Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2025; 6(2): 01-07

www.mathematicaljournal.com
Received: 01-05-2025
Accepted: 04-06-2025

Karani Sammy
Department of Mathematics and Computer Science, University of Eldoret, P.O. Box, Eldoret, Kenya

Denis King'ang'i
Department of Mathematics and Computer Science, University of Eldoret, P.O. Box, Eldoret, Kenya

Linety Muhati
Department of Mathematics and Computer Science, University of Eldoret, P.O. Box, Eldoret, Kenya

Corresponding Author:
Karani Sammy
Department of Mathematics and Computer Science, University of Eldoret, P.O. Box, Eldoret, Kenya

On some operator equivalences in Hilbert spaces

Karani Sammy, Denis King'ang'i and Linety Muhati

DOI: https://www.doi.org/10.22271/math.2025.v6.i2a.214

Abstract
This paper is on some classes of operators which happen to be invariant under unitary and metric equivalences. In particular, we show that the classes of $n-\theta$ Operators, α-supraposinormal operators and $\mathcal{S}-$ operators are both isometrically and coisometrically and hence unitarily invariant while the class of supraposinormal is invariant under both unitary and metric equivalences.

Keywords: Unitary equivalence, metric equivalence and supraposinormal operators

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert Space and $\mathfrak{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. An operator $X \in \mathfrak{B}(\mathcal{H})$ is said to be a quasiffinity if it is both injective and has dense range. Also, two operators A and $\mathcal{B}$ are said to be similar if there exists an invertible operator S such that $A = S^{-1}\mathcal{B}S$, where S^{-1} denotes the inverse of S. If there exists an isometry U, with its adjoint U^* such that $A = U\mathcal{B}U^*$, then A and $\mathcal{B}$ are said to be isometrically equivalent and if U is unitary, then A and $\mathcal{B}$ are said to be unitarily equivalent. We also have that if A belongs to a class, say $\mathcal{M}$ of operators and $A = U\mathcal{B}U^*$, for any other operator $\mathcal{B}$, where U is unitary, then $\mathcal{M}$ is invariant under unitary equivalence. Thus, any operator $\mathcal{B}$ which is unitarily equivalent to operator A belonging to $\mathcal{M}$, also belongs to $\mathcal{M}$.
Two operators A and $\mathcal{B}$ are said to be quasisimilar if there exist two quasiaffinities say X and Y such that $AX = X\mathcal{B}$ and $AY = Y\mathcal{B}$. Also, A and $\mathcal{B}$ are said to be almost similar if there exists an invertible operator $\mathcal{N}$such that $A^*A = \mathcal{N}^{-1}(\mathcal{B}^*\mathcal{B})\mathcal{N}$ and $A^* + A = \mathcal{N}^{-1}(\mathcal{B}^* + \mathcal{B})\mathcal{N}$. The properties of unitary equivalence, similarity, quasisimilarity and almost similarity have been studied by various authors who to a large extent related them to equality of spectra.
However, recently the consideration of two operators belonging to the same class when they are under some operator equivalence has been investigated by a number of authors. For example, Nzimbi et.al [11] showed that if an operator T is normal and is unitarily equivalent to any other operator S, then S is also normal. This clearly shows that the class of normal operators is invariant under unitary equivalence. In a similar manner Luketero S.W and Khalagai J.M [10] showed that the class of binormal operators is not just invariant under unitary equivalence but also under isometric and co-isometric equivalences. Recently, Karani S.K et.al [9] showed that the classes of $\theta-$ operator and that of posinormal operators are also invariant under isometric and co-isometric equivalences, implying that they are also unitarily invariant. In this paper, we continue with this study in which we also consider the metric equivalence of operators. Thus, two operators A and $\mathcal{B}$ are said to be metrically equivalent if $A^*A = \mathcal{B}^*\mathcal{B}$.

2. Notations, Definitions and Terminologies

- An operator $A \in \mathfrak{B}(\mathcal{H})$ is said to be;
- Self adjoint if $A = A^*$.
- Normal if $A^*A = AA^*$.
- Quasinormal if $AA^*A = A^*AA$ i.e $[A, A^*A] = 0$ where $[,]$ denotes the commutator.
- Hyponormal if $A^*A \geq AA^*$.
- Binormal if $[A^*A, AA^*] = 0$.
- Partial isometry if $A = AA^*A$.
- Isometry if $A^*A = I$, where I is the identity operator.
- Co-isometry if $AA^* = I$.
- Unitary if $A^*A = AA^* = I$.

- $\theta-$ operator if $[A^*A, A^*+A]=0$.
- $n-\theta$ operator if $[A^*A^n, A^*+A^n]=0$.
- Posinormal if $AA^*=A^*PA$, for $P\geq 0$.
- Supraposinormal if $A^*PA=A^*QA$, for $P\geq 0$ and for $Q\geq 0$.
- α-supraposinormal for α positive integer if $A^\alpha QA^*=(A^\alpha)^*PA$, for $Q\geq 0$ and for $P\geq 0$
- $\mathcal{S}-$ operator if $A^2A^{*2}+A^{*2}A^2=2AA^{*2}A$.

We have the following set inclusions of classes of operators;

- { Normal } ⊆ { Quasinormal } ⊆ {Binormal}.
- { Normal } ⊆ { $\theta-$ Operators }⊆{ $n-\theta$ Operators}.
- { Unitary } ⊆ { Isometry } ⊆ {Partial Isometry}.
- { Unitary } ⊆ { Coisometry } ⊆ {Partial Isometry}.
- { Normal } ⊆ { Hyponormal } ⊆ {Posinormal } ⊆ {Supraposinormal } ⊆
- { α-Supraposinormal}.

3. Preliminary Results

The following theorems highlights the basic fundamental properties of unitary equivalence in which this paper endeavor to build on;

Theorem 3.1 ([11], Theorem 2.1)
If an operator $A\in\mathfrak{B}(\mathcal{H})$ is normal and $\mathcal{B}\in\mathfrak{B}(\mathcal{H})$ is unitarily equivalent to A, then $\mathcal{B}$ is also normal.

The class of normal operators as analyzed by nzimbi et.al, (2013) [11] has ideal spectral decomposition which characterizes widely linear operators. From the class inclusion above, Luketero S.W and J.M Khalagai (2020) [10] extended the study to the classes of binormal operators as shown in the next theorem.

Theorem B ([10], theorem 3.1)
Let $\mathcal{S},\mathcal{T}\in\mathfrak{B}(\mathcal{H})$ be such that $\mathcal{S}$ is binormal and $\mathcal{S}=U\mathcal{T}U^*$, where U is an isometry. Then $\mathcal{T}$ is also binormal.

Consequently, Karani et.al, (2021), further studied the unitary invariance of
$\theta-$ operator and posinormal operators as discussed in theorem 3.3 and theorem 3.4 below.

Theorem 3.3 ([9], theorem 3.1)
Let $\mathcal{S},\mathcal{T}\in\mathfrak{B}(\mathcal{H})$ be such that $\mathcal{S}$ is a $\theta-$ operator and either
i) $\mathcal{S}=U\mathcal{T}U^*$, where U is an isometry or
ii) $\mathcal{S}=U^*\mathcal{T}U$, where U is a coisometry.

Then $\mathcal{T}$ is also a $\theta-$ operator.

Theorem 3.4 ([9], theorem 3.5)
Let $\mathcal{S},\mathcal{T}\in\mathfrak{B}(\mathcal{H})$ be such that $\mathcal{S}$ is posinormal and either
i) $\mathcal{S}=U\mathcal{T}U^*$, where U is an isometry or
ii) $\mathcal{S}=U^*\mathcal{T}U$, where U is a coisometry.

Then $\mathcal{T}$ is also posinormal operator.
The above mentioned results on normal, binomial, $\theta-$ operator and posinormal form the basis for discussion on their respective larger classes which include; $n-\theta$ operators, α-supraposinormal operators and $\mathcal{S}-$operators. Unitary invariance of these classes are discussed in the following section. Also, metric equivalence of supraposinormal operators is discussed.

4. Main Results

Theorem 4.1
Let $A,\mathcal{B}\in\mathfrak{B}(\mathcal{H})$ with A $n-\theta$ Operator and $\mathcal{B}$ be such that
i) $\mathcal{B}=UAU^*$, where U is an isometry, then $\mathcal{B}$ is also $n-\theta$ operator.
ii) $\mathcal{B}=U^*AU$, where U is a coisometry, then $\mathcal{B}$ is also $n-\theta$ operator.

Proof
i) Since A is $n-\theta$ Operator implies $A^*A^n(A^*+A^n)=(A^*+A^n)A^*A^n$ i.e.

$$A^*A^nA^*+A^nA^*A^n=A^*A^*A^n+A^nA^*A^n \quad (1)$$

Now, $\mathcal{B}=UAU^*$ implies

$\mathcal{B}^*=UA^*U^*$. Thus, $\mathcal{B}^n=(UAU^*)^n=UAU^*UAU^*UAU^*\ldots\ldots UAU^*$

$$= \underbrace{UAAAU^* \ldots\ldots UAU^*}_{\text{n times}}$$

$$= UA^nU^*.$$

Therefore, $\mathcal{B}^*\mathcal{B}^n = UA^*U^*UA^nU^*$

$= UA^*A^nU^*$ and

$\mathcal{B}^* + \mathcal{B}^n = UA^*U^* + UA^nU^*.$

Thus, $\mathcal{B}^*\mathcal{B}^n(\mathcal{B}^* + \mathcal{B}^n) = UA^*A^nU^*(UA^*U^* + UA^nU^*)$

$= UA^*A^nU^*UA^*U^* + UA^*A^nU^*UA^nU^*)$

$$= UA^*A^nA^*U^* + UA^*A^nA^nU^* \qquad (2)$$

Also,

$(\mathcal{B}^* + \mathcal{B}^n)\mathcal{B}^*\mathcal{B}^n = (UA^*U^* + UA^nU^*)UA^*U^*UA^nU^*$

$= UA^*U^*UA^*A^nU^* + UA^nU^*UA^*U^*UA^nU^*$

$$= UA^*A^*A^nU^* + UA^nA^*A^nU^* \qquad (3)$$

Combining (2) and (3) then using (1) we have

$U(A^*A^nA^* + A^*A^nA^n)U^* = U(A^*A^*A^n + A^nA^*A^n)U^*$. i.e

$\mathcal{B}^*\mathcal{B}^n(\mathcal{B}^* + \mathcal{B}^n) = (\mathcal{B}^* + \mathcal{B}^n)\mathcal{B}^*\mathcal{B}^n$. Hence, $\mathcal{B}$ is also $n - \theta$ operator.

ii) We also have that A is $n - \theta$ operator implies

$$A^*A^nA^* + A^nA^*A^n = A^*A^*A^n + A^nA^*A^n. \qquad (1)$$

Now, $\mathcal{B} = U^*AU$ implies

$\mathcal{B}^* = U^*A^*U$. Thus, $\mathcal{B}^n = (U^*AU)^n = U^*AUU^*AUU^*AU \ldots\ldots U^*AU$

$$= \underbrace{U^*AAAU \ldots\ldots U^*AU}_{\text{n times}}$$

$= U^*A^nU$. Therefore,

$\mathcal{B}^*\mathcal{B}^n = U^*A^*UU^*A^nU = U^*A^*A^nU$ and $\mathcal{B}^* + \mathcal{B}^n = U^*A^*U + U^*A^nU$. Consequently,

$\mathcal{B}^*\mathcal{B}^n(\mathcal{B}^* + \mathcal{B}^n) = U^*A^*A^nU(U^*A^*U + U^*A^nU)$

$$= U^*A^*A^nA^*U + U^*A^*A^nA^nU \qquad (2)$$

and

$(\mathcal{B}^* + \mathcal{B}^n)\mathcal{B}^*\mathcal{B}^n = (U^*A^*U + U^*A^nU)U^*A^*A^nU$

$$= U^*A^*A^*A^nU + U^*A^nA^*A^nU. \qquad (3)$$

Again, combining (2) and (3) then using (1) gives

$\mathcal{B}^*\mathcal{B}^n(\mathcal{B}^* + \mathcal{B}^n) = (\mathcal{B}^* + \mathcal{B}^n)\mathcal{B}^*\mathcal{B}^n$. Hence, $\mathcal{B}$ is also $n - \theta$ operator.

Remark 4.2

The theorem above shows that the class of $n - \theta$ operators is invariant under both isometric and coisometric equivalences. Consequently, it is invariant under unitary equivalence.

Corollary 4.3. ([9] theorem 3.1)

The class of $\theta-$operators is invariant under isometric and coisometric equivalences and hence invariant under unitary equivalence.

Proof
This follows easily from the theorem 4.1 above since the class of θ operators is a subclass of $n-\theta$ operators.

Theorem 4.4
Let $A, \mathcal{B} \in \mathfrak{B}(\mathcal{H})$ with A α-supraposinorm al and $\mathcal{B}$ is such that;
i) $A = U\mathcal{B}U^*$, where U is an isometry, then $\mathcal{B}$ is also α-supraposinormal.
ii) $A = U^*\mathcal{B}U$, where U is coisometry, then $\mathcal{B}$ is also α-supraposinormal.

Proof
i) Since A is α-supraposinormal, we have $A^\alpha QA^* = (A^\alpha)^*PA$ (1)
Where, for $P \geq 0$ and for $Q \geq 0$. But $A = U\mathcal{B}U^*$ implies $A^* = (U\mathcal{B}U^*)^* = U\mathcal{B}^*U^*$. Thus,

$$A^\alpha = U\mathcal{B}U^*U\mathcal{B}U^*U\mathcal{B}U^* \dots . U\mathcal{B}U^*$$

$$= U\underbrace{\mathcal{B}\mathcal{B}\mathcal{B}U^* \dots . U}_{\alpha \text{ times}}\mathcal{B}U^*$$

$$= U\mathcal{B}^\alpha U^*.$$

Also, $(A^\alpha)^* = (U\mathcal{B}^\alpha U^*)^* = U(\mathcal{B}^\alpha)^*U^*$. Thus from (1) we have

$$A^\alpha QA^* = U\mathcal{B}^\alpha(U^*QU)\mathcal{B}^*U^* \quad (2)$$

and $(A^\alpha)^*PA = U(\mathcal{B}^\alpha)^*U^*PU\mathcal{B}U^*$

$$= U(\mathcal{B}^\alpha)^*(U^*PU)\mathcal{B}U^*. \quad (3)$$

Now, combining (2) and (3) and using (1) we have;

$$U\mathcal{B}^\alpha(U^*QU)\mathcal{B}^*U^* = U(\mathcal{B}^\alpha)^*(U^*PU)\mathcal{B}U^*. \quad (4)$$

In (4) pre-multiply by $U^*$and post-multiplying by U to give

$$\mathcal{B}^\alpha(U^*QU)\mathcal{B}^* = (\mathcal{B}^\alpha)^*(U^*PU)\mathcal{B}, \quad (5)$$

Where $U^*QU \geq 0$ and for $U^*PU \geq 0$. Hence, $\mathcal{B}$ is also α-supraposinormal.

ii) Similarly A is α-supraposinormal implies $A^\alpha QA^* = (A^\alpha)^*PA$ (1)
But $A = U^*\mathcal{B}U \Longrightarrow A^* = (U^*\mathcal{B}U)^* = U^*\mathcal{B}^*U$. Thus also
$A^\alpha = U^*\mathcal{B}UU^*\mathcal{B}UU^*\mathcal{B}U \dots U^*\mathcal{B}U$

$$= U^*\underbrace{\mathcal{B}\mathcal{B}\mathcal{B}U \dots U^*}_{\alpha \text{ times}}\mathcal{B}U\backslash$$

$$= U^*\mathcal{B}^\alpha U.$$

Also, $(A^\alpha)^* = (U^*\mathcal{B}^\alpha U)^* = U^*(\mathcal{B}^\alpha)^*U$. Now from (1) we have

$$A^\alpha QA^* = U^*\mathcal{B}^\alpha UQU^*\mathcal{B}^*U$$

$$= U^*\mathcal{B}^\alpha(UQU^*)\mathcal{B}^*U. \quad (2)$$

Also, $(A^\alpha)^*PA = U^*(\mathcal{B}^\alpha)^*UPU^*\mathcal{B}U$

$$= U^*(\mathcal{B}^\alpha)^*(UPU^*)\mathcal{B}U. \quad (3)$$

i.e. $U^*\mathcal{B}^\alpha(UQU^*)\mathcal{B}^*U = U^*(\mathcal{B}^\alpha)^*(UPU^*)\mathcal{B}U.$ (4)

Where, $UQU^* \geq 0$ and for $UPU^* \geq 0$. In (4), pre-multiply by U and post-multiply by U^* to give

$$\mathcal{B}^{\alpha}(UQU^*)\mathcal{B}^* = (\mathcal{B}^{\alpha})^*(UPU^*)\mathcal{B}. \tag{5}$$

From(5), we conclude that $\mathcal{B}$ is also α-supraposinormal operator.

Remark 4.5
Theorem 3.4 above shows that the class of α-supraposinormal is invariant both isometric and co-isometric equivalence.

Corollary 4.6
The class of supraposinormal is invariant under both isometric and coisometric equivalence and consequently invariant under unitary equivalence.

Proof
Trivially, the class of supraposinormal is a subclass of α-supraposinormal.

Theorem 4.7
Let $A, \mathcal{B} \in \mathfrak{B}(\mathcal{H})$ be such that A is a $\mathcal{S}$ $-$operator and either
i) $A = U\mathcal{B}U^*$, where U is an isometry or
ii) $A = U^*\mathcal{B}U$, where U is a coisometry.

Then $\mathcal{B}$ is also a $\mathcal{S} -$ operator.

Proof
i) Since A is a $\mathcal{S}$ $-$operator we have $\mathrm{A}^2\mathrm{A}^{*2} + \mathrm{A}^{*2}\mathrm{A}^2 = 2\mathrm{A}\mathrm{A}^{*2}\mathrm{A}$. But $\mathrm{A} = \mathrm{U}\mathcal{B}\mathrm{U}^*$ implies $\mathrm{A}^* = \mathrm{U}\mathcal{B}^*\mathrm{U}^*$. Therefore,

$$\begin{aligned} A^2A^{*2} + A^{*2}A^2 &= (U\mathcal{B}U^*)^2(U\mathcal{B}^*U^*)^2 + (U\mathcal{B}^*U^*)^2(U\mathcal{B}U^*)^2 \\ &= U\mathcal{B}U^*U\mathcal{B}U^*U\mathcal{B}^*U^*U\mathcal{B}^*U^* + U\mathcal{B}^*U^*U\mathcal{B}^*U^*U\mathcal{B}U^*U\mathcal{B}U^* \\ &= U\mathcal{B}\mathcal{B}\mathcal{B}^*\mathcal{B}^*U^* + U\mathcal{B}^*\mathcal{B}^*\mathcal{B}\mathcal{B}U^* \\ &= \mathrm{U}\mathcal{B}^2\mathcal{B}^{*2}\mathrm{U}^* + \mathrm{U}\mathcal{B}^{*2}\mathcal{B}^2\mathrm{U}^* \text{ and} \end{aligned}$$

$$\begin{aligned} 2AA^{*2}A &= 2U\mathcal{B}U^*(U\mathcal{B}^*U^*)^2U\mathcal{B}U^* \\ &= 2U\mathcal{B}U^*U\mathcal{B}^*U^*U\mathcal{B}^*U^*U\mathcal{B}U^* \\ &= 2U\mathcal{B}\mathcal{B}^*\mathcal{B}^*\mathcal{B}U^* \\ &= U2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}U^*. \end{aligned}$$

Thus, $U\mathcal{B}^2\mathcal{B}^{*2}U^* + U\mathcal{B}^{*2}\mathcal{B}^2U^* = U2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}U^*$. (1)

Now pre-multiplying by U^* *and post-multiplying by* U *in* (1) *gives* $U^*U\mathcal{B}^2\mathcal{B}^{*2}U^*U + U^*U\mathcal{B}^{*2}\mathcal{B}^2U^*U = U^*U2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}U^*U$, *which is simply* $\mathcal{B}^2\mathcal{B}^{*2} + \mathcal{B}^{*2}\mathcal{B}^2 = 2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}$. *Hence* $\mathcal{B}$ *is also an* $\mathcal{S}$ $-$*operator*.

(ii) *Since* A *is a* $\mathcal{S} -$ *operator we have* $A^2A^{*2} + A^{*2}A^2 = 2AA^{*2}A$. *But* $A = U^*\mathcal{B}U$ *implies* $A^* = U^*\mathcal{B}^*U$. *Therefore,*

$$\begin{aligned} A^2A^{*2} + A^{*2}A^2 &= (U^*\mathcal{B}U\,)^2(U^*\mathcal{B}^*U)^2 + (U^*\mathcal{B}^*U)^2(U^*\mathcal{B}U\,)^2 \\ &= U^*\mathcal{B}UU^*\mathcal{B}UU^*\mathcal{B}^*UU^*\mathcal{B}^*U + U^*\mathcal{B}^*UU^*\mathcal{B}^*UU^*\mathcal{B}^*UU^*\mathcal{B}^*U \\ &= U^*\mathcal{B}\mathcal{B}\mathcal{B}^*\mathcal{B}^*U + U^*\mathcal{B}^*\mathcal{B}^*\mathcal{B}^*\mathcal{B}^*U \\ &= U^*\mathcal{B}^2\mathcal{B}^{*2}U + U^*\mathcal{B}^{*2}\mathcal{B}^2U \textit{ and} \end{aligned}$$

$$\begin{aligned} 2AA^{*2}A &= 2U^*\mathcal{B}UU^*\mathcal{B}^*UU^*\mathcal{B}^*UU^*\mathcal{B}U \\ &= 2U^*\mathcal{B}\mathcal{B}^*\mathcal{B}^*\mathcal{B}U \\ &= U^*2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}U. \end{aligned}$$

Thus, $U^*\mathcal{B}^2\mathcal{B}^{*2}U + U^*\mathcal{B}^{*2}\mathcal{B}^2U = U^*2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}U$. (1)

Now pre-multiplying by U and post-multiplying by U^* in (1) *give*

$UU^*\mathcal{B}^2\mathcal{B}^{*2}UU^* + UU^*\mathcal{B}^{*2}\mathcal{B}^2UU^* = UU^*2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}UU^*$,
which is simply

$\mathcal{B}^2\mathcal{B}^{*2} + \mathcal{B}^{*2}\mathcal{B}^2 = 2\mathcal{B}\mathcal{B}^{*2}\mathcal{B}$. Hence $\mathcal{B}$ is also *a* $\mathcal{S}$ −operator.

Remark 4.8
The theorem above shows that the class of $\mathcal{S}$ −operators is invariant both isometrically and coisometrically and hence unitarily invariant.

Remark 4.9
In what follows, we now delve into the theory of metric equivalence with some classes of operators. Note that if two operators A and $\mathcal{B}$ are metrically equivalent i.e. $A^*A = \mathcal{B}^*\mathcal{B}$, then there exists a partial isometry U such that $A = U\mathcal{B}$ (see [11] theorem 2.1). In fact a corollary (corollary 2.6) to the same theorem in [11] asserts that if A and $\mathcal{B}$ are normal operators which are metrically equivalent then U above turns out to be unitary. However, if we take U to be an isometry, we can obtain the following results;

Theorem 4.10
Let $A, \mathcal{B} \in \mathfrak{B}(\mathcal{H})$ be metrically equivalent operators with A supraposinormal. Then $\mathcal{B}$ is also supraposinormal.

Proof
Let $A = U\mathcal{B}$, where U is an isometry. Since A and $\mathcal{B}$ are metrically equivalent, we have $A^*A = \mathcal{B}^*\mathcal{B}$. Also, A is supraposinormal implies $AQA^* = A^*PA$, where $Q, P > 0$. Now, $A = U\mathcal{B}$ implies $A^* = (U\mathcal{B})^* = \mathcal{B}^*U^*$.

Thus, $AQA^* = U\mathcal{B}Q\mathcal{B}^*U^*$.

Also, $A^*PA = \mathcal{B}^*U^*PU\mathcal{B}$.

i.e. $U\mathcal{B}Q\mathcal{B}^*U^* = \mathcal{B}^*U^*PU\mathcal{B} \Longrightarrow \mathcal{B}(UQU^*)\mathcal{B}^* = \mathcal{B}^*(U^*PU)\mathcal{B}$.

Thus $\mathcal{B}$ is also supraposinormal operator since $U^*PU \geq 0$ and $UQU^* \geq 0$.

Theorem 4.11
Let $A \in \mathfrak{B}(\mathcal{H})$ be a partial isometry and $\mathcal{B} \in \mathfrak{B}(\mathcal{H})$ be metrically equivalent to A. Then $\mathcal{B}$ is also a partial isometry.

Proof
Since A is a partial isometry, it follows that $A = AA^*A - - - - (1)$. Now, $\mathcal{B}$ is metrically 9equivalent to A implies $A^*A = \mathcal{B}^*\mathcal{B}$ and also $A = U\mathcal{B}$, where U is an isometry. From (1), we have $U\mathcal{B} = U\mathcal{B}\mathcal{B}^*\mathcal{B}$ i.e. $U\mathcal{B} - U\mathcal{B}\mathcal{B}^*\mathcal{B} = 0$

$U(\mathcal{B} - \mathcal{B}\mathcal{B}^*\mathcal{B}) = 0$.

Since U is one-one, we get $\mathcal{B} = \mathcal{B}\mathcal{B}^*\mathcal{B}$. Hence $\mathcal{B}$ is also a partial isometry. Thus, the class of partial isometries is invariant under metric equivalence.

Corollary 4.12
The class of isometries and coisometries are all invariant under metric equivalence.

Proof
Trivially, isometries and coisometries are subclasses of partial isometries.

5. Conclusion and Recommendation
Similar to other known operator equivalences, unitary equivalence and by extension metric equivalence are very fundamental in solving not only classical moment problems and linear systems but also operator interpolation problems. Theorems 4.1, 4.4 and 4.7 highlights the properties of unitary equivalences on the stated new classes of operators.
A study can also be done using the properties of almost similarity and metro-similarity on the classes of operators discussed in this paper.

6. Acknowledgement
The authors are grateful to Professor Jairus Mutekhele Khalagai for his useful comments and suggestions during preparation of this paper.

7. Competing Interests
The authors have no competing interests to declare.

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