Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2025; 6(1): 195-201

www.mathematicaljournal.com
Received: 18-01-2025
Accepted: 23-02-2025

Shankar Kandel
Central, Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal

Chet Raj Bhatta
Central, Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal

Corresponding Author:
Shankar Kandel
Central, Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal

A review on development of the hardy inequality

Shankar Kandel and Chet Raj Bhatta

DOI: https://www.doi.org/10.22271/math.2025.v6.i1c.194

Abstract
This article is discussion of the development of the Hardy Inequality. We detail the inequality in both the discrete and continuous cases, as well as notable work, by Hardy and other mathematicians at the time that contributed to its development. Also, necessary and sufficient conditions are given for a weighted Norm inequality for the sum of two dimensional Hardy-type integral operators with not necessarily non-negative coefficients. Furthermore we give necessary and sufficient conditions for certain multi-dimensional over spherical cones. The inequalities involve adjoint Hardy operator. A mixed norm inequality has also been characterized.

Keywords: Hardy's inequality, monotonicity, convergence; hardy operator, adjoint hardy operator, higher dimensional inequalities

Introduction
The Hardy Inequality is a fundamental result in the field of analysis, particularly in the study of functional spaces and partial differential equations. Its prehistory traces back to the early 20th century when mathematicians began to explore inequalities that relate the various norms of functions [1].
The Hardy inequality, named after the British mathematician G.H. Hardy, is a pivotal result in real analysis that establishes a relationship between (L^p) norms of a function and its integral properties. This inequality serves cornerstone in various mathematical disciplines, including functional analysis, PDEs and number theory. To fully appreciate its implications and applications, it is essential to delve into its historical motivation and intellectual climate from which it emerged.

Key developments leading to the Hardy inequality included
i) **Fundamental work:** The groundwork was laid by mathematicians like Holder and Minkowski, who established fundamental inequalities in analysis.
ii) **Development of Functional Spaces:** The rise of functional analysis introduced the study of various function spaces (like L^p spaces), emphasizing the importance of norms and integrability.
iii) **Hardy's Contribution:** G.H. Hardy formally introduced the inequality in his work during 1920s specifically addressing the behaviour of functions and their integrability properties over given domains [1].

The Hardy inequality provides estimate for integrals and sums involving positive functions and plays a crucial role in various areas of mathematics such as probability theory, harmonic analysis and partial differential equations prehistoric discussions related to the inequalities often revolved around the convergence of series, bounds of integrals and ratio of functions. Hardy's work was inspired by earlier results and methods from mathematical analysis, particularly those developed by mathematicians like Holder and Minkowski, who laid groundwork for understanding inequalities involving integrals and norms.
The development of popular Hardy inequality in both discrete and continuous forms during 1906 to 1928 has its own history which is known as prehistory of Hardy inequality [1]. Hardy-type inequalities have attracted a lot of interest during all the years from the dramatic prehistory to a still very active research.
This paper is devoted to this inequality in two dimensions. Hardy-type inequalities for various integral operators in two dimensions have been studied in this article [2].

Several introductory remarks are in order

i) Standards forms of Hardy's inequalities (1) and (2) [3, see page-1] that can be found in many textbooks on analysis and were highlighted first in the famous book Inequalities by Hardy, Littlewood, and Polya [4].

ii) The constant $\left(\frac{p}{p-1}\right)^p$ in both (1) and (2) is sharp constant it cannot be replaced with a smaller number such that (1) and (2) remain true for all revelant sequences and functions respectively. [3, see page 1]

iii) Inequalities (1) and (2) [3, see page 1] together with the statement (ii) imply the important information that the discrete Hardy operator $'h'$ and continous Hardy operator $'H'$, defined by,

$h(\{a_n\}) = \{\frac{1}{n}\sum_{k=1}^{n} a_k\}$ and $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$, maps the spaces l_p into l_p and L_p into L_p, respectively $(p > 1)$, and each has norm $p'\left(\frac{p}{(p-1)}\right)$. Here as usual the spaces l_p and $L_p$are the Lebesgue spaces consisting of all sequences $a = \{a_n\}$ of real numbers and all (eqivalence classes modulo equality almost everywhere) measurable functions $'f'$ on $(0, \infty)$, respectively, such that

$||a||_{l_p'} = (\sum_{n=1}^{\infty} |a_n|^p)^{\frac{1}{p}} < \infty,$

$||f||_{L_p'} = (\int_0^{\infty} |f(x)|^p dx)^{\frac{1}{p}} < \infty.$

We mention that the spaces l_p and L_p were introduced and investigated in 1910 by F. Riesz [5]. The results obtained from Hardy inequality links the behaviour of functions at small values and emphasizes the importance of understanding how functions behave in different limits. Over time, Hardy's inequalities has been organized and adapted, leading to modern interpretations applications in various branches of mathematics. The prehistory of this inequality demonstrates the evolution of mathematical thought regarding inequalities the importance of analysis informing rigorous mathematical frameworks.

The main aim of this article "A Review on Development of The Hardy Inequality" is to explore the historical development and foundational aspects of the Hardy inequality, a significant result in mathematical analysis. This article describes the contribution of various mathematician's leading up to this in equality, providing context and background to its formulation. He emphasizes the intellectual journey that led to the inequalities discovery, highlighting its implications in real analysis and its applications in various mathematical fields. The article serves both a historical account and as an exposition of the mathematical concepts involved.

Necessary and sufficient conditions are given for a weighted norm inequality for the sum of two dimensional Hardy-type integral operators with not necessarily non-negative coefficients. In this paper we also give the necessary and sufficient conditions for certain multidimensional Hardy inequalities over spherical cones. The inequalities involve adjoint Hardy operator. A Mixed norm inequality has also been characterized. Thus the Hardy inequality represents a significant intersection of analysis, functional spaces and applications across mathematics.

Historical Motivations

Researchers are highly motivated to develop Hardy's inequality because of its crucial role in providing sharp estimates in various areas of analysis and its potential to solve challenging problems and advance theoretical understanding in various mathematical fields, leading to stronger results and new research avenues. Some important factors of historical motivations are given below:

i) Early Inequalities: The root of the Hardy inequality can be traced back to classical inequalities studied in the 19th and early 20th centuries. Mathematicians like Holder's and Minkowski laid the foundational framework with their work on normed spaces and inequalities, emphasizing the importance of understanding how different norms interplay.

ii) Functional Analysis Development: The development of functional analysis during the early 1900s played a crucial role in shaping the environment where the Hardy inequality could flourish spaces examination of convergence properties led mathematicians to investigate inequalities that hold between different types of function spaces.

iii) G.H. Hardy's Contributions: In the 1920s, G.H. Hardy was among those who pushed forward the study of such inequalities. His work, Motivated by a blend of pure mathematics and practical applications, introduced the Hardy inequality as a tool to analyze particular types of functions, particularly in the context of series and integrals [1].

iv) Application and Generalizations: Following Hardy's work, the inequality was shown to here numerous applications in different areas, including number theory, Quantum Mechanics and PDEs.It becomes a tool for studying convergence properties of sequences and functions.

v) David Hilbert Contributions: In this section we briefly discuss about an inequality discovered by David Hilbert in the early 1900s (see his paper [6] from 1906) called the HILBERT INEQUALITY. This inequality is closely related to the discrete Hardy in equality (1) (see page-1 of [3]).

2. Preliminaries

Holder's, Minkowski's and Hilbert's inequalities are considered preliminary to Hardy's inequality because they provide essential tools and techniques used in its proof and generalizations. Hardy's inequality often involves integral manipulations and estimations of norms, which are precisely the areas where these preliminary inequalities excel. The proofs of Hardy-type inequalities frequently rely on applications of Holder's inequality to bound certain terms and Minkowski's inequality to handle norms. Hilbert's inequality, with its focus on double integrals, can be particularly useful in proving more complex variants of Hardy's inequality. In essence, these inequalities provide the foundational building blocks and analytical machinery necessary to establish and extend Hardy's inequality. In the following section we state different important inequalities frequently used in analysis and their verification with their suitable example.

2.1. Holder's Inequality [3]

2.1.1. Statement: Holder's Inequality states that ``for functions $f \in L^p(x)$ and $g \in L^q(x)$ on a measure space $(X, \sum, \mu)$ with conjugate exponents p' and q',

$\left(\frac{1}{p} + \frac{1}{q} = 1\right), ||fg||_1 \leq \left||f|\right|_p \left||g|\right|_q$ ".

This means:

$\int_X |f(x)g(x)|d\mu(x) \leq (\int_X |f(x)|^p d\mu(x)^{1/p} (\int_X |g(x)|^q d\mu(x)^{1/q}$.

2.1.2. Example: For the function $f(x) = x$ and $g(x) = x^2$ on $[0,1]$ with $p = 2, q = 2$ demonstrates the Holder's inequality.
Solution: Let's consider the case where $x = [0,1]$ with Lebesgue measure, $p = 2$ and $q = 2$ (the Cauchy-Schwarz inequality is a special case of Holder's Inequality).

Let $f(x) = x$ and $g(x) = x^2$

Step (i) Calculate $||f||_2$:$||f||_2 = (\int_0^1 x^2 dx)^{1/2} = \frac{1}{\sqrt{3}}$

Step (ii) Calculate $||g||_2$:$||g||_2 = (\int_0^1 x^4 dx)^{1/2} = \frac{1}{\sqrt{5}}$

Step (iii) Calculate $||fg||_1$:$||fg||_1 = (\int_0^1 |x * x^2| dx)^{1/2} = \frac{1}{4}$

Step (iv) Verify the inequality: $||fg||_1 = \frac{1}{4}, ||f||_2 ||g||_2 \approx 0.258$

Hence, Holder's inequality holds in this example.

2.2. Minkowski's Inequality [3]

2.2.1. Statement: The Minkowski inequality for $L_p$spaces $(p \geq 1)$ states:
``$||f + g||_p \leq ||f||_p + ||g||_p$ Where $||f||_p = (\int_X |f(x)|^p d\mu(x))^{1/p}$ for $1 \leq p < \infty$ and $||f||_\infty = \text{ess sup}_{x \in X}|f(x)|$ for $p = \infty$", where ess sup means essential supremum.

2.2.2. Examples

(1) Real number $(p = 2)$
Solution: Let us consider the
Simplest case: The space of all real numbers
$\mathbb{R}$ with the Euclidean norm $(p = 2)$. This is equivalent to the triangle inequality.

Let, $f = 3$ and $g = 4$. Then, $\left||f + g|\right|_2 = |3 + 4| = 7$

$||f||_2 = |3| = 3$ and $||g||_2 = |4| = 4$

The inequality holds, $7 \leq 3 + 4$.

(2) Function on an Interval $(p = 1)$
Solution: Consider the L^1 space of function on the interval $[0,1]$ with Lebesgue measure. Let $f(x) = x, g(x) = 1 - x$.
Then, $||f||_1 = \int_0^1 x dx = \frac{1}{2}$

$||g||_1 = \int_0^1 (1 - x) dx = \frac{1}{2}$

$||f + g||_1 = \int_0^1 (x + 1 - x) dx = 1$

The inequality holds: $1 \leq \frac{1}{2} + \frac{1}{2}$.

2.3 Hilbert Inequality [6]

2.3.1. Statement: The Hilbert inequality states that ``for any two sequences of complex numbers $\{a_n\}$ and $\{b_n\}$ such that $\sum |a_n|^2 < \infty$ and $\sum |b_n|^2 < \infty$ (meaning they are in the Hilbert space I^2), the following inequality holds:
$|\sum \frac{a_n b_n}{n}| \leq \pi * (\sum |a_n|^2)^{1/2} * (\sum |b_n|^2)^{1/2}$ ".

2.3.2. Example: Let us consider two simple sequences

$a_n = \frac{1}{n}$ for n = 1,2,3,… N, $b_n = \frac{1}{n}$ for n = 1,2,3,… N.

For these sequences, $\sum |a_n|^2$ and $\sum |b_n|^2$ are both convergent (though not to a finite value as N approaches infinity). The Hilbert inequality gives us an upper bound for:

$$|\sum(\frac{1}{n^2}) \leq \pi * (\sum(\frac{1}{n^2}))^{1/2} * (\sum(\frac{1}{n^2}))^{1/2} = \pi * \sum(\frac{1}{n^2})$$

The sum $\sum(\frac{1}{n^2})$ is a well-known convergent series (the Basel problem, which sums to $\frac{\pi^2}{6}$).

Therefore, the Hilbert inequality provides an upper bound related to $\frac{\pi^3}{6}$.

Note that this example is simplified, a rigorous application would involve careful consideration of the limits as N approaches infinity. The inequality holds for infinite sequence as well, provided the sums converge.

2.4 Hardy Inequality [3]

2.4.1 Statement: ``Hardy's inequality, for non-negative function f on $(0, \infty)$ and $p > 1$, is defined by the relation:

$$\int_0^\infty \left(\frac{1}{x}\right) * [\int_0^x f(t)dt]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx \text{ ".}$$

2.4.2. Example: Let us consider a simple example with $p = 2$ and $f(x) = e^{(-x)}$ for $x \in (0, \infty)$, Then f(x) is non-negative and integrable.

We need to evaluate both sides of Hardy's inequality:

$$\text{R.H.S.} = \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx = (\frac{2}{2-1})^2 \int_0^\infty e^{-2x} dx = 2$$

$$\text{L.H.S.} = \int_0^\infty \left(\frac{1}{x}\right) * [\int_0^X e^{-t} dt]^2 dx.$$

The inner integral is $1 - e^{-x}$. The outer integral is more challenging to evaluate analytically. However, numerical methods can be used to approximate the value of the L.H.S. The result will be less than or equal to R.H.S. confirming Hardy's Inequality.

3. Main Results

The main results of Development of Hardy's Inequality are break down into four sections (Intermediate results, proof of Hardy Inequality, The Operator T, The Operator H_E^*).

3.1. Intermediate Results

This section deals about the necessary and sufficient condition for the convergence of integral.In the course of proving the final form of each of the discrete and continuous cases of the Hardy

Inequality, Hardy and his contemporaries proved various related results. In 1915 Hardy developed an article[7].The main result of this article are presented here:

Theorem 3.1.1. [7]: Let $a > 0, f(x)$ be non-negative and integrable on (a, ∞), and denote $F(x) = \int_a^x f(t)dt$. The convergence of the any of the integrals $\int_a^\infty \frac{f(x)F(x)}{x} dx, \int_a^\infty (\frac{F(x)}{x})^2 dx, \int_a^\infty \int_a^\infty \frac{f(x)f(y)}{x+y} dxdy$ implies that of the others.

Theorem 3.1.2. [7]: The convergence of a $\int_a^\infty f(x)^2 dx$ for f non-negative implies that of $\int_a^\infty (\frac{F(x)}{x})^2 dx$.

Corollary 3.1.1. [7]: The convergence of $\int_a^\infty f(x)^2 dx$ for f non-negative implies that of the integrals in Theorem 3.1.1.

Theorem 3.1.3. [7]: The convergence of $\sum a_n^2$ when $a_n \geq 0$ implies that of $\sum_{n=0}^\infty (\frac{A_n}{n})^2$ where $A_n = \sum_{k=1}^n a_k$.

Theorem 3.1.4. [7] Let $a > 0, f(x), g(x)$ be non-negative and integrable on (a, ∞), and denote $F(x) = \int_a^x f(t)dt, G(x) = \int_a^x g(t)dt$. The following hypotheses are equivalent,

i) $\int_a^\infty \frac{f(x)G(x)}{x} dx$ and $\int_a^\infty \frac{f(x)g(x)}{x} dx$ are convergent,

ii) $\int_a^\infty \frac{f(x)G(x)}{x^2} dx$ is convergent,

iii) $\int_a^\infty \frac{f(x)g(y)}{x+y} dxdy$ is convergent.

Corollary 3.1.2. [7]: The convergence of $\int_a^\infty f(x)^2 dx$ and $\int_a^\infty g(x)^2 dx$ implies that of the integrals in Theorem 3.1.4.

From the motivation of above article [7], Hardy developed an another article [8].

This article included a proof of the continuous case of the Hardy Inequality for $p = 2$, with the sharp constant 4, although there was no mention of it being a sharp constant.

3.2. Proof of Hardy Inequality

Hardy's proofs of (1) and (2) (see page 1 of [3])can be done by using concept of Intermediate results and A note on hardy inequality [9]. Related theorem in continuous and discrete case are given below.

Theorem 3.2.1. (See page 6 and 7 of [3]) (Theorem of Hardy Inequality in Continuous case)
Suppose that $f(x) \geq 0, p > 1$, that $'f'$Lebesgue integrable over any finite interval (0,x) and $F(x) = \int_0^x f(t)dt$; and that $f \in L^p(R^+)$ Then,

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx.$$

In his 1925 paper, Hardy utilizes the continuous inequality to prove a more general version of discrete inequality.

Theorem 3.2.2. (See page 8-10 of [3]) (Theorem of Hardy Inequality in Discrete case)
Suppose $a_n > 0$ and $\lambda > 0$ for all $n > 0$ that

$A_n = \lambda_1 a_1 + \cdots + \lambda_n a_n$, that $\Lambda n = \lambda_1 + \cdots + \lambda_n$, and that $\sum \lambda_n a_n^p$ is convergent. Then, $\sum_{n=1}^\infty \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right) \sum_{n=1}^\infty \lambda_n a_n^p$.

3.3. The operator T [2]: In this section, Necessary and sufficient condition are given for a weighted norm inequality for the sum of two dimensional Hardy-type integral operators with not necessarily non- negative coefficients.

Definition 3.3.1(see page 247 and 248 of [2]) (The operator 'T'): A common way to define the operator 'T' in a Hardy-type inequality is as follows:
$Tf(x) = \int_a^x k(x,y)f(y)dy$, where f(y) is the function being transformed. k(x,y) is a kernel function, often positive and playing a crucial role in the inequality. x and y are variables of integration. In weighted Hardy- type inequalities, the kernel function k(x,y) might incorporate weight function, such as w(x) or w(y), to modify the integral.

Theorem 3.3.1. [2]: Suppose $1 < p \leq q < \infty$ and that u, v be weight functions on R_+^2. The inequality (1) $\left||Tf|\right|_{q,u} \leq K\left||f|\right|_{p,v}$ holds for a constant $K > 0$ if and only if $\max\{A_1, A_2, A_3, A_4, A_5, A_6\} < \infty$ where

$$A_1 = \sup_{y_1,y_2>0}\left(\int_{y_1}^\infty \int_{y_2}^\infty u(x_1,x_2)|\phi_1(x_1,x_2)|^q dx_1 dx_2\right)^{1/q} * \left(\int_0^{y_1} \int_0^{y_2} \left|\psi_1(x_1,x_2)\right|^{p'} v^{1-p'}(x_1,x_2)dx_1 dx_2\right)^{1/p'}$$

$$A_2 = \sup_{y_1,y_2>0}\left(\int_0^{y_1} \int_0^{y_2} \left(\int_0^{x_1} \int_0^{x_2} \left|\psi_1(t_1,t_2)\right|^{p'} v^{1-p'}(t_1,t_2)dt_1 dt_2\right)^q * u(x_1,x_2)|\phi_1(x_1,x_2)|^q dx_1 dx_2\right)^{1/q} /$$
$$\left(\int_0^{y_1} \int_0^{y_2} \left|\psi_1(t_1,t_2)\right|^{p'} v^{1-p'}(t_1,t_2)dt_1 dt_2\right)^{1/p}$$

$$A_3 = \sup_{y_1,y_2>0}\left(\int_{y_1}^\infty \int_{y_2}^\infty \left(\int_{x_1}^\infty \int_{x_2}^\infty u(t_1,t_2)\left|\psi_1(t_1,t_2)\right|^q dt_1 dt_2\right)^{p'} * |\psi_1(x_1,x_2)|^{p'} v^{1-p'}(x_1,x_2)dx_1 dx_2\right)^{1/p'} /$$
$$\left(\int_{y_1}^\infty \int_{y_2}^\infty u(x_1,x_2)|\psi_1(x_1,x_2)|^q dx_1 dx_2\right)^{1/q'},$$

$$A_4 = \sup_{y_1,y_2>0}\left(\int_{y_1}^\infty \int_{y_2}^\infty \left|\psi_2(t_1,t_2)\right|^{p'} v^{1-p'}(t_1,t_2)dt_1 dt_2\right)^{1/p'} * \left(\int_0^{y_1} \int_0^{y_2} u(t_1,t_2)|\psi_1(t_1,t_2)|^q dt_1 dt_2\right)^{1/q},$$

$$A_5 = \sup_{y_1,y_2>0}\left(\int_0^{y_1} \int_0^{y_2} \int_0^{x_1} \int_0^{x_2} u(t_1,t_2)|\phi_2(t_1,t_2)|^q dt_1 dt_2\right)^{p'} *$$
$$|\psi_2(x_1,x_2)|^{p'} v^{1-p'}(x_1,x_2)dx_1 dx_2\Big)^{1/p'} / \left(\int_0^{y_1} \int_0^{y_2} u(x_1,x_2)|\phi_1(x_1,x_2)|^q dx_1 dx_2\right)^{1/q'},$$

$$A_6 = \sup_{y_1,y_2>0} \int_{y_1}^\infty \int_{y_2}^\infty \left(\int_{x_1}^\infty \int_{x_2}^\infty \psi_2(t_1,t_2)|^{p'} v^{1-p'}(t_1,t_2)dt_1 dt_2\right)^q * u(x_1,x_2)|\phi_1(x_1,x_2)|^q dx_1 dx_2\Big)^{1/q} /$$
$$\left(\int_{y_1}^\infty \int_{y_2}^\infty \left|\psi_2(x_1,x_2)\right|^{p'} v^{1-p'}(x_1,x_2)dx_1 dx_2\right)^{1/p}.$$

Lemma 3.3.1[2]: Suppose $1 < p \leq q < \infty$ and that u, v be weight functions on R_+^2. The inequality (2) $\left||T_1 f|\right|_{q,u} \leq K\left||f|\right|_{p,v}$ holds for a suitable positive constant K if and only if $\max\{A_1, A_2, A_3\} < \infty$.

Lemma 3.3.2[2]: Suppose $1 < p \leq q < \infty$ and that u, v be weight functions on R_+^2. The inequality (3) $\left||T_2 f|\right|_{q,u} \leq K\left||f|\right|_{p,v}$ holds for a constant $K > 0$ only if $\max\{A_4, A_5, A_6\} < \infty$.

3.4. The Operator H_E^* [10]: In this section we discuss about necessary and sufficient conditions for a certain multi-dimensional Hardy inequalities over spherical cones. The inequalities involve adjoint hardy operator and weight functions.

Some basic definition are mentioned below

Definition 3.4.1 (Measurable function) [10]

In measure theory, a measurable function is a function between two measurable spaces that preserves the structure of the spaces, meaning the pre-image of any measurable set is also measurable.

Definition 3.4.2 (Weight function) [2]

In the context of Hardy's inequality, a weight function is a non-negative, measurable function, often denoted as $w(x)$, that modifies the integral or sum, giving certain elements more "Weight" or influence in the result than others.

Definition 3.4.3 (Hardy operator) [10]

The Hardy operator, often denoted as H, is typically defined as:

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt, \text{for } x > 0.$$

This operator takes a function f and integrates it from 0 to x, then divides by x.

Definition 3.4.4 (Adjoint Hardy operator) [10]

The adjoint of the Hardy operator H, defined as:

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt, \text{ is given by}$$

$$(H * g)(x) = \int_0^\infty \frac{g(t)}{t}dt.$$

Theorem 3.4.1 [10]: Let $1 < p < \infty, 0 < q < \infty, E, S_{N_x}, B_N$ be as defined above (see page 225 of [10]) and w, w_0 weight functions on E. Then the inequality,

$$\left(\int_E w_0(x)(H^*g)^q(x)\,dx\right)^{1/q} \le C\left(\int_E w(x)f^p(x)\,dx\right)^{1/p} \quad (1)$$

holds for all $f \ge 0$ if and only if,

$$\int_0^\infty w_0(x)(H^*g)^q(x)\,dx\Big)^{1/q} \le C\left(\int_0^\infty w(x)f^p(x)\,dx\right)^{1/p} \quad (2)$$

holds for all $g \ge 0$ with

$w_0(x_0) = \int_{B_N} w_0(x_0x')x_0^{N+1}dx', x_0 > 0$ and (3)

$$w(x_0) = \left(\int_{B_N} w^{1-p'}(x_0x')x_0^{N-1}dx'\right)^{1-p}, x_0 > 0. \quad (4)$$

Theorem 3.4.2 (see page 229 and 230 of [10]) Let w_0, w be weight functions on $(0, \infty)$.

i) For $1 < p \le q < \infty$, the inequality

$\left(\int_0^\infty \int_0^\infty g(t)dt\right)^q w_0(x)\,dx\Big)^{1/q} \le C\left(\int_0^\infty w(x)f^p(x)\,dx\right)^{1/p}$ holds for $g \ge 0$ if and only if

$$B^* = \sup_{x>0}\left(\int_0^x w_0(y)dy\right)^{1/q}\left(\int_x^\infty w^{1-p'}(y)dy\right)^{1/p'} < \infty.$$

Moreover, if C is the best possible constant then

$$B^* \le C \le \left(1+\frac{q}{p'}\right)^{1/q}\left(1+\frac{p'}{q}\right)^{1/p'}B^* \quad (A)$$

ii) For $1 < p < q < \infty, p > 1$, the inequality holds for $g \ge 0$ if and only if

$$A^* = \left(\int_0^\infty\left(\int_0^x w_0(y)dy\right)^{r/q}\left(\int_x^\infty w^{1-p'}(y)dy\right)^{r/q'}w^{1-p'}(x)dx\right)^{1/r} < \infty. \quad (B)$$

Where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Theorem 3.4.3.(see page 230 of [10]): Let the assumptions of the Theorem 3.4.1 be satisfied.

a) For $p \le q$, the inequality (1) holds if and only if (A) holds with w_0 and w given, respectively by (3) and (4).

b) For $q < p$, the inequality (1) holds if and only if (B) holds with w_0 and w given, respectively by (3) and (4).

Conclusion

The conclusion of the prehistory of Hardy inequalities centers on the emergence of Hardy's original inequality and its immediate aftermath. While earlier results hinted at the inequality's core concept, Hardy's 1920 paper provided the first rigorous formulation and proof, establishing a foundational result in analysis. The subsequent years saw refinements and extensions including generalizations to different setting and the development of related inequalities, solidifying Hardy's inequality's place as a cornerstone of mathematical analysis and its application. The prehistory culminates not in a single event, but in the establishment of a powerful and influential result that continues to inspire research and find applications across various mathematical fields.

Acknowledgement

We acknowledge University Grants Commission (UGC) Nepal, for financial support for the prepration of this article. Also I acknowledge with thanks my supervisor Prof.Dr.Chet Raj Bhatta for his guidance and encouragement to write this article.

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