible. In this case the proof follows from corollary 3.4

I.H Sheth and J.M Khalagai [3] proved the following result

Theorem A [4]

Let T and S be operators satisfying the equation $TST^* = S$ with T left invertible. Then T is invertible under any of the following conditions

1. $0 \notin \overline{W(S)}$
2. $0 \notin \overline{W(ReS)}$
3. $0 \notin \overline{W(ImS)}$

Remark 3.6

We note that it is clear that theorem 3.1 and corollary 3.4 together constitute an improvement of theorem (A) above in the sense that we have removed closure on the numerical range wherever it appears. Thus, we have relaxed the conditions $0 \notin \overline{W(S)}$ or $0 \notin \overline{W(ReS)}$ or $0 \notin \overline{W(ImS)}$ to those of $0 \notin W(S)$ or $0 \notin W(ReS)$ or $0 \notin W(ImS)$ respectively.

We now bring into our discussions the following equation which was considered by J.P. Williams [5] which is as follows $ST = T^*S$ for $T, S \in$ B(H)
Specifically, he proved the following result.

Theorem (B) [5]
Let $T, S \in$ B(H) satisfy the equation $ST = T^*S$ with T hyponormal and $0 \notin \overline{W(S)}$. Then T is self adjoint.
We require the following theorems in our further discussions

Theorem (C) [1] **Putnam Fuglede property**
Let $T, S \in$ B(H) satisfy the equation $TX = XS$ then we have $T^*X = XS.^*$ Also M.R Embry proved the following result.

Theorem (D) [2]
Let A and E be operators such that $AE = -EA$ where either A or E is normal and $0 \notin W(A)$ then $E = 0$
In view of the two theorems above we have the following result.

Theorem 3.7
Let $T, S \in$ B(H) satisfy the equation $ST = T^*S$ with $0 \notin W(S)$ where T and T^* satisfy Putnam Fuglede property. Then T is self adjoint.

Proof
Now $ST = T^*S$ implies $ST^* = TS$ i.e. $S(T - T^*) = (T^* - T)S$ i.e. $S(T - T^*) = -(T - T^*)S$. We now show that $T - T^*$ is normal indeed let $L = T - T^*$. Then we have $L^* = T^* - T$
$LL^* = (T - T^*)(T^* - T) = TT^* - T^2 - T^{*2} + T^*T$ (i)
Also $L^*L = (T^* - T)(T - T^*) = T^*T - T^{*2} - T^2 + TT^*$ (ii)
From (i) and (ii) $LL^* = L^*L$. Thus, $T - T^*$ is normal. Since $0 \notin W(S)$ we have by theorem (D) above that $T - T^* = 0 \Longrightarrow T = T^*$. Hence T is self adjoint.

Corollary 3.8
Let T and T^* be operators satisfying Putnam Fuglede property and such that $ST = T^*S$ with either $0 \notin W(ReS)$ or $0 \notin W(ImS)$. Then T is self adjoint.

Proof
$ST = T^*S$ implies $ST^* = TS$ i.e. $ST - ST^* = T^*S - TS$
i.e. $S(T - T^*) = (T^* - T)S$ (i)
Taking adjoints gives $(T^* - T)S^* = S^*(T - T^*)$ (ii)
Adding (i) and (ii) gives
$(S + S^*)(T - T^*) = (T^* - T)(S + S^*)$
i.e. $ReS(T - T^*) = (T^* - T)ReS$
i.e. $ReS(T - T^*) = -(T - T^*)ReS$
Similarly, $ImS(T - T^*) = -(T - T^*)ImS$
Now by theorem 3.7 above $T - T^* = 0$ i.e. $T = T^*$. Hence T is self adjoint.

Remark 3.9
We note that theorem 3.7 improves theorem (B) of J.P Williams above in two aspects. In the first case it has relaxed the condition $0 \notin \overline{W(S)}$ to $0 \notin W(S)$. Secondly, we have dropped the condition that T is hyponormal and replaced it with operators satisfying Putnam Fuglede property. This property includes quite a number of classes of operators which contain hyponormal operators as a subclass in fact the following corollary shades more light on this aspect.

Corollary 3.10
Let $T, S \in$ B(H) satisfy the equation $ST = T^*S$ with either $0 \notin W(S)$ or $0 \notin W(ReS)$ or $0 \notin W(ImS)$. Then T is self adjoint provided T and T^* belong to each of the following classes of operators
1. M-hyponormal
2. W-hyponormal
3. Log-hyponormal

Proof
The result follows from the fact that each of the classes above satisfies the Putnam Fuglede property.
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References

1. Berberian SK. Note on a theorem of Fuglede and Putnam. Proceedings of the American Mathematical Society. 1959;10(2):175-82.
2. Embry M. Similarities involving normal operators on Hilbert space. Pacific Journal of Mathematics. 1970;35(2):331-6.
3. Sheth IH, Khalagai JM. on the operators equation AH=KA. Mathematics today. 1987;5:29-36.
4. Khalagai JM. On the Operator Equation TST∗=S. Unitary Solutions. Kenya J. Science Technology Series A. 1985;6(2):157-63.
5. Williams JP. Operators similar to their adjoints. Proceedings of the American Mathematical Society. 1969;20(1):121-3.
6. Kipruto E, Muhati L, Meli C. On numerical range and spectral properties of some classes of operators in hilbert spaces. 2020.