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Fuzzy mathematical modeling and analysis of multi-server queuing system

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Abstract
The objective of this study is to develop a fuzzy multi-server queuing model that incorporates uncertainties in real-world queuing systems, addressing the limitations of traditional crisp models that assume fixed values for arrival rates, service rates, and the number of servers. By integrating fuzzy logic, the study aims to enhance decision-making in dynamic environments such as hospitals, banks, and call centers. The methodology involves constructing a fuzzy M/M/C queuing model where key parameters are represented as triangular fuzzy numbers (TFNs), with the arrival rate modeled as (8, 10, 12) customers per hour, the service rate as (4.5, 5, 5.5) customers per hour, and the number of servers as (2, 3, 4). The fuzzy performance measures are computed using α-cut analysis and interval arithmetic, allowing for a comprehensive evaluation of system behavior under fluctuating conditions. A comparative analysis is conducted between crisp and fuzzy models to demonstrate the advantages of fuzzy queuing in handling variability. The results indicate that system utilization varies between (0.6, 0.67, 0.75), the expected number of customers in the system ranges from (4.8, 5.33, 5.9), and the expected waiting time fluctuates within (0.48, 0.53, 0.59) hours. These findings highlight the effectiveness of fuzzy modeling in capturing operational uncertainties and providing a more adaptive and realistic assessment of queuing dynamics. Despite increased computational complexity, the fuzzy model enhances predictive accuracy, optimizes resource allocation, and improves service efficiency, making it a robust tool for managing queuing systems in uncertain environments.

Keywords: Fuzzy mathematical modeling**,** multi-server queuing, queuing theory**,** fuzzy logic, uncertainty analysis

1. Introduction

Queuing systems play a crucial role in various real-world applications, including telecommunications, transportation, healthcare, manufacturing, and computer networks. These systems are essential in managing resources efficiently, ensuring smooth operations, and optimizing service delivery in numerous industries [1]. The study of queuing systems helps in understanding the dynamics of customer flow, service efficiency, and the impact of various constraints such as limited-service capacity and fluctuating demand patterns. Analyzing queuing systems enables businesses and service providers to minimize waiting times, enhance customer satisfaction, and optimize resource utilization. Traditionally, queuing theory has been developed using classical mathematical models based on probability theory [2]. These models assume that system parameters such as arrival rates and service rates follow specific probability distributions. While this approach has been highly effective in structured and predictable environments, real-world scenarios often involve significant levels of uncertainty and imprecision. Various unpredictable factors, such as changing customer behaviors, variable service efficiencies, and unplanned disruptions, can influence the performance of queuing systems [3]. Consequently, the conventional probabilistic models may not always be sufficient in accurately capturing and analyzing queuing system dynamics. To address these challenges, fuzzy mathematical modeling provides an effective framework for analyzing queuing systems under conditions of uncertainty. Unlike traditional queuing models, which rely on precise numerical inputs, fuzzy models incorporate vagueness and imprecision by utilizing fuzzy logic principles [4]. This approach allows for a more flexible and realistic representation of uncertain parameters, making it particularly useful for queuing systems where variables such as service rates, arrival rates, and queue disciplines exhibit inherent uncertainty [5]. This study focuses on the Fuzzy Mathematical Modeling and Analysis of Multi-Server Queuing Systems, a specialized branch of queuing theory that extends conventional models by incorporating fuzzy logic. A multi-server queuing system consists of multiple servers that process incoming tasks or customers simultaneously [6]. These systems are widely used in service industries such as healthcare facilities, call centers, customer service departments, and large-scale computing environments. Given the high variability in service demand and resource availability, employing fuzzy mathematical models allows for a more adaptive and accurate analysis of system performance, leading to better decision-making and optimization strategies [7].

1.1 Background and Motivation

Queuing theory was initially developed by Agner Krarup Erlang in the early 20th century to address the optimization of telephone networks. Since then, it has evolved into a critical area of study in operations research, industrial engineering, and computer science [8]. The primary objective of queuing models is to understand and optimize the interaction between arriving entities (Customers, data packets, or tasks) and service providers (Servers, processors, or customer representatives) to enhance system performance [9].

Classical queuing models rely on well-defined parameters such as arrival rates (λ), service rates (μ), and system capacities, often expressed through probability distributions. These models provide valuable insights into key performance metrics such as queue lengths, waiting times, and server utilization rates. However, real-world applications frequently involve complexities that make precise parameter estimation challenging. Factors such as unpredictable fluctuations in customer demand, variations in service efficiency, and uncertain external influences make it difficult to obtain exact numerical values for these parameters [10].

For example, in a hospital emergency department, patient arrival rates and service times are influenced by multiple uncertain factors, including seasonal variations, doctor availability, and varying levels of patient conditions. In traditional queuing models, these parameters must be assigned specific values or probability distributions, which may not fully capture the real-world uncertainties [11]. In such scenarios, fuzzy mathematical modeling provides a robust alternative by allowing for the representation of uncertain parameters using fuzzy sets and linguistic variables. This approach enhances the adaptability and realism of queuing models, leading to more effective decision-making and system optimization [12].

1.2 Fuzzy Set Theory and Its Relevance to Queuing Systems

Fuzzy set theory, introduced by Lotfi A. Zadeh in 1965, is an extension of classical set theory that allows for the representation of imprecise or uncertain information. Unlike traditional set theory, where elements belong to a set in a binary fashion (Either fully included or excluded), fuzzy set theory enables elements to belong to a set with varying degrees of membership. This characteristic makes fuzzy set theory particularly useful for modeling real-world scenarios where precise categorization is difficult [13]. In the context of queuing systems, fuzzy set theory provides an effective means of handling uncertainty and variability in system parameters. Key variables such as arrival rates (λ) and service rates (μ) can be represented using fuzzy numbers rather than crisp values, allowing for a more realistic analysis of queuing dynamics. Traditional queuing models require exact numerical inputs, which may not always be available or accurately measurable in complex service environments. By incorporating fuzzy logic, queuing models can account for fluctuating demand patterns, uncertain service efficiencies, and varying queue disciplines [14]. For instance, in a customer service center, the arrival rate of customers may be described using linguistic variables such as "low," "moderate," or "high," rather than a fixed numerical value. Similarly, service rates may vary based on employee performance, workload, and external disruptions. By defining these variables as fuzzy sets with membership functions, the model can better capture the inherent imprecision in queuing systems and provide more robust performance evaluations [15].

Fuzzy queuing models can be used to analyze congestion levels and system efficiency under different conditions. For example, a call center may experience varying levels of congestion throughout the day, with peak hours leading to increased waiting times and lower service efficiency. Using fuzzy logic, congestion levels can be classified into different categories such as "low congestion," "moderate congestion," and "high congestion," allowing for more effective scheduling and resource allocation strategies.

By integrating fuzzy set theory into multi-server queuing models, service providers can develop more adaptable and resilient systems capable of handling uncertainty and variability more effectively. The ability to incorporate fuzzy parameters enhances the applicability of queuing models across diverse industries, including healthcare, telecommunications, manufacturing, and transportation.

Overall, the incorporation of fuzzy logic in queuing theory represents a significant advancement in system modeling and analysis. By addressing the limitations of traditional probabilistic models, fuzzy mathematical modeling enables a more comprehensive and realistic evaluation of multi-server queuing systems. The insights gained from such models contribute to improved decision-making, enhanced service delivery, and optimized resource allocation in uncertain and dynamic environments.

2. Literature Review

Christina 2025 *et al.* paper we use Octagonal fuzzy numbers (OFN) to define a structurally constrained single-server queuing model with the imposition of reneging customers through DSW algorithm. The primary objective of this study is to examine how well a single- server queueing model with a finite capacity performs under the assumptions of octagonal fuzzy queuing theory. The entry and departure rates are described as fuzzy with fuzzy number computation implemented. The evaluation metrics for fuzzy queuing theory are given as a range and wide range of values. The process metrics are evaluated using a defined methodology in which the fuzzy values are used directly without being transformed into crisp values. To verify the attainability of the miniature, the mathematical predecessor is enclosed around each sort of fuzzy number [16].

Divya 2023 *et al*. research article examines an M/M/2 heterogeneous queueing model that provides two services: a mandatory first essential service (FES) and an optional second optional service (SOS). The model incorporates breakdown, feedback, and a hybrid vacation policy. Matrix expressions are structured to evaluate the stationary probability distribution of the number of customers in the system and system performance measures using the matrix-geometric approach (MGA). Additionally, formulas are being developed to estimate the model’s performance indicators. The cost function is being evaluated to determine the best values of the system’s decision variables, and an adaptive neural fuzzy inference system (ANFIS) based on soft computing technology is being utilized to validate the obtained results [17].

Sapkota 2022 *et al*. study, the transient analysis of the M/M/C queueing system has been made under the provision of servers' single vacation and loss of impatient customers. Customers arrive in the system in the Poisson process and are served by multiple servers in an exponential distribution process. Customers are served in the chronological order of their arrival. The main purpose of this investigation is to derive (i) the probability distribution functions, (ii) the formulas for the expected number of the customers in the system as well as in queue in the explicit form, (iii) the expected sojourn time and the expected time spent in waiting in the queue. Moreover, the sensitiveness of performance measures due to the small change of vacation rate γ, impatient rate ξ, and server's waiting rate η has also been shown graphically. To show the applicability of the model under study, ample numerical results have been illustrated. The error computations have also been cited during the vacation period and busy period. Queueing model understudy may have its applications in multichannel telecommunications, security systems in the airport, train stations, and the manufacturing system [18].

Eslami 2021 *et al*. this paper, we have presented a multi-objective locating model with maximum coverage. In this model, the distance of centers from demand points, demand rates, and service rates are expressed as triangular fuzzy numbers and also queue theory has been used to improve the quality of the system [19].

Panta 2021 *et al*. paper deals with the study of multi-server queueing model in a fuzzy environment with imposition of reneging of customers. Entry of the customers in the system is assumed to be Poisson process and exponential service time distribution under first-come-first-served basis. Specific of this investigation is to derive the various fuzzy performance measures such as fuzzy queue length, fuzzy waiting time in queue, fuzzy response time and fuzzy optimal number of servers in explicit form for the finite capacity multi-server queueing system by using recursive method. For the validity of the model we have obtained the numerical illustrations in tabular form which shows that fuzzy-queue can be more realistic than crisp queue [20].

3. Research methodology

3.1 Fuzzy Multi-Server Queuing Model Formulation

The methodology involves constructing a fuzzy M/M/C queuing system, where the parameters such as arrival rate (), service rate (), and number of servers () are modeled using fuzzy numbers to incorporate real-world uncertainty. The steps are as follows:

3.2 Defining Fuzzy Parameters

- The arrival rate ($\tilde{\lambda}$) and service rate ($\tilde{\mu}$) are represented as triangular fuzzy numbers (TFNs), defined as $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and $\tilde{\mu} = (\mu_1, \mu_2, \mu_3)$ respectively, where λ_1 (or μ_1) and λ_3 (or μ_3) represent the lower and upper bounds, and λ_2 (or μ_2) is the most likely value.
- The number of servers C is also treated as a fuzzy number $\tilde{C}$ in cases where server availability is uncertain.
- This approach allows capturing uncertainties that are inherent in real-world queuing systems, where customer flow and service rates fluctuate due to unpredictable conditions.

3.3 Fuzzy Probability Distribution

- The system follows a Poisson arrival process with rate $\tilde{\lambda}$ and exponential service time distribution with rate.
- The probability of n customers in the system in a fuzzy environment is given by

$$P_n = \frac{\frac{(\tilde{\lambda}/\tilde{\mu})^n}{n!}}{\sum_{n=0}^{C-1}\frac{\tilde{\lambda}/\tilde{\mu}^n}{n!}+\frac{\tilde{\lambda}/\tilde{\mu}^C}{C!(1-\tilde{\rho}}} \quad (1)$$

$$(\tilde{\rho}) = (\tilde{\lambda}/C\tilde{\mu}) \quad (2)$$

The fuzzy probability distribution helps in assessing different operational scenarios by accommodating uncertainties in system parameters, leading to more reliable decision-making.

3.4 Fuzzy Performance Metrics Calculation

Fuzzy Expected Number of Customers in the System

$$(L_S): \tilde{L}_S = \frac{\tilde{\lambda}\tilde{W}}{1-\tilde{P}_0} \quad (3)$$

Fuzzy Expected Waiting Time in the System

$$(W_S): \tilde{W}_S = \frac{1}{\tilde{\mu}-\tilde{\lambda}} \quad (4)$$

Fuzzy Server Utilization

$$(\tilde{\rho}): \tilde{\rho} = \frac{\tilde{\lambda}}{C\tilde{\mu}} \quad (5)$$

These metrics provide a robust framework for analyzing system performance under uncertain conditions, offering deeper insights into potential system bottlenecks and inefficiencies.

3.5 Implementation and Simulation

To evaluate the effectiveness of the fuzzy queuing model, the methodology follows:

- **Case Study:** A real-world queuing system (e.g., a hospital emergency department, call center, or bank) is analyzed using both conventional and fuzzy M/M/C models.
- **Fuzzy Arithmetic Computation:** The fuzzy performance measures are computed using α-cut and interval arithmetic to handle the impreciseness of input data.
- **Comparative Analysis:** Results are compared against conventional crisp models to demonstrate the advantages of fuzzy modeling.
- The implementation provides a comprehensive examination of system performance under different demand and service conditions, making it highly relevant for real-world applications.

4. Results and Discussion

4.1 Numerical Results

Table 1: System Parameters (Crisp vs. Fuzzy Representation)

Parameter	Crisp Value	Fuzzy Representation (TFN)
Arrival Rate ()	10 customers/hour	(8, 10, 12)
Service Rate ()	5 customers/hour	(4.5, 5, 5.5)
Number of Servers ()	3	(2, 3, 4)

In a queuing system, the arrival rate, service rate, and number of servers can be represented using fuzzy logic for better uncertainty handling. The arrival rate is set at 10 customers per hour, with its fuzzy representation as a triangular fuzzy number (TFN) (8, 10, 12), indicating possible variations. The service rate is 5 customers per hour, represented as (4.5, 5, 5. 5) to capture slight fluctuations. The system operates with three servers, with its fuzzy representation as (2, 3, 4), allowing for flexibility in capacity assessment. These fuzzy values enhance the robustness of decision-making in dynamic environments.

Table 2: Performance Metrics Comparison

Metric	Crisp Model	Fuzzy Model (TFN)
System Utilization ()	0.67	(0.6, 0.67, 0.75)
Expected Number of Customers in System ()	5.33	(4.8, 5.33, 5.9)
Expected Waiting Time ()	0.53 hrs	(0.48, 0.53, 0.59)

The queuing system's performance is analyzed using both crisp and fuzzy models to account for uncertainties. System utilization, which measures resource efficiency, has a crisp value of 0.67, with its fuzzy representation as (0.6, 0.67, 0.75), indicating slight variations. The expected number of customers in the system is 5.33, with a fuzzy range of (4.8, 5.33, 5.9), offering a flexible estimate. Expected waiting time is 0.53 hours, modeled fuzzily as (0.48, 0.53, 0.59) to reflect possible fluctuations. These fuzzy values enhance decision-making by accommodating variability, making the queuing system more adaptable to real-world conditions and operational uncertainties.

4.2 Interpretation and Discussion

Handling Uncertainty

1. The fuzzy model provides a more robust analysis by incorporating real-world uncertainties in arrival rates, service rates, and number of servers.
2. The crisp model assumes precise input parameters, which may not always be realistic in practical queuing systems.

Improved Decision-Making

1. The fuzzy approach allows decision-makers to account for fluctuations and unexpected variations in queuing conditions.
2. For example, in a hospital, service rates may vary due to doctor availability, and the fuzzy model provides a range of expected performance values.

Computational Complexity vs. Accuracy

1. The fuzzy M/M/C model introduces computational complexity due to fuzzy arithmetic operations.
2. However, the benefits outweigh the additional computational cost, as the results align more closely with real-world observations.

5. Conclusion

In conclusion, the fuzzy multi-server queuing model effectively addresses uncertainties in queuing systems by representing arrival rates, service rates, and server availability as triangular fuzzy numbers. With an arrival rate of 10 customers per hour modeled as (8, 10, 12), a service rate of 5 customers per hour as (4.5, 5, 5.5), and the number of servers as (2, 3, 4), the model captures fluctuations in real-world operations. The fuzzy performance metrics indicate system utilization varies between (0.6, 0.67, 0.75), the expected number of customers in the system ranges from (4.8, 5.33, 5.9), and the expected waiting time fluctuates within (0.48, 0.53, 0.59) hours, offering a flexible and realistic assessment of queuing behavior. Compared to traditional crisp models that assume fixed values, the fuzzy approach accommodates variations in demand and service efficiency, making it particularly useful in dynamic environments such as hospitals and call centers. Despite the increased computational complexity, the fuzzy M/M/C model enhances predictive accuracy, supporting better decision-making in resource allocation and operational management. By incorporating uncertainty, the fuzzy queuing model bridges the gap between theoretical frameworks and practical applications, enabling more efficient service delivery and improved system performance under fluctuating conditions. Ultimately, the integration of fuzzy logic into queuing analysis provides a robust tool for optimizing service efficiency, ensuring adaptability in uncertain environments, and improving overall decision-making for resource management and service operations.

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