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The connection between module theory and act theory concerns generalizations of extending act over monoids

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Abstract
Generalizations of injective acts have long been a source of interest. However, no specific research has been conducted on generalizations of extending acts. This is what prompted us to propose a new generalization of extending acts, the prime-extending act. This is a generalization of the author's previous work, as well as that of J. Ahsan and Liu Zhongkui. Furthermore, because the theory of acts is a generalization of module theory, this study not only generalized the theory of acts but also Tamadher A. Ibrahiem's work in module theory. Around the same time, the idea of R-prime acts was established. Several aspects and features related to these concepts were discussed. Some results about extending acts are applied to prime-extending acts. Furthermore, certain novel features of prime-extending and R-prime acts are discussed and achieved. An S-act A is prime-extending if all non-zero proper sub-acts of A are ⋂-large in the prime retract. Furthermore, the concept of R-prime is introduced, with an S-act being considered R-prime if each proper retract is prime. It was clarified that prime-extending acts and extending acts will be the same under the R-prime act condition. Finally, it was determined that a semi-simple act and prime-extending acts are equivalent if and only if the S-act is R-prime.

Keywords: Extending acts, prime sub-acts, prime-extending acts, prime retract sub-acts

Introduction

S denotes a monoid that includes zero elements 0 throughout this paper, and each right S-act includes both unity and zero elements, then it indicated by M_S . An S-act M_S which contain zero is a non-empty set along with a mapping ($f: M \times S \to M$, $f(m, s) \mapsto ms$)characterized by these properties: ((1) $m \cdot 1 = m$), and associative property ((2) $m(st) = (ms)t$, for all $m \in M$ and $s, t \in S$),and 1is the identity element of S [1]. An S-act (B_S) is a retract of S-act (A_S) if and only if there exists a sub-act (C) of (A_S) and S-epimorphism ($f: A_S \to C$)where ($B_S \cong C$) and ($f(c) = c$) for each $c \in C$[2, P.84]. An S-homomorphism f that transfers an S-act A_S into an S-act B_S is considered split if there is an S-homomorphism g that maps B_S toward A_S ensuring that ($fg = 1_N$[3]). A sub-act B of A_S is deemed large (or essential) in A_S if and only if every homomorphism$f: A_S \to C_S$, where C_S is S-act with restriction to B_S is 1-1 then, whenever f is itself one to one [4].

In this case, we assert that A_S is essential extension of B_S. In [4], Berthiaume demonstrated that each S-act possesses a maximal essential extension that is injective and is unique up to S-isomorphism in A_S. A sub-act B_S of A_S that is not equal to zero is an intersection large if for every sub-act C (not equal to zero) of A_S, ($C \cap B \neq \Theta$), and it will be denoted that B_S is ⋂-large in A_S[5]. In [6], the authors demonstrated that each large sub-act of A_S is ⋂-large, though, the alternative is not always true. An S-act A_S (that not equal to zero) in a monoid S is termed reversible (⋂-reversible) if all sub-act of A_S (that not equal to zero) is large (⋂-large).

It is evident that any reversible act (not equal to zero) is a ⋂-reversible act. Though, the alternative is not always true. Besides, they coincide when ($\psi_A = i$) [7]. An S-act A_S is considered decomposable if it has two sub-acts B and C with ($A_S = B \cup C$) and ($B \cap C = \Theta$). Thereby, ($B \cup C$) is termed decomposition of A_S. Otherwise, A_S is described as indecomposable ([2], p.65). An S-act A_S is considered simple if it includes no sub-act other than A_S itself. An S-act A_S is considered Θ-simple if it includes no sub-acts other than A_S and one element sub-act ([8], p.13]. A semi-simple S-act A_S is one in which every sub-act of A_S is either a retract or a union of simple sub-acts [9]. A sub-act B of a right S-act A_S is considered fully invariant if ($f(B) \subseteq B$) for each endomorphism f of A_S and A_S is termed duo if each sub-act of A_S is fully invariant [10].

Let A_S, B_S represent two S-acts, A_S is termed B-injective if for any an S-monomorphism $\alpha: C \longrightarrow B_S$ where C is a sub-act of B_S, each S-homomorphism $\beta : C \longrightarrow A_S$, can be extended to an S-homomorphism $\sigma : B_S \longrightarrow A_S$. An S-act A_S is referred to as injective if and only if it is B-injective for every S-acts B_S[11]. A sub-act B of an S-act A_S is considered a closed if there is no proper $\cap$-large extension in A_S that uniquely satisfies ($B \hookrightarrow^{\cap L} C_S \neq \hookrightarrow A_S$ as $B = C_S$), where C_S represents any sub-act of A_S. For additional details regarding acts, refer to the subsequent references [12, 13, 14, 15]. In this research, we utilize a different generalization of extending act, specifically prime-extending acts. Certainly, the motivation behind our study is that although generalizations of injective acts have captivated attention for numerous years, there has been a lack of focused research on the generalizations of extending acts. In 2017 and 2020, the author suggested extending acts and their generalizations. Approximately at that time, the specific issue of when extending acts will transform into prime extending acts remains unaddressed. In this paper, we can address those issues.

2. Literature review

It is widely recognized that extensive research on extending modules has been thoroughly conducted in a monograph by Dung *et al.* [16] and in an earlier book by Mohammed and Müller [17]. Regarding the generalizations of this concept, it is widely recognized that numerous authors have identified different generalizations of extending modules, as demonstrated below:

In [18], Haibat K. Mohammadali and Yokcell A. Sadiq investigated the generalization of extending modules known as strongly-pseudo-extending modules. An R-module N is termed strongly-pseudo-extending if each submodule is essential in a pseudo stable direct summand of N. Aside from that, numerous characterizations of this idea were presented.

In [19], Tamadher A. Ibrahiem introduced another generalization of extending modules, termed prime-extending modules. An R-module N is referred to as a prime-extending module if any proper submodule (not equal to zero) B of A is essential in a prime direct summand of A. Furthermore, the characteristics of this module were elucidated, and its relationship with other categories of injectivity was discussed

Fully extended modules were presented by Inaam M. A. Hadi and *et al.* in [20]. The notion of quasi-closed submodules, which is not as robust as closed submodules, was introduced. With this concept, the authors define fully extending modules: an R-module N is considered fully extending if each quasi-closed submodule of N serves as a direct summand. The authors showed that this kind of module exceeds the category of extending modules in strength.

Additionally, in [21], Burcu Ungor and Sait Halicioglu demonstrated modules that were strongly extending. A module N is termed to as strongly extending if for every submodule A of N, there exists a decomposition $N = A \oplus B$ where B is a strongly large submodule of A. A submodule B of a module A is deemed strongly large in A when, for every $a \in A$ and every right ideal I of R, if $aI \neq 0$ then $a(a^{-1}B)I \neq 0$

S. Ebrahimi Atani, and *et al.* presented strongly extending modules in their paper "on strongly extending modules" in [22]. A module M is termed strongly extending if every submodule of N is an essential within a fully invariant direct summand of N. The authors investigated the behavior of the class of strongly extending modules regarding the preservation of this property in direct summands and direct sums, showcasing various traits of these modules, including the strongly summand intersection property and the weakly co-Hopfian property. Moreover, several modules were given on commutative Dedekind domains.

Muna Abbas Ahmed and Maysaa R. Abbas introduced semi-extending modules in [23]. An R-module N is termed to as semi-extending if each submodule of N is semi-essential in a direct summand. Different characterizations of semi-extending modules are presented. Additionally, the connections between semi-extending modules and other associated notions, for example CLS modules and FI-extending modules, are examined

Saad A. Al-Saadi and *et al.* conducted an in-depth study of strongly uniform extending modules in [24]. An R-module M is considered as a strongly-uniform extending (or M has (1-SC1) condition) if each uniform submodule of M is essential in a stable (fully invariant) direct summand of M. It serves as a more robust version of uniform extending modules and a generalization of strongly extending modules. This study examines the characteristics of strongly uniform extending modules and presents comparable attributes to strongly extending modules.

Additionally, in [25], the authors proposed a further generalization of extending modules using weakly supplemented submodules in their article titled Weakly Supplement Extending Modules. An R-module (M) is called a weakly supplemented extending if every submodule of (M) is essential in a weakly supplemented submodule of (M). Various properties of weakly supplemented extending modules were discovered. They demonstrated that (M) is a weakly supplemented extending module if and only if every closed submodule is a weakly supplemented submodule of (M). Furthermore, it clarifies the link between the weakly supplementary extending module and other known module categories, including the lifting module, weakly supplemented module, and supplement extending module. Furthermore, this study investigated the circumstances under which a direct sum of weakly supplemented extending modules remains weakly supplemented.

The authors studied T-Stable extending and Strongly T-Stable Extending modules in [30]. An R-module M is a t-stable extending if every stable submodule of (M) is t-essential in a direct summand. A ring (R) is referred to as a right t-stable extending if (R) is seen as a right t-stable extending (R)-module. An R-module (M) is called strongly t-stable extending if every stable submodule N of (M) is t-essential in a stable direct summand. Various characteristics of each of these notions were explored. It was shown that a direct sum of strongly t-stable extending modules is also strongly t-stable extending. However, under certain conditions, a direct sum of strongly t-stable extending remains strongly t-stable extendable. Furthermore, it was verified that, under certain conditions, a stable submodule of t-stable extending (strongly t-stable extending) has the property.

Saad Abdulkadhim Al-Saadi and Darya Jaber Abdul-kareem, presented classes of modules that are strictly more powerful than N-extending modules, specifically strongly N-extending modules in [27]. A module A is considered as a strongly B-extending when B is a module if for every sub-module C of A with $C \in C(B, A) =$

$\{X \subseteq A | \exists K \subseteq B, \exists f \in Hom(K, A) \text{where } f(K) \text{is essential in } X\}$

is an essential within a stable direct summand of A. Certain properties of these classes were provided.

In [28] Muna Abbas Ahmed and *et al.* proposed a combination of two extensions of modules to form new generalizations called Almost Semi extending Modules. An R-module (M) is

termed almost semi extending if every submodule of (M) is semi essential within a pure submodule of (M). A submodule (N) of an (R)-module (M) is called a pure submodule if (IM∩N=IN) for all finitely generated ideals (I) of (R) [21, P. 30]. Several instances were also presented. The characteristics of almost semi extending modules are investigated. Furthermore, the connection with other concepts was established.

Regarding act theory, which is a generalization of module theory, and the importance of extending act. The author explored its generalizations in references [9, 29, and 30].

In [31], Ahsan, J., and Zhongkui, L. investigated prime and semiprime acts on monoids with zero, stating that an s-act is prime if its zero subact (θ) is prime. Specifically, the monoid (S) is considered prime if the zero ideal (0) of (S) is prime when viewed as an (S)-subact of (S_S).

Ultimately, this paper offers a new generalization of the research conducted by Tamadher A. Ibrahiem and J. Ahsan alongside Liu Zhongkui [19, 31].

Results

3. Prime-Extending acts

This section presents definition of the prime-extending acts, and provides examples, along with several fundamental properties and characterizations of this notion

Definition (3.1): [31] A sub-act B of the right S-act A is considered a prime sub-act of A if, for every $x \in A$ and $y \in S$, the inclusion $xSy \subseteq B$ means that either $x \in B$ or $y \in J_B = \{s \in S | As \subseteq B\}$. A right S-act A is referred to as prime if the sub-act (θ) is prime. Specifically, a monoid S is considered prime if the zero ideal (0) of S qualifies as a prime sub-act within S_S.

Definition (3.2): An S-act (N) is referred to as a prime-extending act if for each nonzero proper sub-act (H) of (N) is ⋂-large within a prime retract of (N).

Examples and Remarks (3.3)

1. It is obvious that each prime-extending act constitutes an extending act however, the reverse is generally not the case. In this case, it is clear that both Q and Z are extending acts that are not prime-extending acts. The sole prime sub-act of the Z-act Q is (Θ). Therefore, every sub-act (not equal to zero) of Q is not ⋂-large in prime retract. Additionally, it can be confirmed that the sub-act (3) is ⋂-large within Z, but Z does not serve as a prime retract sub-act of Z.

2. The Z-act Z_{12} is not prime-extending because $(\bar{2})$ serves as a ⋂-large sub-act of $(\bar{2})$ and is a prime sub-act, yet it is not retracted in Z_{12}. Additionally, $(\bar{2})$ is a ⋂-large sub-act of Z_{12} and Z_{12} is not a prime retract sub-act of Z_{12}. Because Z_{12} is not a proper sub-act of Z_{12}.

3. The Z-act Z_6 is an act serve as a prime-extending. Given that $(\bar{2})$ and $(\bar{3})$are the only non-zero proper sub-acts of Z_6. Moreover, $(\bar{2})$ is a ⋂-large sub-act of $(\bar{2})$ while $(\bar{3})$ is a ⋂-large sub-act of $(\bar{3})$. Additionally, $(\bar{2})$ and $(\bar{3})$ are both prime.

4. Each ⋂-reversible act is not a prime-extending. To demonstrate this: Assume that A is a non-zero sub-act of ⋂-reversible S-act N. The sole retract of N is the zero sub-act Θ and N itself. Therefore, there is no prime sub-act B of N where A is ⋂-large in B. Therefore, N does not a prime-extending act.

Proposition (3.4): An S-act $N_S$is a prime-extending if and only if each a closed sub-act of $N_S$is a prime retract of N_S .

Proof: ⟹) Suppose that N_S is a prime-extending act. Let A be a closed sub-act of N_S. Given that N_S is a prime-extending, there is a prime retract B of N_S for which A is ⋂-large in B. Since A is a closed in N_S, it follows that A = B. Therefore, A is a prime retract of N_S.

⟸) Let A be a sub-act of N_S . Therefore, according to Zorn's lemma, there is a closed sub-act (B) of N_S for which (A) is ⋂-large in (B). Given that (B) is a closed in N_S, it follows that B is a prime retract of N_S by hypothesis. Consequently, A is ⋂-large within a prime retract of N_S. Therefore, N_S is a prime-extending act.

Theorem (3.5): The statements that follow are identical for the S-act N_S:

1. N_S is a prime-extending S-act,
2. Each a closed sub-act of N_S is a prime retract,
3. If A is a prime retract of $E(N_S)$, then it follows that $A \cap N_S$is a prime retract of N_S.

Proof: (1⟶ 2) According to proposition (3.4).

(2⟶3) Define $E(N_S) = A \dot{\cup} B$, with B being a sub-act of $E(N_S)$. Assume $A \cap N_S$ is ⋂-large in H, with H is a sub-act of N_S and subsequently of $E(N_S)$. Let $h \in H$. Then $h \in E(N_S)$, leading to $h \in A$ or $h \in B$ (indicating that h = a or h = b). Now, assume $h \notin A$ and $h = b \neq \Theta$. Given that $N_S$is ⋂-large in $E(N_S)$, it follows that there is $\Theta \neq s \in S$ such that hs =bs$\in N_S$. However, $\Theta \neq b \in B$ and, therefore, (bs)∈(B). Consequently, (bs)$\in (N_S \cap B)$. Conversely, $(A \cap N_S)$ is ⋂-large in (H) and (B) is ⋂-large in (B), making $(A \cap N_S \cap B)$ is ⋂-large in $(H \cap B)$. Yet, $(N_S \cap A \cap B) = \Theta$, implying $(H \cap B) = \Theta$ and thus $(bs) = \Theta$ leading to a contradiction. Consequently, $(A \cap N_S)$ is closed in $N_S$and thus by (2), it serves as a prime retract of N_S.

(3⟶ 1) Assume that A is a sub-act of N_S and B is the relative complement of A. Thereby, according to lemma (2.6) in[24], $A \dot{\cup} B$ is a ⋂-large sub-act of N_S. Given that N_S is ⋂-large in $E(N_S)$, it follows that $(A \dot{\cup} B)$ is ⋂-large in $E(N_S)$ according to lemma (3.1) in [3], hence $E(A) \dot{\cup} E(B) = E(A \dot{\cup} B) = E(N_S)$. As E(A) is a retract of $E(N_S)$, it follows from (3) $(E(A) \cap N_S)$is a prime retract of N_S. However, if (A) is ⋂-large in E(A) and N_S is ⋂-large in N_S, then (A) = $(A \cap N_S)$ is ⋂-large in $(E(A) \cap N_S)$. Therefore, N_S is a prime-extending act.

Propositions (3.6), (3.7) and (3.8) demonstrate the conditions under which the sub-acts of a prime-extending act remains also prime extending:

Proposition (3.6): The closed sub-act of a prime-extending-act remains a prime-extending.

Proof: Assume that (A) is a closed sub-act of the prime-extending act N_S. Assume that B is closed subact of A. According to lemma (2.4) in [9], B is closed in N_S. However, N_S is a prime-extending act, thus according to proposition (2.5), B serves as a prime retract of N_S. Given that B is a sub-act of A, it leads to the fact that B is a prime retract of A. Consequently, (A) is prime-extending act.

Proposition (3.7): Each sub-act A of the prime-extending act N_S qualifies as a prime-extending act if it meets this criterion: The criterion states that the intersection of A with any prime retract of N_S is a prime retract of A.

Proof: Suppose that A is sub-act of N_S, and B is sub-act of (A). Because that N_S is a prime-extending act, there exists a prime retract C of N_S where B is $\bigcap$-large in C. However, $B \subseteq A\bigcap C \subseteq C$. Therefore, according to lemma (3.1) in [3], B is ($\bigcap$-large) within $A\bigcap C$, and by assumption $A\bigcap C$ is prime retract of A. Therefore, A is a prime-extending.

Proposition (3.8): Each fully invariant sub-act of an extending S-act is a prime-extending.

Proof: Suppose that N_S is prime-extending S-act and that A is a fully invariant sub-act of N_S. If B is sub-act of A, thereby (B) is also a sub-act of N_S. Given that N_S is a prime-extending, there is a prime retract C of N_S where B is $\bigcap$-large within C. This indicates that $N_S = C\dot{\cup}K$, where K represents any sub-act of N_S. Given that A is fully invariant, it implies that $A = (A\bigcap C)\dot{\cup}(A\bigcap K)$. This means that $A\bigcap C$ is prime retract of A. Because B is $\bigcap$-large within C and A is $\bigcap$-large in (A), it implies to $(B = A\bigcap B)$ is $\bigcap$-large in $(A\bigcap C)$. Therefore, A is prime-extending.

Proposition (3.9): Suppose that $N_S = N_1\dot{\cup}N_2$, where both $N_1$and $N_1$are prime-extending acts. N_S is considered as a prime-extending if and only if each closed sub-act A of N_S that intersects either N_1 or N_2 trivially (i.e. $A\bigcap N_1 = \Theta$ or $A\bigcap N_2 = \Theta$) is prime retract of N_S.

Proof: $\Longrightarrow$) The results can be obtained According to proposition (3.4).

$\Longleftarrow$) Let each closed sub-act A of N_S, where $A\bigcap N_1 = \Theta$ or $A\bigcap N_2 = \Theta$ be considered a prime retract of N_S. Assume that B is closed sub-act of N_S. Thereby, there is a complement C within (B) where $B\bigcap N_2$is $\bigcap$-large in C, and since B is a closed in N_S, (C) is also closed in N_S according to lemma (2.4) in [9]. Because$(A\bigcap N_2)\bigcap N_1$ is ($\bigcap$-large) in $C\bigcap N_1$, it implies that N_1 is $\bigcap$-large in N_1. Consequently, $C\bigcap N_1 = \Theta$ (as $B\bigcap(N_2\bigcap N_1) = B\bigcap\Theta = \Theta$ indicating that Θ is $\bigcap$-large in Θ.Therefore, by assumption, $N_S = C\dot{\cup}C'$ for some sub-act $C'$of N_S, and C serves as a prime retract of N_S. Now, $B = B\bigcap N_S = B\bigcap(C\dot{\cup}C') = C\dot{\cup}(B\bigcap C')$. Consequently, $B\bigcap C'$ is a closed in N_S(as $B\bigcap C'$is closed in B). Additionally, $(B\bigcap C')\bigcap N_2 = \Theta$, therefore by assumption $B\bigcap C'$ is a prime retract of N_S and consequently of C'(as $B\bigcap C' \subseteq C'$. Hence, $C' = (B\bigcap C')\dot{\cup}A$ with (A) being sub-act C'.Therefore, $N_S = C\dot{\cup}C' = C\dot{\cup}((B\bigcap C')\dot{\cup}A) = C\dot{\cup}(B\bigcap C'))\dot{\cup}A = B\dot{\cup}A$. This implies that B is a prime retract of N_S. Thereby, N_S is a prime-extending act.

The subsequent proposition provides a criterion for when the direct sum of the prime-extending acts is prime-extending:

Proposition (3.10): Assume that $N_S = \dot{\bigcup}_{i=1}^{n} N_i$ represents a finite direct sum of relatively injective acts N_i. Therefore, N_S is a prime-extending act if and only if all N_i are prime-extending.

Proof: $\Longrightarrow$)An requirement is evident (as retract(closed) of a prime-extending is a prime-extending).

$\Longleftarrow$)Assume that every N_i is a prime-extending act and all N_i are relatively injective acts. By induction on n, it suffices to demonstrate that N_S is a prime-extending when n = 2. Suppose that $B \subseteq N_S$ is a closed and $B\bigcap N_1 = \Theta$. According to proposition (2.12) in [9], there is a sub-act $N'$of N_S where $N_S = N_1\dot{\cup}N'$and $B \subseteq N'$. It is obvious that $N' \cong N_2$, so N' is a prime-extending. It is simple to demonstrate that B is a closed sub-act of N'. Given that $N'$is a prime-extending, according to proposition (3.4) B is a prime retract of N' and thus it is a prime retract of N_S, consequently, $N'$is prime retract of N_S. Similarly, every closed set $C \subseteq N_S$ where $C\bigcap N_2 = \Theta$ is prime retract of N_S. As stated in proposition (3.9) N_S will serve as a prime-extending act.

4. R-prime acts

In this section, we observed that the notions of prime-extending acts and extending acts are identical when the condition (each retract sub-act is a prime) is satisfied. This prompts us to present and examine this condition in relation to the category of semi-simple, $\bigcap$-reversible and various others types of acts.

Definition (4.1): An s- act N is termed as an R-prime act if each proper retract of N qualifies as a prime. Furthermore, we refer to a monoid S as an R-prime monoid if S operates as R-prime under S-act. For instance, the Z-act Z_3 qualifies as an R-prime act since Z_3 lacks any non-zero proper retract, whereas the Z-act Z_{12} fails to be an R-prime act because $(\overline{4})$ serves as a retract of Z_{12} but is not prime.

- **Proposition (4.2):** Let A be a prime act in R. Thereby, A is a prime-extending act if and only if (A) is an extending.
- **Proof:** Suppose that (A) is an extending act, and B non-zero proper sub-act of A. Thus, (B) is essential in retract (C) of (A). Given that A is R-prime, C is a prime retract of A. Hence, A is a prime-extending act. The opposite is evident.
- **Corollary (4.3):** An S-act A is considered as a prime-extending if and only if A is both extending and an R-prime act.
- **Proof:** By proposition (4.2).
- **Proposition (4.4):** Assume that A is a semi-simple S-act. A is a prime-extending act if and only if A is an R-prime act.

Proof: $\Longrightarrow$) Let A be a prime-extending act and (B) is proper retract of (A). Consequently, (B) is closed sub-act of A. Given that A is a prime-extending, it follows that B is prime by theorem (3.5). Thus, A qualifies as an R-prime act.

$\Longleftarrow$) Assume that B is a proper sub-act of A. Because that A is a semi-simple act, so B is a retract of A. But A is R-prime act, so B is a prime in A. Hence, B is prime retract of A, and B is essential in B. Thus, A is prime-extending act.

This proposition demonstrates the transitive nature of the prime property

- **Proposition (4.5):** [31] Each non-zero sub-act (B) of prime S-act A is itself prime S-act.
- **Preposition (4.6):** Each retract of an R-prime act is an R-prime act.
- **Proof:** Suppose that (B) is retract of R-prime act (A). Therefore, (B) is prime retract of A. Since A is an R-prime act. Assume that (C) is retract of (B). Since (B) is prime, it follows that C is also prime of B according to proposition (4.5). Consequently, C will be a prime retract of B. This indicates that B is an R-prime act.

5. Conclusion

In our research, we introduced new ideas referred to as prime extending acts and R-prime acts. We achieved multiple intriguing outcomes, which included different new characterizations for these concepts. The connection between these classes and other injectivity classes is accepted. In a

similar way, we identified the criteria for the identical of these classes.

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