E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2025; 6(2): 29-34

www.mathematicaljournal.com
Received: 16-05-2025
Accepted: 19-06-2025

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Commutativity in normal matrices: When does AB = BA

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DOI: https://www.doi.org/10.22271/math.2025.v6.i2a.218

Abstract
The strong commutativity of self-adjoint and ordinary unbounded operator has been demonstrated by Devinatz, Nussbaum, and Neumann. This paper demonstrates a similar spirit.

Keywords: Commutativity, unbounded operators, normal operators, self-adjoint operators

Introduction
We start by assuming that every operator's operation is liner. It is assumed that bounded operators are defined on Hilbert space as a whole. Unbounded operators will be referred to as densely defined because they are supported to have dense domains. Refer to [12, 24, 25] for general references on unbounded operator theory.
However, let's review a few notations that will be covered later. If *A* and *B* are Two operators having domains that are not bound $D(A)$ and $D(B)$ In response, *B* is called, and *A* is extended, and we write $A \subset B$, if $D(A) \subset D(B)$ and if *A* and *B* coincide on $D(A). If\ A \subset B$, then $B^* \subset A^*$.
The definition of the product *AB* of two unbounded operators, *A* and *B*, is

$$BA(x) = B(Ax) for\ x \in D(BA)$$

Where;

$$D(BA) = \{x \in D(A): Ax \in D(B)\}$$

Recall too that the unbounded operators *A*, defined on a Hilbert space *H*, is said to be invertible if there exists an everywhere defined (i.e. on the whole of *H* bounded operator *B*, which then be designated by $\boldsymbol{A^{-1}}$, Such that

$$BA \subset \mathrm{AB} = I$$

Where the standard identification operator is *I*. The definition used in this study is this one. It may be found, for example, in [3] or [11].
A limitless operator If *A*'s graph is closed and symmetric, then *A* is considered closed. if $A \subset A^*$; self-adjoint if $A = A^*$ (Therefore, self-adjoint operators are automatically closed based on known facts); usual if it's closed and $AA^* = A^*A$ $(this\ that\ D(AA^*) = D(AA))$.
Unbounded operators' commutativity has to be treated carefully. First, review the definition of two unbounded (self-adjoint) operators that strongly commute (see e.g.) [23].
Definition. Let *A* and *B* be self-adjoint operators that are unbounded. If every projection in their corresponding projection-valued measures commutes, we say that *A* and *B* strongly commute.
This in fact equivalent to saying that $e^{itA}\, e^{isB} = e^{isB}\, e^{itA}\ for\ s, t \in \mathbb{R}$.
The definition for unbounded normal operators will be the same.
Nelson [19] shown that over a common domain *D*, there exists a pair of two essentially self-adjoint operators *A* and *B* such that
1) $A: D \longrightarrow D, B: D \longrightarrow D,$
2) ABx=BAx for all x ϵ D,
3) bute^itA and e^isB do not commute, i.e. There is no significant commute between A and B.

Fuglede [7] demonstrated a similar outcome based on the first case. Thus, a form of expression AB=BA, even though it is really powerful since it suggests that D(AB)=D(BA), does not imply that A and B commute substantially.
Some findings (see, for example,) [7, 19] provide circumstances that suggest substantial commutativity of $\bar{A}$ and $\bar{B}$. As an example, we have

Theorem 1 [21]: Let D be a dense linear manifold that is contained in the domains, and let A and B be two semi-bounded operators in a Hilbert space H of AB, BA, A^2 and B^2 such that ABx=BAx for all x∈D. If the restriction of (A+B)^2 to D is Basically, if A and B are basically self-adjoint, then and $\bar{A}$ and $\bar{B}$ strongly commute.
Let's now review Devinatz-Nussbaum's (and von Neumann's) findings about strong commutativity:
Devinatz-Nussbaum-Von Neumann, Theorem 2 [4] and see also) [20]. If a self-adjoint operator is present A such that A_(⊂) BC,where B and C strongly commute if B and C are self-adjoint.

Corollary 1: Let A, B and C Be Self-adjoint operators with no bounds. Then

A_(⊂) BC ⇒A=BC.

Corollary 1 was improved in the following way in [20].
Gantmaher-Krein's work (see) [8] initiated the investigation into the normality of the product of two bounded normal operators in 1930. Papers by [26, 25] and [13] came next [10]. said that,
"In relation to certain polarisation optics concerns, the normality of operators in the Pauli algebra representations becomes interesting (see) [21].
In quantum optics, same issues can occur (see) [2].
The author has made many successful efforts to extend the prior to the situation in which at least one operator is unbounded. For example, see [16] and [18]. One of the most crucial factors in determining the normalcy of the letter's output. In fact, the following remarkable finding—which few people are aware of—demonstrates the strong interest in examining the (NPUNO) topic:

Theorem 2: [5] Devinatz-Nussbaum. If unbounded normal operators A, B, and N obey N=AB=BA, then A and B strongly commute.
In this paper we weaken the condition AB=BA to AB⊂BA, say, and still derive results on the strong commutativity of A and B (in unbounded normal and self-adjoint setting).
To show that a product is normal, we need its adjoints and closedness [26]. So, let's go over the existing research on those two ideas:

Theorem 3: Assume that the unbounded operator A is densely defined.
1) (BA)^*=A^* B^* if B is bounded.
2) A^* B^*⊂(BA)^* for any densely unbounded B and if BA is densely defined.
3) Both AA^* and A^* A are self-adjoint whenever A is closed.

Lemma 1 [30]: If A is inversely invertible and B and A are both densely defined A-1 in B(H), then (BA) ^*=A^* B^*.
Lemma 2: If one of the following happens, the product AB (in this sequence) of two densely defined closed operators A and B is closed:
1) A is invertibele,
2) B is Bounded,

We recommend that the reader consult [1, 20, 21, 25, 25] for relevant work on strong commutativity. The interested reader might refer to [9, 12, 14], and [22] for similar product studies, as well as the additional literature listed therein.

Main Results
First, we provide a strong commutativity conclusion for unbounded normal operators. However, the first result we have on (NPUNO) is as follows:

Theorem 4: Let A and B be two unbounded normal operators that confirm AB⊂BA. If B is Invertible, then BA and _AB are Both are typical anytime AB is densely defined [26].

Proof. Because B is invertible,

AB⊂BA⇒A⊂BAB^(-1)⇒B^(-1) A⊂AB^(-1).

According to the Fuglede theorem [6], we get

B^(-1) A^*⊂A^* B^(-1).

Multiplying by B from left to right yields

A^* B⊂BA^* so that AB^* ⊂B^* A

By Lemma 1. Next

$(BA)^* BA\subset(AB)^* BA=B^* A^* BA\subset B^* BA^* A.$

But BA is closed, hence $(AB)^* BA$ is self-adjoint. Since $B^* B$ and A^*A thus self-adjoint, as Corollary 1 provides us with $(BA)^* BA=B^* BA^* A$.

It is possible to demonstrate this using very similar reasons

$BA(BA)^*=BB^* AA^*.$

Consequently, given that A and B are normal, we get

$(AB)^* BA=BA (BA)^*,$

Determining if BA is normal.
To prove that $\overline{AB}$ is normal, note first that since B is invertible, we have

$AB\subset BA\Rightarrow A^* B\subset B^* A^*.$

Since B is invertible, B^* too is invertible. Thus, by the first part of the proof, $B^* A^*$ is normal and so is $(AB)^*=B^* A^*$. Hence its adjoint $(AB)^{**}= \overline{AB}$ stays normal.
The hypothesis $AB\subset BA$ is essential, as seen in the example that follows:
Example. Let's define A and B, respectively, by

$Af(x)=f'$ and $Bf(x)= e^{|x|} f(x)$

On

$D(A)=H^1 (\mathbb{R})$ and $D(B)=\{f\in L^2 (\mathbb{R}): e^{|x|} f L^2 (\mathbb{R})\}$

Where $H^1 (\mathbb{R})$ is the standard Sobolev space.
Because it is unitarily equivalent, operator A is known to be normal (via the $L^2 (\mathbb{R})$- Fourier transform) to a complex-valued function multiplied by an operator.
Regarding B, it is invertible, self-adjoint, and densely defined. The fact that AB and BA do not coincide on any dense set is likewise readily apparent.
Finally, let us show that $N:=BA$ (It is clearly closed.) is not normal. We have

$Nf(x)=e^{|x|} f'(x)$ defined on $D(N)=D(BA)$ hence

$D(N)=\{f\in L^2 (\mathbb{R}):f'\in L^2 (\mathbb{R}),e^{|x|} f'\in L^2 (\mathbb{R})\}=\{f\in L^2 (\mathbb{R}):e^{|x|} f'\in L^2 (\mathbb{R}))\}.$

To compute N^*, We need to start with the adjoint of N for $C_0^\infty (\mathbb{R}^*)$ functions, then continue as described in [14]. We discover that

$N^* f(x)=e^{|x|} (+\bar{f}(x)-f' (x))$
On

$D(N)^*=\{f\in L^2 (\mathbb{R}):e^{|x|} f\in L^2 (\mathbb{R}),e^{|x|} f'\in L^2 (\mathbb{R})\}.$

Then we may easily get that

$NN^* f(x)=e^{|x|} (-f(x) +\bar{2}f' (x)-f'' (x))$

and

$N^* Nf(x)=e^{2|x|} (-f(x) +\bar{2}f' (x)-f'' (x))$

Shown that N is not typical.

Corollary 2: Assume that A and B are two unbound normal operators that confirm $AB\subset BA$. If B is invertible, then $\overline{AB}=BA$ In cases when AB is well-defined.
Proof. Since BA is closed, we have

$AB\subset BA\Rightarrow \overline{(AB)}\subset BA.$

But both (AB)¯and BA are normal operators are maximum normal (see, for example,) [24], thus
(AB)¯=BA

Corollary 3: Let A and B be two normal operators that are not bounded. If B is invertible and BA= AB, Consequently, whenever AB is densely defined, A and B strongly commute.
Proof. According to Theorem 5, both BA and AB are normal this time. A and B strongly commute according to Theorem 3.
Note, the above conclusion may have been expressed as follows: Let A and B be two unbounded normal operators. If B is reversible and AB⊂BA and AB is closed, then A and B strongly commute whenever AB is densely defined.
Our findings (cf.) [17] are an application of the strong commutativity of unbounded normal operators:
Proposition 1. Let A and B be two strongly commuting unbounded normal operators. Then A+B is essentially normal.
Remark. It is important to remember that if a ubounded closeable operator has a normal closure, it is considered fundamentally normal (though this word obviously has a different meaning in banch algebras).
Proof. According to the spectrual theorem, because A and B are normal, we may write,

A=l_C zdE_A (z)and B=l_C z^' dF_B (z^'),

Where E_B anf F_B Indicate which spectral measurements are involved. We have substantial commutativity, which means that

E_B (I) F_B (J)=F_B (J)E_B (I)

For all Borel sets I and J in C. Hence

E_(A,B) (z,z^')=E_A (z) F_B (z^')

Describes a spectral measure with two parameters. Thus

C=l_C l_C (z+z^')dE_(A,B) (z,z^')

Describes a typical operator in a way that C=(A+B)¯. Therefore, A+B is essentially normal.
We then address the situation involving two unbounded self-adjoint operators. We begin with the following intriguing outcome.

Corollary 4: Consider two unbounded normal operators, A and B. If B is invertible and BA= AB (where AB is densely). Then A+B is essentially normal.
Proposition 2. Let A and B be two unbounded self-adjoint operators such that B is invertible. If AB⊂BA, then A and B strongly commute whenever AB is densely defined.
Proof. By Theorem 5 and Corollary 3, we have (AB)¯=BA. Hence

(BA)^*=([AB]^*)¯=(AB)^*=B^* A^*=BA.

That is BA is self-adjoint. Thus, and because AB⊂BA, Corollary 2 Yields AB=BA. Thus, and by theorem 2, A and B strongly commute.
For self-adjoint operators, our finding is comparable to that of Propsition 1

Corollary 5: (cf.) [1]. Let A and B be two self-adjoint, unbounded operators that make B inveritable. If AB⊂BA, then A+B is seft-adjoint in essence. Morever for all t∈R:

e^(it((A+B))¯)=e^itA e^itB=e^itB e^itA.

Proof. A and B strongly commute according to Proposition 2. Apply a similar proof to that of proposition 1 after that. Simply apply the Trotter product formula, which may be found in [23, 26], for the final equation that is shown.
It was mentioned in [5] that in the situation of unbounded normal operators, there is no analogue for Theorem 2. It is sufficient to take the product of two non-commuting unitary operators, which is unitary nevertheless, since the counterexample is rather straightforward.

Theorem 6: Assume that A and B are two unbounded self-adjoint operators, with B being invertible and positive. Consider the normal operator C to be unbounded. If AB⊂C, AB is self-adjoint (when it is dense), therefore A and B strongly commute AB=C.
The following outcome is required to demonstrate it:
Lemma 3. In a Hilbert space H, let S and T be two self-adjoint operators. Suppose that S is bounded and that it commutes with T, i.e. ST⊂TS. Then f(S)T⊂Tf(S) for all continuous functions with real values that are specified on σ(S). In particular, we have S 1/2⊂TS 1/2, if S is position.
Proof. We may easily show that I P is a real polynomial, then P(S)T⊂TP(S).
Now let f be a continuous function on σ(S). Then (by the polynomial density defined on the compact σ(S) in the Given a collection of continuous functions that obey the supremum norm, a series (S_n) of polynomials that uniformly converge to f in order that

(lim@n⟶∞)||P_n (s)-||B_((H))=0.

Let x∈D(f)(S)T)=D(T). Let t∈D(T) and x=f(S)t. Setting x_n=S_n (S)t, we see that

Tx _n=TP_n (S)t=P_n (S)Tt⟶f((S)Tt.

Since x_n→x, by the closedness of T, we get

x=f(S)t∈D(T) and T_x=Tf(S)t=f(S)Tt,

That is, we have proved that f(S)T⊂Tf(S).
We now provide the evidence for Theorem 6:
Proof. Since AB⊂C and B may be inverted, we get

C⊂(AB)^* A^* B^*=BA or B^(-1) C^*⊂A.

Additionally, it is evident that AB⊂C implies that A⊂CB^(-1). The Fuglede-Putnum theorem (see [6] and [22]) so permits us to write because C is normal

B^(-1) C^*⊂CB^(-1)⇒B^(-1) C⊂ C^* B ^(-1).

So

B^(-1) AB⊂BAB^(-1).

Multiplaying was left, although B^(-1), then multiplaying on the right by B^(-1) give us
Since B^(-1) is Given that Lemma 3 is limited and positive, we may state that B^(-1) and A commute. Hence

B^(-1) A⊂AB^(-1) or just AB⊂BA.

Because A strongly commutes with B and BA is self-adjoint according to Proposition 2, Corollary 2, we have AB⊂BA. Because self-adjoit operators are maximally normal, we thus conclude that AB=C.

References

1. Devinatz AE, Nussbaum J. On the permutability of self-adjoint operators. Annals of Mathematics. 1955;62(2):199-203.
2. Devinatz AE, Nussbaum J. On the permutability of normal operators. Annals of Mathematics. 1957;65(1):144-152.
3. Fuglede B. A commutativity theorem for normal operators. Proceedings of the National Academy of Sciences. 1950;36(1):35-40.
4. Fuglede B. Conditions for two selfadjoint operators to commute or to satisfy the Weyl relation. Mathematica Scandinavica. 1982;51(1):163-178.
5. Gantmaher FR, Krein MG. O normalnyh operatorah v ermitovom prostranstve [Russian]. Izvestiya Fiziko-Matematicheskogo Obshchestva pri Kazanskom Universitete. 1929-1930;1(3):71-84.
6. Gesztesy F, Goldstein JA, Holden H, Teschl G. Abstract wave equations and associated Dirac-type operators. Annali di Matematica Pura ed Applicata. 2007;186(1):163-190.
7. Gheondea A. When are the products of normal operators normal? Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie. 2009;52(100)(2):129-150.
8. Gohberg S, Goldberg S, Kaashoek MA. Basic classes of linear operators. Basel: Birkhäuser Verlag; 2003.
9. Jorgensen PET. Unbounded operators: perturbations and commutativity problems. Journal of Functional Analysis. 1980;39(3):281-307.
10. Kaplansky I. Products of normal operators. Duke Mathematical Journal. 1953;20(2):257-260.
11. Mortad MH. An application of the Putnam-Fuglede theorem to normal products of selfadjoint operators. Proceedings of the American Mathematical Society. 2003;131(10):3135-3141.
12. Mortad MH. On some product of two unbounded self-adjoint operators. Integral Equations and Operator Theory. 2009;64(3):399-408.
13. Mortad MH. On the closedness, the self-adjointness and the normality of the product of two unbounded operators. Demonstratio Mathematica. 2012;45(1):161-167.
14. Hong Y, Horn RA. A characterization of unitary congruence. Linear and Multilinear Algebra. 1989;25(2):105-119.
15. Horn RA, Johnson CR. Matrix analysis. New York: Cambridge University Press; 1985.
16. Horn RA, Johnson CR. Topics in matrix analysis. New York: Cambridge University Press; 1991.
17. Ikramov KD. Hamiltonian square roots of skew-Hamiltonian matrices revisited. Linear Algebra and Its Applications. 2001;325(1-3):101-107.
18. Jacobson N. An application of E.R. Moore’s determinant of a Hermitian matrix. Bulletin of the American Mathematical Society. 1939;45(10):745-748.
19. Conway JB. A course in functional analysis. New York: Springer-Verlag; 1985. (Graduate Texts in Mathematics; vol. 96)
20. Fuglede B. A commutativity theorem for normal operators. Proceedings of the National Academy of Sciences of the United

States of America. 1950;36(1):35-40.
21. Guesba M, Nadir M. On n-power-hyponormal operators. Global Journal of Pure and Applied Mathematics. 2016;12(1):473-479.
22. Griffiths DJ. Introduction to quantum mechanics. United States: Pearson Education; 2017.
23. Hooft G. The cellular automaton interpretation of quantum mechanics. In: Blanchard P, Coecke B, Dieks D, editors. Fundamental Theories of Physics. Cham: Springer; 2016. p. 1-211. (Vol. 185).
24. Jibril AAS. On n-power normal operators. Arabian Journal for Science and Engineering. 2008;33(2A):247-251.
25. Jibril AAS. On 2-normal operators. Dirasat. 1996;23(2):190-194.
26. Nadir M, Smati A. Closedness and skew self-adjointness of Nadir's operator. Australian Journal of Mathematical Analysis and Applications. 2018;15(1):1-5.