Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2025; 6(2): 16-21

www.mathematicaljournal.com
Received: 07-05-2025
Accepted: 12-06-2025

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Simulation of a new circular distribution using IDF transformation

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DOI: https://www.doi.org/10.22271/math.2025.v6.i2a.216

Abstract
This research introduces a new compound circular probability distribution, the "Cardiac-Cardiac Circular Distribution," obtained by combining the Cardiac Circular Distribution using the inverse transformation method as a basis for proposing new circular probability distributions. Compound distribution are distinguished by their flexibility and high accuracy in representing statistical data, as they possess more parameters compared to standard probability distributions. Some mathematical properties of the proposed distribution, such as location, variance, and cumulative function, are derived. The maximum likelihood method is employed to estimate the parameters of the Cardiac-Cardiac Circular Distribution, and a simulation is made using the statistical criterion of mean squared error (MSE) to study the behavior of the new circular distribution in circular coordinate by testing more than one model.

Keywords: Cardiac circular distribution, maximum likelihood method

1. Introduction
Circular data arises when the values of a random variable are distributed along the circumference of a unit circle and are measured as angles ranging between 0° and 360°. The support for circular data is the unit circle, whereas for linear data, it is the real number line. Circular data can be modeled using three fundamental geometric representations:

1. **Full Circular Models:** For data spanning 0°-360° (e.g., wind directions, animal migration paths) [2].
2. **Semi-Circular Models** and **Half-circular Models:** For 0°-180° ranges (e.g., bilateral symmetry in biology) [8, 9].

The choice of model depends on both the data's physical constraints and its periodicity. For instance, semi-circular distributions like the Projected Normal distribution are used when data exhibits axial symmetry ($\theta \equiv \theta + 180°$), while full circular distributions (e.g., von Mises) are preferred for unimodal directional data.
This research employs the Inverse Distribution Function (IDF) method to construct a novel circular probability distribution by transforming and combining existing distributions, such as the Cardiac Circular Distribution and the Laplace Circular Distribution. The IDF approach leverages the quantile functions of these base distributions to generate the proposed "Spider Circular Distribution", ensuring mathematical tractability and flexibility in modeling directional data. Key properties of the new distribution-including its circular mean, variance, and concentration parameters-are analytically derived. For parameter estimation, we use the maximum likelihood method, with emphasis on the IDF's role in enabling efficient simulation and inference with several sets of parameters [7].

The IDF transformation can be represented as

$\Theta = F^{-1}(U)$, $U \sim Uniform(0,1)$

Where F^{-1} is the quantile function of the base distribution. For circular distributions, this ensures:

$\Theta \bmod 2\pi \in [-\pi, \pi)$.

2. Circular Statistics

Circular statistics is a specialized branch of statistics that deals with data representable as points on the circumference of a unit circle. It is applied to such data to distinguish it from conventional linear data used in many analyses. Circular data is utilized in various fields, including biology, medicine, image analysis, earth sciences, physics, political studies, and astronomy. In two-dimensional space, any point can be represented either by its Cartesian coordinates (x, y) or its polar coordinates (r, θ), where r is the distance from the circumference to the origin. In circular analysis, the focus is solely on direction, so vectors of unit length are considered. Thus, any point on the circle can be represented as (sin θ, cos θ). Examples of circular data include directions measured using tools such as compasses, protractors, or wind vanes. Such directions are typically recorded as angles expressed in degrees or radians, either clockwise or counter clockwise from a reference point on the circle, known as the zero direction.

A circular probability distribution is a distribution where the total probability is concentrated along the circumference of a unit circle. Since each point on the circumference represents a direction, this distribution assigns probabilities to different directions or defines a directional distribution. The range of a circular random variable can be (0, 2π) or (-π, π). Circular distributions are primarily of two types: discrete, assigning probabilities to a countable number of directions, or continuous (with respect to the Lebesgue measure).

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For continuous circular data, the random variable θ has a probability density function with the following properties

1. $f(\theta) \geq 0$
2. $\int_0^{2\pi} f(\theta)\, d\theta = \int_{-\pi}^{\pi} f(\theta) d\,\theta = 1$
3. $f(\theta + 2k\pi) = f(\theta)\, for\, k \in Z, -\infty < \theta < \infty,$
4. $F(\theta + 2\pi) - F(\theta) = 1, -\infty < \theta < \infty$

3. Construction of the Cardioid-Cardioid Distribution Using the Inverse Distribution Function (IDF) Method

3.1 The General IDF Formula

- The Inverse Distribution Function (IDF) method for circular distributions involves
- Let: $\emptyset \in [0,2\pi]$ be random variable with known circular pdf $f_{\emptyset}(\emptyset)$
- Define a monotonic and invertible transformation

$\theta = T(\emptyset)$

Then the transformed variable θ has pdf

$$f_{\theta}(\,\theta) = f^{*}(\theta) = f_{\emptyset}(\,T^{-1}(\theta)).\left|\frac{d}{d\theta}\, T^{-1}(\theta)\right|$$

3.2 Cardiac Circular Distribution

The Cardiac distribution is a symmetric unimodal probability distribution concentrated on the unit circle. Its probability density function (PDF) is defined for θ ∈ [-π, π] as

Probability Density Function (PDF)

$$f(\theta) = \frac{1}{2\pi}\,[\,1 + 2\rho\cos(\theta - \mu)]$$

Its cdf is

$$F(\theta) = \frac{1}{2\pi}\,[\,\theta + 2\rho\sin(\theta - \mu)]$$

Where $\in [0,2\pi], -\frac{1}{2} \leq \rho \leq \frac{1}{2}$, This CDF is monotonic and invertible numerically

We define two such distributions:

$f_1(\theta, \mu_1, \rho_1\,)$ and $f_2(\theta, \mu_2, \rho_2\,)$

Now to constrict the new circular distribution (Cardioid-Cardioid) using IDF-Based Transformation

$\theta = T(\emptyset)$

Then the transformed variable θ has pdf

$$f_\theta(\theta) = f^*(\theta) = f_\emptyset(T^{-1}(\theta)).\left|\frac{d}{d\theta}T^{-1}(\theta)\right|$$

We treat

$$\theta = T(\emptyset) = F(\emptyset)\ and\ thus: T^{-1}(\theta) = \emptyset = F^{-1}(\theta)$$

Now, using the IDF formula: $f^*(\theta) = f_\emptyset(T^{-1}(\theta)).\left|\frac{d}{d\theta}T^{-1}(\theta)\right|$

Let $\emptyset = F^{-1}(\theta)$ such that F is the cdf of Cardioid distribution then

$$f^*(\theta) = \frac{1}{2\pi}[1 + 2\rho_1\cos(F^{-1}(\theta) - \mu_1)].\left|\frac{d}{d\theta}F^{-1}(\theta)\right|$$

The final explicit pdf

$$f_{cc}(\theta) = \frac{1}{2\pi[1 + 2\rho_1\rho_2 cos(\mu_2 - \mu_1)]}.\frac{1}{2\pi}[[1 + 2\rho_1\cos(\theta - \mu_1)] + [1 + 2\rho_2\cos(\theta - \mu_2)] + 2\rho_1\rho_2 cos(\mu_2 - \mu_1) + 2\rho_1\rho_2 cos(2\theta - \mu_1 - \mu_2)]$$

While the final explicit cdf is given by

$$F_{cc}(\theta) = \frac{1}{2\pi[1 + 2\rho_1\rho_2 cos(\mu_2 - \mu_1)]}.[[1 + 2\rho_1\rho_2\cos(\mu_2 - \mu_1)]\theta + [2\rho_1\sin(\theta - \mu_1)] + 2\rho_2 sin(\theta - \mu_2) + \rho_1\rho_2 sin(2\theta - \mu_1 - \mu_2)]$$

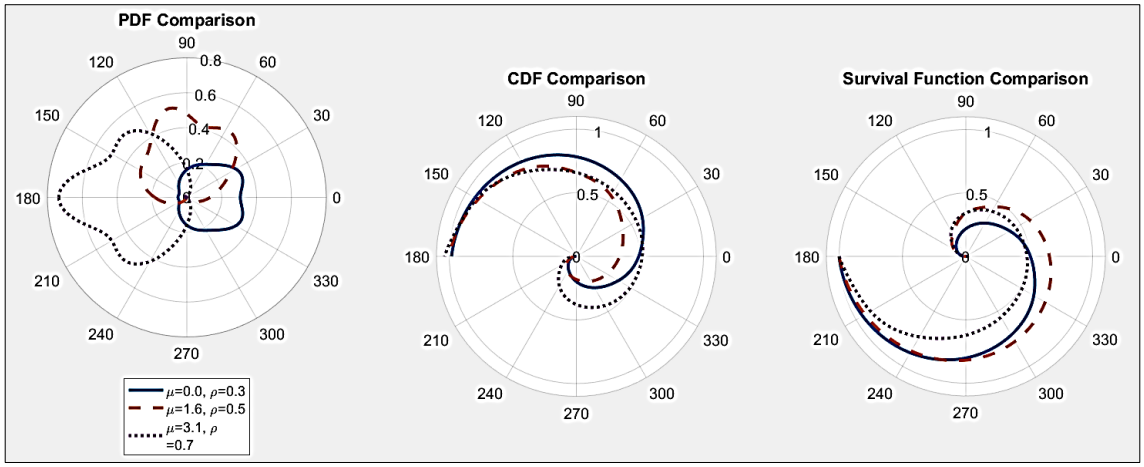

Fig 1: The shape of new distribution at different values of parameters

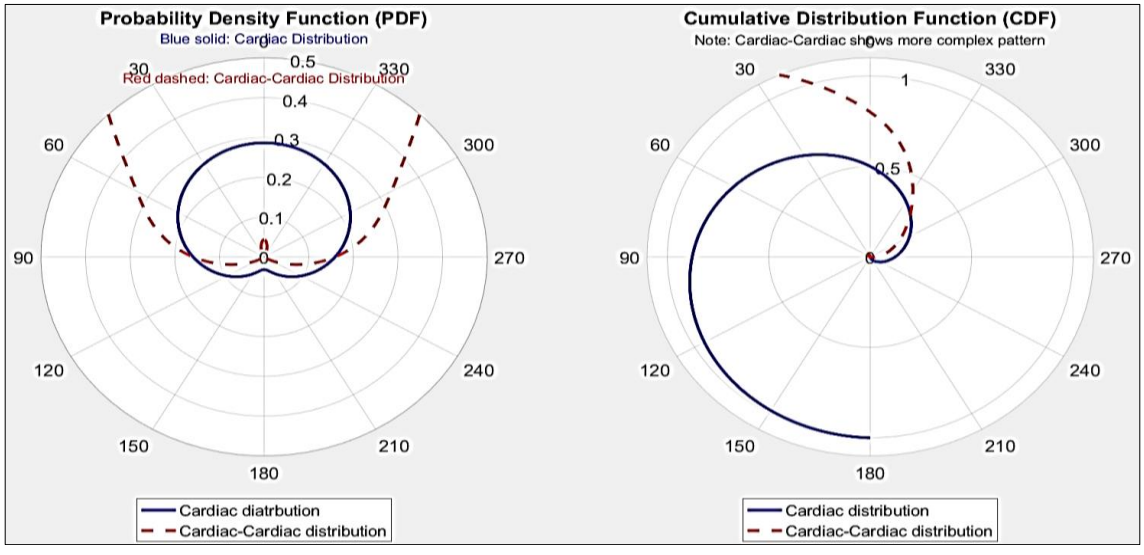

Fig 2: Comparison of Cardiac and Cardiac-Cardiac distribution.

4. Maximum Likelihood Estimation (MLE) for the Proposed Distribution

The Maximum Likelihood Estimation (MLE) method is employed to estimate the parameters (μ, ρ**)** of the proposed Cardiac-Cardiac distribution**.** The likelihood function is constructed based on the probability density function (PDF) of the distribution ()
For a sample of n independent observation $(\theta_1, \theta_2, \dots, \theta_n)$, the likelihood function $L(\mu.\rho)$ and log-likelihood function $l(\mu.\rho)$ are given by

$$L(\mu.\rho) = \prod_{i=1}^{n} f(\theta_i; \mu.\rho)$$

$$l(\mu.\rho) = \prod_{i=1}^{n} \ln f(\theta_i; \mu.\rho)$$

The MLE estimate $\hat{\mu}$ and $\hat{\rho}$ are obtained by maximizing the log-likelihood function

$$(\hat{\mu}, \hat{\rho}) = arg \max_{\mu,\rho} l(\mu.\rho)$$

This is achieved by solving the partial derivatives of l with respect to μ and ρ:$\frac{\partial l}{\partial \mu} = 0, \frac{\partial l}{\partial \rho} = 0$
Since these derivatives lead to nonlinear equations, hence numerical techniques are used.

5. Simulation

In this section, the new distribution will be tested by simulating different models of parameters with estimating by MLE method where the variation in these parameters helps analyze the behavior of the distribution function and the performance of estimation methods. This simulation is run over size of sample n= 25, 50, 75,100 and 150. And according to the MSE values such that the following tables show the results for this process where:

- n: Sample size
- $\hat{\mu_1}$, $\hat{\rho_1}$, $\hat{\mu_2}$, $\hat{\rho_2}$: Estimated parameters
- MSE $\hat{\mu_1}$, MSE $\hat{\rho_1}$, etc.: Mean Squared Error of each parameter
- Sur. MSE: MSE of the estimated survival (reliability) function

Experiment 1

Table 1: "Simulation Study: MLEs and Associated MSEs for the Cardioid-Cardioid Model (1)

n	$\hat{\mu}_1$	mse $\hat{\mu}_1$	$\hat{\rho}_1$	mse $\hat{\rho}_1$	$\hat{\mu}_2$	mse $\hat{\mu}_2$	$\hat{\rho}_2$	mse $\hat{\rho}_2$	Sur._MSE
25	1.286194	0.250797	0.5	0.01	1.8554	0.250795	-0.5	0.0025	0.004828
50	1.286193	0.250796	0.5	0.01	1.855399	0.250796	-0.5	0.0025	0.004828
75	1.286193	0.250796	0.5	0.01	1.855399	0.250796	-0.5	0.0025	0.004828
100	1.286193	0.250796	0.5	0.01	1.855399	0.250796	-0.5	0.0025	0.004828
150	1.286193	0.250796	0.5	0.01	1.8554	0.250796	-0.5	0.0025	0.004828

Table (1) illustrates the maximum likelihood estimates (MLEs) of the parameters for the Cardioid-Cardioid distribution, as well as their respective mean squared errors (MSEs) across various sample sizes. The estimated values for all four parameters remain remarkably constant and consistent as the sample size improves.

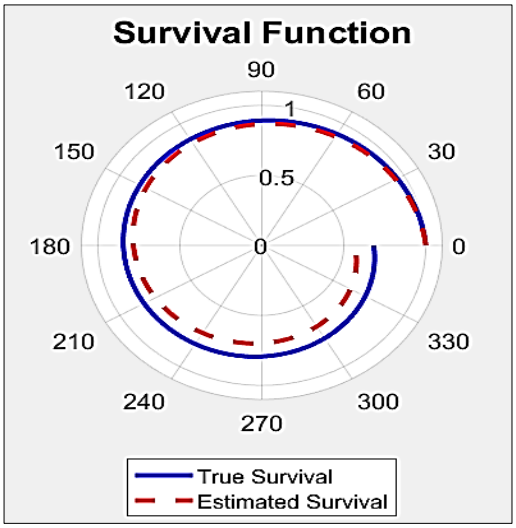

Fig 3: The survival functions curves for the Cardioid-Cardioid Model (1)

Experiment 2

Table 2: "Simulation Study: MLEs and Associated MSEs for the Cardioid-Cardioid Model (2)

n	$\widehat{\mu}_1$	mse $\widehat{\mu}_1$	$\widehat{\rho}_1$	mse $\widehat{\rho}_1$	$\widehat{\mu}_2$	mse $\widehat{\mu}_2$	$\widehat{\rho}_2$	mse $\widehat{\rho}_2$	Sur._MSE
25	0.277971	0.000485	-0.02346	0.457707	0.731743	0.004659	0.175967	0.767317	0.004408
50	0.290934	8.22E-05	0.129848	0.688647	0.856204	0.003159	0.124287	0.679449	0.005659
75	0.30141	1.99E-06	0.155208	0.731381	0.847178	0.002226	0.128018	0.685613	0.00592
100	0.295934	1.65E-05	0.142121	0.709168	0.850959	0.002597	0.118812	0.670453	0.005788
150	0.294787	2.72E-05	0.139389	0.704573	0.851727	0.002676	0.120627	0.673429	0.005761

1. According to the results of model (2)

μ_1 estimates show consistent deviation from the true value ($\pi/4 \approx 0.785$), ranging between 0.278-0.301, indicating significant estimation bias. ρ_1 estimates gradually improve with sample size but fail to converge to the true value (-0.2), suggesting model sensitivity to this parameter. Also Estimation Accuracy: μ_1 demonstrates strong convergence with MSE decreasing from 0.000485 to 2.72E-05 as sample size increases from 25 to 150. Concentration parameters (ρ_1 and ρ_2) show persistently high MSE (0.67-0.76) even at n=150, revealing

In general, for two models: Survival function MSE remains stable around ~0.005 across all sample sizes, suggesting

- Consistent Survival estimation performance
- Possible model robustness in Survival prediction
- Diminishing returns beyond n=75

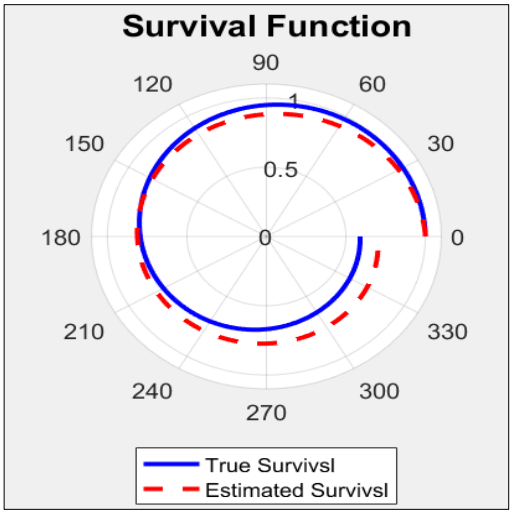

Fig 4: The survival functions curves for the Cardioid-Cardioid Model (2)

6. Conclusion

This research developed and analyzed a novel Cardiac-Cardiac circular distribution**,** constructed through the composition of two cardiac distributions using the inverse transform method. The proposed distribution offers enhanced flexibility for modeling directional data with complex multimodal behaviors. The maximum likelihood estimation (MLE) method demonstrated consistent convergence for location parameters (μ_1, μ_2), with MSE decreasing significantly as sample size increased (e.g., μ_1 MSE dropped from 0.000485 to 2.72E-05 for n=25 to n=150). Concentration parameters (ρ_1, ρ_2) proved more challenging to estimate, maintaining high MSE (0.67-0.77) even for large samples, suggesting inherent identifiability challenges in the mixture structure.

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