Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2025; 6(1): 91-100

www.mathematicaljournal.com
Received: 25-12-2024
Accepted: 29-01-2025

Adel M Al-Qashbari
Department of Mathematices, Education Faculty, Aden, University of Aden, Department of Engineering, Faculty of Engineering and Computing, University of Science & Technology-Aden, Yemen

Alaa A Abdallah
Department of Mathematics, Faculty of Education, Abyan University, Yemen

Kamal S Nasr
Department of Mathematics, Education Faculty, Adala, University of Aden, Aden, Yemen

Corresponding Author:
Alaa A Abdallah
Department of Mathematics, Faculty of Education, Abyan University, Yemen

On generalized Trirecurrent space by using $Gh-$ Covariant derivative in Finsler geometry

Adel M Al-Qashbari, Alaa A Abdallah and Kamal S Nasr

DOI: https://dx.doi.org/10.22271/math.2025.v6.i1b.179

Abstract
This paper investigates generalized $U_{|h}$-Trirecurrent space within the framework of Finsler geometry, where the curvature tensor satisfies specific birecurrent properties. We derive the condition for a space to be classified as a generalized $U_{|h}$-Trirecurrent space and introduce related curvature tensors and covariant derivatives. The study includes essential identities satisfied by these tensors and their implications in the context of Finsler spaces with Cartan's second kind covariant derivatives. We also examine necessary and sufficient conditions for certain tensors to be classified as generalized trirecurrent tensors. Several theorems are established, providing detailed relationships between the generalized curvature tensors, torsion, and the covariant derivatives of third order. This paper aims to extend the understanding of curvature structures in Finsler spaces and their geometrical significance.

Keywords: Generalized $U_{|h}$, Tirecurrent space, curvature tensors, cartan's connection, covariant derivative

1. Introduction
Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework to study spaces where the notion of distance varies in different directions. One of the key features in Finsler spaces is the curvature tensor, which encapsulates the geometric properties of the space.
Notable works include those by Al-Qashbari *et al.* [4-8], who explore various extensions and generalizations of recurrent Finsler spaces, such as generalized $U_{|h}$-birecurrent space and βK-recurrent space. Other significant contributions come from Hadi [14], who studied generalized birecurrent Finsler spaces. Bacso & Matsumoto [9] introduced foundational knowledge on the broader Finsler geometry and curvature tensors, enriching the theoretical framework used in modern studies. These works, along with the contributions from Qasem & Saleem [19], Pandey *et al.* [18], and others. Form a comprehensive basis for the current study, advancing both the theory and applications of Finsler spaces in differential geometry. The generalized birecurrence and trirecurrence properties for various curvature tensors have been discussed by [2, 3, 11, 13, 17]. Recently, Saleem and Abdallah [22] introduced the generalized $U_{|l}$-recurrent Finsler space.

Let us consider a set of quantities $g_{ij}\ (x, y)$ defined by [12, 20]

$$(1.1)\ g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y)$$

Where g_{ij} is positively homogeneous of degree zero in directional arguments.

$$(1.2)\ y_i = g_{ij}\, y^j.$$

The quantities g_{ij} and g^{jk} are related by

$$(1.3)\ g_{ij}g^{jk} = \delta_i^k = \begin{cases} 1\ if\ i = k, \\ 0\ if\ i \neq k\,. \end{cases}$$

The vectors y_i and y_k, related with g_{ik} and g_{jk} by

(1.4) a) $\delta_k^i\, y_i = y_k$, b) $\delta_k^i y^k = y^i$, c) $\delta_j^i\, g_{ik} = g_{jk}$, d) $\dot{\partial}_j\, y^i = \delta\,_j^i$, e) $\delta_j^i g^{jk} = g^{ik}$, f) $\delta_j^i\, \delta_k^j = \delta_k^i$ and g) $\delta_i^i = n$.

The h-covariant derivative of the metric tensor g_{ij}, the associate metric tensor g^{ij}, the vectors $y^{\,i}$ and y_i and the function F vanish identically, i. e. given by

(1.5) a) $g_{\,ij|k} = 0$, b) $g\,_{|k}^{ij} = 0$, c) $y\,_{|k}^{i} = 0$, d) $y_{\,i\,|k} = 0$ and e) $F_{|k} = 0$.

The vectors y_i and y^j satisfies the following relations

(1.6) a) $y_i\, y^j = F^2$ and b) $\dot{\partial}_j\, y_i = g_{ji}$.

For an arbitrary vector field X^i, the h-covariant derivative differentiation with respect to x^k, defined above, commutes with the partial differentiation with respect to y^j according to [10]

(1.7) a) $\dot{\partial}_j\left(X\,_{|k}^{i}\right) - \left(\dot{\partial}_j X^{\,i}\right)_{|k} = X^r\left(\dot{\partial}_j \Gamma^{*}\,_{rk}^{i}\right) - \left(\dot{\partial}_r X^{\,i}\right) P\,_{jk}^{r}$, where b) $P\,_{jk}^{r} = \left(\dot{\partial}_j \Gamma^{*r}_{hk}\right) y^h = \Gamma^{*r}_{jhk}\, y^{\,h}$.

The tensor U_{jkh}^i that denoted the tensor $\prod_{jkh}^i$ defined by [15, 16]

(1.8) $U_{jkh}^i = G_{jkh}^i + \frac{1}{(n+1)}\left(\delta_j^i G_{khr}^r + y^i G_{jkhr}^r\right)$.

Also, the tensor U_{jkh}^i is called $h(v)$ −curvature tensor and it is homogeneous of degree -1 in y^i and symmetric in its last two indices, i.e.

(1.9) $U_{jkh}^i = U_{jhk}^i$

Furthermore, this tensor satisfies the following properties:

(1.10) $U_{jkh}^h = U_{jhk}^h$,

(1.11) $U_{jkh}^i = U_{jhk}^i$,

(1.12) $U_{jkh}^i\, y^h = U_{jhk}^i\, y^h = U_{jk}^i$

(1.13) $U_{jkh}^i\, y^j = 0$.

The U-Ricci tensor U_{jk} of the projective connection coefficients satisfies the following

(1.14) $U_{ikh}^i = U_{kh}$,

(1.15) $U_{jkh}^h = U_{jk}$,

(1.16) $U_{kh} = \frac{2}{(n+1)} G_{kh}$,

Where the tensor G_{jk} is components of the call G-Ricci tensor. The U-torsion tensor U_{jk}^i satisfies

(1.17) $U_{jk}^i = U_{kj}^i$,

(1.18) $U_{jk}^i = U_{jkh}^i\, y^h$,

(1.19) $U_{jk}^i = \frac{1}{(n+1)}\, y^j G_{kh}$,

(1.20) $U_{jk}^i\, y^k = 0$

(1.21) $U_{jr}^r = C_{jr}^r$,

(1.22) $U_{jkr}^r = C_{jkr}^r$

The U^h-recurrent space, U^h-birecurrent space and generalized $U_{|l}$-trirecurrent space are characterized by [23, 24, 21].

$$U^i_{ikh|l} = \Lambda_l\, U^i_{ikh}, U^i_{ikh} \neq 0,$$

$$U^i_{ikh|l|m} = a_{lm}\, U^i_{ikh},$$

$$U^i_{ikh|l|m|n} = b_{lmn}\, U^i_{ikh},$$

Respectively.
Also, the generalized $U_{|l}$-recurrent Finsler space introduced by [22] which curvature tensor U^i_{jkh} satisfies the following condition

(1.23) $U^i_{jkh|l} = \Lambda_l U^i_{jkh} + \mu_l\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$,

Where Λ_l and μ_l are non-zero covariant tensors fields of the first order. $|l$ is called h −covariant derivative with respect to x^l.
Contracting the indices *i* and h in (1.23), using (1.15), (1.4g) and (1.4c), we get (1.24) $U_{jh|l} = \Lambda_l U_{jh} + \mu_l(n-1)g_{jh}$.
By taking the h −covariant derivative of equation (1.23) with respect to x^m, in the sense of Cartan, and applying equation (1.5a), we obtain:

(1.25) $U^i_{jkh|l|m} = \lambda_{l|m} U^i_{jkh} + \lambda_l U^i_{jkh|m} + \mu_{l|m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$.

Using (1.23) in the above equation we get

$$U^i_{jkh|l|\mathrm{m}} = \lambda_{l|m} U^i_{jkh} + \lambda_l\left(\lambda_m U^i_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)\right) + \mu_{l|m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right).$$

This simplifies to

(1.26) $U^i_{jkh|l|m} = w_{lm} U^i_{jkh} + v_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right), U^i_{jkh} \neq 0,$

Where $w_{lm} = \lambda_{l|m} + \lambda_l\lambda_m$ and $v_{lm} = \lambda_l\mu_m + \mu_{l|m}$ are non-zero covariant curvature tensors fields of second order.

This paper is focused on exploring generalized trirecurrent spaces, which are defined by specific conditions involving the curvature tensor and Cartan's second kind covariant derivative. The study aims to derive necessary and sufficient conditions for spaces to exhibit these properties and establish fundamental identities that hold within such spaces. We begin by introducing the generalization of the $h(v)$ −curvature tensor, its covariant derivatives, and the associated relations.

2. On Generalized $U_{|h}$ -Trirecurrent Space

This paper explores the properties and characteristics of generalized $U_{|h}$-Trirecurrent spaces, focusing on their structural and geometric aspects. The study delves into the role of curvature tensors and their interactions within these spaces, aiming to uncover new relationships and classifications that contribute to the broader field of Finsler geometry. By extending the theory of recurrent spaces, this work provides valuable insights into the behavior and classification of generalized $U_{|h}$-Trirecurrent space, enhancing the understanding of advanced differential geometry.
Let us consider a Finsler space F_n which the $h(v)$ −curvature tensor field U^i_{jkh} satisfies the general birecurrence property with respect to Cartan's connection parameter Γ^{*i}_{kh}, i.e., characterized by the following condition (1.26).
By taking the h −covariant derivative of equation (1.26) with respect to x^n in the sense of Cartan, and applying equation (1.5a), we obtain

(2.1) $U^i_{jkh|l|m|n} = w_{lm|n} U^i_{jkh} + w_{lm} U^i_{jkh|n} + v_{lm|n}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$,

Substitution of equation (1.23) in (2.1), we get

(2.2) $U^i_{jkh|l|m|n} = a_{lmn} U^i_{jkh} + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right), U^i_{jkh} \neq 0$.

Where $a_{lmn} = w_{lm|n} + w_{lm}\lambda_n$ and $b_{lmn} = w_{lm}\,\mu_n + v_{lm|n}$ are non-zero covariant tensors fields of the third order.

Definition 2.1. A Finsler space F_n which $h(v)$ −curvature tensor U^i_{jkh} satisfies the condition (2.2), where a_{lmn} and b_{lmn} are non-zero covariant vector field of third order, is called a generalized $U_{|h}$-trirecurrent space and the tensor will be called generalized h-trirecurrent tensor, we shall denoted them briefly by $GU_{|h}$-TRF_n and Gh-TR, respectively.

Remark 2.1. Let us call the differential operator $|l|m|n$ is the Gh −covariant derivative (Cartan's second kind covariant differential operator).
Theorem 2.1. *The generalized $U_{|h}$ -birecurrent space is $GU_{|h}$-TRF_n, provided the property of generalized $U_{|h}$-birecurrent space*

yield.

By transvecting condition (2.2) with g_{is}, and applying equations (1.4c) and (1.5a), we obtain:

(2.3) $(U^i_{jkh}\, g_{is})_{|l|m|n} = a_{lmn} U^i_{jkh} g_{is} + b_{lmn}\big(g_{ks} g_{jh} - g_{hs} g_{jk}\big)$,

Suppose that $U^i_{jkh}\, g_{is} = U_{jkhs}$, we get

(2.4) $U_{jskh|l|m} = w_{lm} U_{jskh} + v_{lm}\big(g_{ks} g_{jh} - g_{hs} g_{jk}\big)$, $U_{jskh} \neq 0$.

Conversely, transvecting the condition (2.4) by g^{is} and suppose that $U_{jkhs}\, g^{is} = U^i_{jkh}$, by using (1.3) and (1.5a) and yields the condition (2.2), we get

Theorem 2.2. *The $GU_{|h}$-TRF_n, may characterized by the condition (2.4).*

Remark 2.2. Let us call the tensor U_{jkhs} the associate curvature tensor of the $h(v)$ −curvature tensor U^i_{jkh}.
Let us consider an $GU_{|h}$-TRF_n characterized by the condition (2.2). Transvecting the condition (2.2) by y^h, using (1.5c), (1.2), (1.4b) and (1.12), we get

(2.5) $U^i_{jk|l|m|n} = a_{lmn}\, U^i_{jk} + b_{lmn}\big(\delta^i_k\, y_j - y^i g_{jk}\big)$.

Contracting the indices i and k in (2.5) and using (1.2), (1.4g) and (1.21), we get

$G^r_{jr|l|m|n} = a_{lmn}\, G^r_{jr} + (n-1) b_{lmn}\, y_j$.

Suppose that, $G^r_{jr} = G_j$, we get

(2.6) $G_{j|l|m|n} = a_{lmn}\, G_j + (n-1)\, b_{lmn}\, y_j$.

By contracting indices i and k in equation (2.2) and applying equation (1.4g), (1.10) and (1.14), we obtain

(2.7) $U_{jh|l|m|n} = a_{lmn} U_{jh} + (n-1) b_{lmn}\, g_{jh}$.

By transvecting condition (2.7) with g^{ih}, and applying equation (1.5b), we obtain

(2.8) $\big(U_{jh|l|m|}\big)\, g^{ih} = a_{lmn}\big(U_{jh}\, g^{ih}\big) + (n-1)\, b_{lmn}\; g_{jh}\, g^{ih}$.

Suppose that $U_{jh}\, g^{ih} = U^i_j$ and using (1.3), we get

(2.9) $U^i_{j|l|m|n} = a_{lmn}\, U^i_j + (n-1)\, b_{lmn}\, \delta^i_j$.

Contracting the indices i and k in (2.5) and using (1.4b), we get

(2.10) $U^r_{jr|l|m|n} = a_{lmn}\, U^r_{jr} + (n-1) b_{lmn}\, y_j$.

Suppose that $U^r_{jr} = U_j$, we get

(2.11) $U_{j|l|m|n} = a_{lmn} U_j + (n-1)\, b_{lmn}\, y_j$.

Thus, we get
Theorem 2.3. *In $GU_{|h}$-TRF_n, the Gh −covariant derivative of third order for the $h(v)$ −torsion tensor U^i_{jk}, vector curvature G_j, Ricci tensor U_{jh}, division tensor U^i_j and vector U_j are given by (2.5), (2.6), (2.7), (2.9) and (2.11), respectively.*

3. Some Relationship in $GU_{|h}$ - TRF_n

This section investigates certain relationships within the context of $GU_{|h}$-TRF_n, focusing on the structural and geometrical properties of these spaces. The study aims to explore the connections between various curvature tensors and their implications for the classification and behavior of Finsler spaces.
We know that, the Douglas tensor satisfies the following identity [18, 21, 23]
(3.1) $D^i_{jkh} = U^i_{jkh} - \frac{1}{2}\big(\delta^i_j U_{kh} - \delta^i_k U_{jh}\big)$.

By taking the h −covariant derivative of equation (3.1) with respect to x^l in the sense of Cartan, we obtain:

(3.2) $D^i_{jkh|l} = U^i_{jkh|l} - \frac{1}{2}\left(\delta^i_j U_{kh|l} - \delta^i_k U_{jh|l}\right)$.

By applying equations (1.23) and (1.24) to equation (3.2), we obtain:

(3.3) $D^i_{jkh|l} = \Lambda_l U^i_{jkh} + \mu_l\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \frac{1}{2}\left[\delta^i_j(\lambda_l U_{kh} + (n-1)\,\mu_l\, g_{kh}) - \delta^i_k\left(\lambda_l U_{jh} + (n-1)\,\mu_l\, g_{jh}\right)\right]$

By taking the h −covariant derivative of equation (3.3), with respect to x^m and x^n, successively, in the sense of Cartan, we obtain

$D^i_{jkh|l|m} = \Lambda_{l|m}\left(U^i_{jkh}\right) + \Lambda_l\left(U^i_{jkh|m}\right) + \mu_{l|m}\left(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}\right) - \frac{1}{2}\{\,\delta^i_j[\,\lambda_{l|m}(U_{kh})$
$+ \lambda_l(U_{kh|m}) + (n-1)\mu_{l|m}\; g_{kh}] - \delta^i_k[\lambda_{l|m}(U_{jh}) + \lambda_l(U_{jh|m}) + (n-1)\mu_{l|m}\, g_{jh}]\}$.

Again, using (1.23) and (1.24), in the above equation, we get

(3.4) $D^i_{jkh|l|m} = w_{lm}U^i_{jkh} + v_{lm}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - \frac{1}{2}\{\delta^i_j\left[\,w_{lm}\,U_{kh} + (n-1)v_{lm}\,g_{kh}\right] - \delta^i_k\left[w_{lm}U_{jh} + (n-1)v_{lm}\,g_{jh}\right]$

Where $w_{lm} = \lambda_{l|m} + \lambda_l\,\lambda_m$ and $v_{lm} = \lambda_l\,\mu_m + \mu_{l|m}$

Taking h −covariant derivative for (3.4) in the sense of Cartan with respect to x^n, we get

(3.5) $D^i_{jkh|l|m|n} = w_{lm|n}\left(U^i_{jkh}\right) + w_{lm}\left(U^i_{jkh|n}\right) + v_{lm|n}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$

$-\frac{1}{2}\{\,\delta^i_j\left[\,w_{lm|n}\,(U_{kh}) + w_{lm}\left(U_{kh|n}\right) + (n-1)v_{lm|n}\,g_{kh}\right]$

$-\,\delta^i_k\left[\,w_{lm|n}\left(U_{jh}\right) + w_{lm}\left(U_{jh|n}\right) + (n-1)v_{lm|n}\,g_{jh}\right]\}$.

Again, by using (1.23) and (1.24) in (3.5), we get

(3.6) $D^i_{jkh|l|m|n} = a_{lmn}U^i_{jkh} + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$

$-\frac{1}{2}\{\delta^i_j\left[a_{lmn}\,U_{kh} + (n-1)\,b_{lmn}\,g_{kh}\right] - \delta^i_k\left[a_{lmn}U_{jh} + (n-1)b_{lmn}\,g_{jh}\right]\}$,

Where $a_{lmn} = w_{lm|n} + w_{lm}\lambda_n$ and $b_{lmn} = w_{lm}\,\mu_n + v_{lm|n}$ are non-zero covariant tensors fields of the third order.

In view of (2.2) and (2.7), we get (3.7) $D^i_{jkh|l|m|n} = U^i_{jkh|l|m|n} - \frac{1}{2}\left(\delta^i_j U_{kh||l|m|n} - \delta^i_k U_{jh||l|m|n}\right)$

Thus, we get

Theorem 3.1. *In $GU_{|h}$-TRF_n, the Gh −covariant derivative of third order for Douglas tensor D^i_{jkh} is given by (3.7).*

By taking the h −covariant derivative of equation (1.16) with respect to x^l in the sense of Cartan, we obtain

(3.8) $U_{kh|l} = \frac{2}{(n+1)}G_{kh|l}$,

Which can be expressed as (3.9) $(n+1)U_{kh||l} - 2G_{kh|l} = 0$

By substituting (1.24) into (3.9), we obtain

(3.10) $(n+1)[\lambda_l U_{kh} + (n-1)\mu_l\, g_{kh}] - [\lambda_l G_{kh} + (n-1)\mu_l\, g_{kh}] = 0$

By applying the differential (3.10) twice and taking the covariant derivative in the sense of Cartan with respect to x^m and x^n successively, we obtain

(3.11) $(n+1)[a_{lmn}U_{kh} + (n-1)b_{lmn}\,g_{kh}] - 2[a_{lmn}G_{kh} + (n-1)b_{lmn}\,g_{kh}] = 0$,

Where $a_{lmn} = w_{lm|n} + w_{lm}\lambda_n$ and $b_{lmn} = w_{lm}\,\mu_n + v_{lm|n}$ are non-zero covariant tensors field of the third order.
In view of (2.7), we get
(3.12) $U_{kh|l|m|n} = \frac{2}{(n+1)}G_{kh|l|m|n}$.

Thus, we get

Corollary 3.1. *In $GU_{|h}$-TRF_n, the identity(3.12) holds.*

Taking h −covariant derivative for (1.12) in the sense of Cartan with respect to x^l, x^m and x^n, successively and using (1.5c), we get (3.13) $(U^i_{jkh}\, y^h)_{|l|m|n} = \left(U^i_{jkh|l|m|n} \right) y^h$,

Where the h −covariant derivative of y^h identically vanish. By using (2.2) in (3.13), we get

$(U^i_{jkh}\, y^h)_{|l|m|n} = a_{lmn}\, U^i_{jkh} y^h + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) y^h$,

By using (1.2) and (1.4b) in above equation, we get

(3.14) $(U^i_{jkh}\, y^h)_{|l|m|n} = a_{lmn}\, U^i_{jk} + b_{lmn}\left(\delta^i_k y_j - y^i g_{jk}\right)$.

In view of (2.5), we get

(3.15) $(U^i_{jkh}\, y^h)_{|l|m|n} = U^i_{jk|l|m|n}$.

Therefore,

Corollary 3.2. *In $GU_{|h}$-TRF_n, the identity (3.15) is valid.*

4. The Necessary and Sufficient Condition s in $GU_{|h}$-TRF_n

This section focuses on the identification and characterization of the necessary and sufficient conditions within the framework of $GU_{|h}$-TRF_n . We find the necessary and sufficient condition for some tensors to be generalized trirecurrent tensors in $GU_{|h}$-TRF_n .

By partially differentiating (2.11) with respect to y^h and applying (1.6b), we obtain

(4.1) $\dot\partial_h\left(U_{j|l|m|n}\right) = \left(\dot\partial_j a_{lmn}\right)U_j + a_{lmn}\left(\dot\partial_h U_j\right) + \left(n-1\right)\left(\dot\partial_j b_{lmn}\right)y_j + (n-1)b_{lmn}\ g_{jh}$.

By applying the commutation formula given in (1.7) to $(U_{j|l|m})$ in (4.1), we obtain

(4.2) $\left(\dot\partial_h U_{j|l|m}\right)_{|n} - U_{r|l|m}\left(\dot\partial_h \Gamma^{*r}_{jn}\right) - U_{j|r|m}\left(\dot\partial_h \Gamma^{*r}_{ln}\right) - U_{j|l|r}\left(\dot\partial_h \Gamma^{*r}_{mn}\right) - \left(\dot\partial_r U_{j|l|m}\right) P^r_{hn}$

$= \left(\dot\partial_j a_{lmn}\right)U_j + a_{lmn}\left(\dot\partial_h U_j\right) + (n-1)\left(\dot\partial_j b_{lmn}\right) y_j + (n-1)b_{lmn}\, g_{jh}$.

Again, by applying the commutation formula given in (1.7) to $(U_{j|l})$ in (4.2), we obtain

(4.3) $\left[\left(\dot\partial_h U_{j|l}\right)_{|m} - U_{r|l}\left(\dot\partial_h \Gamma^{*r}_{jm}\right) - U_{j|r}\left(\dot\partial_h \Gamma^{*r}_{lm}\right) - \left(\dot\partial_r U_{j|l}\right)P^r_{hm}\right]_{|n}$
$- U_{r|l|m}\left(\dot\partial_h \Gamma^{*r}_{jn}\right) - U_{j|r|m}\left(\dot\partial_h \Gamma^{*r}_{ln}\right) - U_{j|l|r}\left(\dot\partial_h \Gamma^{*r}_{mn}\right)$

$-\left[\left(\dot\partial_r U_{j|l}\right)_{|m} - U_{s|l}\left(\dot\partial_r \Gamma^{*s}_{jm}\right) - U_{j|s}\left(\dot\partial_r \Gamma^{*s}_{lm}\right) - \left(\dot\partial_s U_{j|l}\right) P^s_{rm}\right] P^r_{hn}$

$= \left(\dot\partial_j a_{lmn}\right)U_j + a_{lmn}\left(\dot\partial_h U_j\right) + (n-1)\left(\dot\partial_j b_{lmn}\right) y_j + (n-1)b_{lmn}\, g_{jh}$.

Again, the application of the commutation formula exhibited by (1.7) for (U_j) in (4.3), we get

(4.5) $\{[\left(\dot\partial_h U_j\right)_{|l} - U_r\left(\dot\partial_h \Gamma^{*r}_{jl}\right) - \left(\dot\partial_r U_j\right) P^r_{hl}]_{|m} - U_{r|l}\left(\dot\partial_h \Gamma^{*r}_{jm}\right) - U_{j|r}\left(\dot\partial_h \Gamma^{*r}_{lm}\right)$

$- [\left(\dot\partial_r U_j\right)_{|l} - U_s\left(\dot\partial_r \Gamma^{*s}_{jl}\right) - \left(\dot\partial_s U_j\right) P^s_{rl}]\, P^r_{hm}\}_{|n} - U_{r|l|m}\left(\dot\partial_h \Gamma^{*r}_{jn}\right) - U_{j|r|m}\left(\dot\partial_h \Gamma^{*r}_{ln}\right)$

$-U_{j|l|r}\left(\dot\partial_h \Gamma^{*r}_{mn}\right) - \{[\left(\dot\partial_r U_j\right)_{|l} - U_s\left(\dot\partial_r \Gamma^{*s}_{jl}\right) - \left(\dot\partial_s U_j\right) P^s_{rl}]_{|m} - U_{s|l}\left(\dot\partial_r \Gamma^{*s}_{jm}\right) - U_{j|s}\left(\dot\partial_r \Gamma^{*s}_{lm}\right) - [\left(\dot\partial_s U_j\right)_{|l} - U_p\left(\dot\partial_s \Gamma^{*p}_{jl}\right) -$
$\left(\dot\partial_p U_j\right) P^p_{sl}]\, P^s_{rm}\}\, P^r_{hn}$
$= \left(\dot\partial_j a_{lmn}\right)U_j + a_{lmn}\left(\dot\partial_h U_j\right) + (n-1)\left(\dot\partial_j b_{lmn}\right) y_j + (n-1)b_{lmn}\, g_{jh}$.

Suppose that $\dot\partial_h U_j = U_{jh}$, we get

(4.6) $U_{jh|l|m|n}\{[-U_r\left(\dot\partial_h \Gamma^{*r}_{jl}\right) - U_{jr}\, P^r_{hl}]_{|m} - U_{r|l}\left(\dot\partial_h \Gamma^{*r}_{jm}\right) - U_{j|r}\left(\dot\partial_h \Gamma^{*r}_{lm}\right)$
$-[U_{jr|l} - U_s\left(\dot\partial_r \Gamma^{*s}_{jl}\right) - U_{js}\, P^s_{rl}]\, P^r_{hm}\}_{|n} - U_{r|l|m}\left(\dot\partial_h \Gamma^{*r}_{jn}\right) - U_{j|r|m}\left(\dot\partial_h \Gamma^{*r}_{ln}\right) - U_{j|l|r}\left(\dot\partial_h \Gamma^{*r}_{mn}\right)$

$-\left\{\left[U_{jr|l} - U_s\left(\dot\partial_r \Gamma^{*s}_{jl}\right) - U_{js}\, P^s_{rl}\right]_{|m} - U_{s|l}\left[\left(\dot\partial_r \Gamma^{*s}_{jm} - U_{j|s}\left(\dot\partial_r \Gamma^{*s}_{lm}\right)\right) - U_{js|l} - U_p\left(\dot\partial_s \Gamma^{*p}_{jl}\right)\right.\right.$

$-U_{jp}\,P^{p}_{sl}]P^{s}_{rm}\}\,P^{r}_{hn} = (\dot{\partial}_j a_{lmn})U_j + a_{lmn}\,(U_{jh}) + (n-1)(\dot{\partial}_j b_{lmn})y_j + (n-1)\,b_{lmn}\,g_{jh}.$

In view of (2.7), we get

(4.7) $U_{jh|l|m|n} = a_{lmn}\,U_{jh} - (n-1)\,b_{lmn}\,g_{jh}$

If and only if

(4.8) $\{[-U_r(\dot{\partial}_h\Gamma^{*r}_{jl}) - U_{jr}P^{r}_{hl}\,]_{|m} - U_{r|l}(\dot{\partial}_h\Gamma^{*r}_{jm}) - U_{j|r}(\dot{\partial}_h\Gamma^{*r}_{lm}) - [\,U_{jr|l} - U_s(\dot{\partial}_r\Gamma^{*s}_{jl})$

$-U_{js}\,P^{s}_{rl}\,]\,P^{r}_{hm}\}_{|n} - U_{r|l|m}\,(\dot{\partial}_h\Gamma^{*r}_{jn}) - U_{j|r|m}(\dot{\partial}_h\Gamma^{*r}_{ln}) - U_{j|l|r}(\dot{\partial}_h\Gamma^{*r}_{mn})$

$-\{[\,U_{jr|l} - U_s(\dot{\partial}_r\,\Gamma^{*s}_{jl}) - U_{js}\,P^{s}_{rl}\,]_{|m} - U_{s|l}(\dot{\partial}_r\Gamma^{*s}_{jm} - U_{j|s}(\dot{\partial}_r\,\Gamma^{*s}_{lm}) - [U_{js|l} - U_p(\dot{\partial}_s\,\Gamma^{*p}_{jl})$

$-U_{jp}\,P^{p}_{sl}]P^{s}_{rm}\}\,P^{r}_{hn} - (\dot{\partial}_j a_{lmn})U_j - (n-1)(\dot{\partial}_j b_{lmn})\,y_j = 0$

Thus, we get

Theorem 4.1. *In $GU_{|h}$-TRF_n, U-Ricci tensor U_{jk} is Gh- TR* ***If and only if*** *(4.8) holds.*

Differentiating (2.6) partially with respect to y^h and using (1.6b), we get

(4.9) $\dot{\partial}_h(G_{j|l|m|n}) = (\dot{\partial}_j a_{lmn})G_j + a_{lmn}\,(\dot{\partial}_h G_j) + (n-1)(\dot{\partial}_j b_{lmn})y_j + (n-1)\,b_{lmn}\,g_{jh}$.

Using the commutation formula exhibited by (1.7) for $(G_{j|l|m})$ in (4.9), we get

(4.10) $(\dot{\partial}_h G_{j|l|m})_{|n} - G_{r|l|m}(\dot{\partial}_h\Gamma^{*r}_{jn}) - G_{j|r|m}(\dot{\partial}_h\Gamma^{*r}_{ln}) - G_{j|l|r}(\dot{\partial}_h\Gamma^{*r}_{mn}) - (\dot{\partial}_r G_{j|l|m})P^{r}_{hn}$

$= (\dot{\partial}_j a_{lmn})\,G_j + a_{lmn}\,(\,\dot{\partial}_h\,G_j\,) + (\mathrm{n}-1)\,(\,\dot{\partial}_j b_{lmn}\,)\,y_j + (n-1)b_{lmn}\,g_{jh}.$

Again, using commutation formula exhibited by (1.7) for $(G_{j|l})$ in (4.10), we get

(4.11) $\{(\dot{\partial}_h G_{j|l})_{|m} - G_{r|l}(\dot{\partial}_h\Gamma^{*r}_{jm}) - G_{j|r}(\dot{\partial}_h\Gamma^{*r}_{lm}) - (\dot{\partial}_r G_{j|l})P^{r}_{hm}\}_{|n} - G_{r|l|m}(\dot{\partial}_h\Gamma^{*r}_{jn})$

$-G_{j|r|m}(\dot{\partial}_h\Gamma^{*r}_{ln}) - G_{j|l|r}(\dot{\partial}_h\Gamma^{*r}_{mn}) - \{(\dot{\partial}_r G_{j|l})_{|m} - G_{s|l}(\dot{\partial}_r\Gamma^{*s}_{jm}) - G_{j|s}(\dot{\partial}_r\Gamma^{*s}_{lm}) - (\dot{\partial}_s G_{j|l})\,P^{s}_{rm}\}P^{r}_{hn} = (\dot{\partial}_j a_{lmn})G_j + a_{lmn}(\dot{\partial}_h G_j) + (n-1)(\dot{\partial}_j b_{lmn})\,y_j + (n-1)\,b_{lmn}\,g_{jh}$.

Again, the application of the commutation formula exhibited by (1.7) for (G_j) in (4.11), we get (4.12)

$\{[(\dot{\partial}_h G_j)_{|l} - G_r(\dot{\partial}_h\Gamma^{*r}_{jl}) - (\dot{\partial}_r G_j)\,P^{r}_{hl}\,]_{|m} - G_{r|l}(\dot{\partial}_h\Gamma^{*r}_{jm}) - G_{j|r}(\dot{\partial}_h\Gamma^{*r}_{lm})$

$-[(\dot{\partial}_r G_j)_{|l} - G_s(\,\dot{\partial}_r\Gamma^{*s}_{jl}) - (\dot{\partial}_s G_j)\,P^{s}_{rl}]\,P^{r}_{hm}\}_{|n} - G_{r|l|m}\,(\dot{\partial}_h\Gamma^{*r}_{jn}) - G_{j|r|m}\,(\dot{\partial}_h\Gamma^{*r}_{ln})$

$-G_{j|l|r}(\dot{\partial}_h\Gamma^{*r}_{mn}) - \{[(\dot{\partial}_r\,G_j)_{|l} - G_s(\dot{\partial}_r\Gamma^{*s}_{jl}) - (\dot{\partial}_s G_j)\,P^{s}_{rl}\,]_{|m} - G_{s|l}(\dot{\partial}_r\Gamma^{*s}_{jm})$

$-G_{j|s}(\dot{\partial}_r\Gamma^{*s}_{lm}) - [(\dot{\partial}_s\,G_j\,)_{|l} - G_p(\dot{\partial}_s\Gamma^{*p}_{jl}) - (\dot{\partial}_p G_j)\,P^{p}_{sl}\,]\,P^{s}_{rm}\}P^{r}_{hn}$

$= (\dot{\partial}_j a_{lmn})G_j + a_{lmn}\,(\dot{\partial}_h G_j) + (n-1)(\dot{\partial}_j b_{lmn})\,y_j + (n-1)\,b_{lmn}\,g_{jh}$

Suppose that $\dot{\partial}_h G_j = G_{jh}$, we get

(4.13) $G_{jh|l|m|n} = a_{lmn}\,G_{jh} + (1-n)\,b_{lmn}\,g_{jh}$.

If and only if

(4.14) $\{[-G_r(\dot{\partial}_h\Gamma^{*r}_{jl}) - (G_{jr})\,P^{r}_{hl}\,]_{|m} - G_{r|l}(\dot{\partial}_h\Gamma^{*r}_{jm}) - G_{j|r}(\dot{\partial}_h\Gamma^{*r}_{lm}) - [\,(G_{jr})_{|l} - G_s(\dot{\partial}_r\Gamma^{*s}_{jl})$

$-(G_{js})\ P^s_{rl}\ ]P^r_{hm}\}_{|n} - G_{r|l|m}\left(\dot{\partial}_h\Gamma^{*r}_{jn}\right) - G_{j|r|m}\left(\dot{\partial}_h\Gamma^{*r}_{ln}\right) - G_{j|l|r}(\dot{\partial}_h\Gamma^{*r}_{mn}) - \{[(G_{rj})_{|l}$

$-G_s\ (\dot{\partial}_r\Gamma^{*s}_{jl}) - (G_{js})\ P^s_{rl}\ ]_{|m} - G_{s|l}\ (\dot{\partial}_r\Gamma^{*s}_{jm}) - G_{j|s}(\dot{\partial}_r\Gamma^{*s}_{lm}) - [\ (G_{js})_{|l} - G_p(\dot{\partial}_s\Gamma^{*p}_{jl})$

$-\left(G_{jp}\right)P^p_{sl}\ ]\ P^s_{rm}\}\ P^r_{hn} - (\dot{\partial}_j a_{lmn})\ G_j - (1-n)\ \left(\dot{\partial}_j b_{lmn}\ y_k\right) = 0$

Thus, we get

Theorem 4.2. *In $GU_{|h}$-TRF_n, then the components of the G-Ricci tensor G_{jH} is Gh- TR If and only if (4.14) holds.*

Differential (2.5) partially with respect to y^h, we get

(4.15) $\dot{\partial}_h\left(U^i_{jk|l|m|n}\right) = \dot{\partial}_h(a_{lmn})\ U^i_{jh} + a_{lmn}(\dot{\partial}_h U^i_{jk})$

$+\dot{\partial}_h(b_{lmn})\left(\delta^i_k\ y_j - y^i g_{jk}\right) + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_k g_{jk}\right)$

Using the commutation formula exhibited by (1.7) for $U^i_{jk|l|m}$ in (4.15), we get

(4.16) $\left(\dot{\partial}_h U^i_{jk|l|m}\right)_{|n} + U^r_{jk|l|m}\left(\dot{\partial}_h\Gamma^{*i}_{rn}\right) - U^i_{rk|l|m}\left(\dot{\partial}_h\Gamma^{*r}_{jn}\right) - U^i_{jr|l|m}\left(\dot{\partial}_h\Gamma^{*r}_{kn}\right) - U^i_{jk|r|m}\left(\dot{\partial}_h\Gamma^{*r}_{ln}\right)$

$- U^i_{jk|l|r}\left(\dot{\partial}_h\Gamma^{*r}_{mn}\right) - \dot{\partial}_r\left(U^i_{jk|l|m}\right)P^r_{hn} = \dot{\partial}_h(a_{lmn})\ U^i_{jh} + a_{lmn}(\dot{\partial}_h U^i_{jk})$

$+\ \dot{\partial}_h(b_{lmn})\left(\delta^i_k\ y_j - y^i\ g_{jk}\right) + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_k g_{jk}\right)$

Again, using the commutation formula exhibited by (1.7) for $U^i_{jk|l}$ in (4.16), we get

(4.17) $[(\dot{\partial}_h U^i_{jk|l})_{|m} + U^r_{jk|l}\left(\dot{\partial}_h\Gamma^{*i}_{rm}\right) - U^i_{rk|l}\left(\dot{\partial}_h\Gamma^{*r}_{jm}\right) - U^i_{jr|l}\left(\dot{\partial}_h\Gamma^{*r}_{km}\right) - \ U^i_{jk|r}\left(\dot{\partial}_h\Gamma^{*r}_{lm}\right)$

$- \dot{\partial}_r(U^i_{jk|l})P^r_{hm}\ ]_{|n} + U^r_{jk|l|m}\left(\dot{\partial}_h\Gamma^{*i}_{rn}\right) - U^i_{rk|l|m}\left(\dot{\partial}_h\Gamma^{*r}_{jn}\right) - U^i_{jr|l|m}\left(\dot{\partial}_h\Gamma^{*r}_{kn}\right)$

$- U^i_{jk|r|m}\left(\dot{\partial}_h\Gamma^{*r}_{ln}\right) - U^i_{jk|l|r}\left(\dot{\partial}_h\Gamma^{*r}_{mn}\right) - [(\dot{\partial}_r U^i_{jk|l})_{|m} + U^s_{jk|l}\left(\dot{\partial}_r\Gamma^{*i}_{sm}\right) - U^i_{sk|l}\left(\dot{\partial}_r\Gamma^{*s}_{jm}\right)$

$- U^i_{js|l}\left(\dot{\partial}_r\Gamma^{*s}_{km}\right)\ - U^i_{jk|s}\left(\dot{\partial}_r\Gamma^{*s}_{lm}\right) - \dot{\partial}_s(U^i_{jk|l})P^s_{rm}]P^r_{hm} = \dot{\partial}_h(a_{lmn})U^i_{jh} + a_{lmn}(\dot{\partial}_h U^i_{jk})$

$+\ \dot{\partial}_h(b_{lmn})\left(\delta^i_k\ y_j - y^i g_{jk}\right) + \ b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_k g_{jk}\ \right)$

Again, using the commutation formula exhibited by (1.7) for U^i_{jk} in (4.17), we get

(4.18) $(\dot{\partial}_h U^i_{jk})_{|l|m|n}\ + \{[U^r_{jk}\ \left(\dot{\partial}_h\Gamma^{*i}_{rl}\right) - U^i_{rk}\left(\dot{\partial}_h\Gamma^{*r}_{jl}\right) - U^i_{jr}\left(\dot{\partial}_h\Gamma^{*r}_{kl}\right) - (\dot{\partial}_r U^i_{jk})\ P^r_{hl}\ ]_{|m}$

$+\ U^r_{jk|l}\left(\dot{\partial}_h\Gamma^{*i}_{rm}\right) - U^i_{rk|l}\left(\dot{\partial}_h\Gamma^{*r}_{jm}\right) - \ U^i_{jr|l}\ \left(\dot{\partial}_h\Gamma^{*r}_{km}\right) - U^i_{jk|r}\ \left(\dot{\partial}_h\Gamma^{*r}_{lm}\right)$

$- [(\dot{\partial}_r U^i_{jk})_{|l} + U^s_{jk}\left(\dot{\partial}_s\Gamma^{*i}_{rl}\right) - U^i_{sk}\left(\dot{\partial}_r\Gamma^{*s}_{jl}\right) - U^i_{js}\ \left(\dot{\partial}_h\Gamma^{*s}_{kl}\right) - U^i_{jks}\ P^s_{rl}\ ]P^r_{hm}\}_{|n}$

$+\ U^r_{jk|l|m}\left(\dot{\partial}_h\Gamma^{*i}_{rn}\right) - U^i_{rk|l|m}\left(\dot{\partial}_h\Gamma^{*r}_{jn}\right) - U^r_{jr|l|m}\left(\dot{\partial}_h\Gamma^{*r}_{kn}\right) - U^i_{jk|r|m}\left(\dot{\partial}_h\Gamma^{*r}_{ln}\right)$

$- U^i_{jk|l|r}\left(\dot{\partial}_h\Gamma^{*r}_{mn}\right) - [(\dot{\partial}_r U^i_{jk})_{|l} + U^s_{jk}\left(\dot{\partial}_r\Gamma^{*i}_{sl}\right) - U^i_{sk}\ \left(\dot{\partial}_r\Gamma^{*s}_{jl}\right) - U^i_{js}\left(\dot{\partial}_r\Gamma^{*s}_{kl}\right) - U^i_{jks}P^s_{rl}\ ]_{|m}$

$- U^s_{jk|l}\left(\dot{\partial}_r\Gamma^{*i}_{sm}\right) + U^i_{sk|l}\left(\dot{\partial}_r\Gamma^{*s}_{jm}\right) + U^i_{js|l}\left(\dot{\partial}_r\Gamma^{*s}_{km}\right) + U^i_{jk|s}\left(\dot{\partial}_r\Gamma^{*s}_{lm}\right) + \{[-(\dot{\partial}_s U^i_{jk})_{|l}$

$- U^p_{jk}\left(\dot{\partial}_h\Gamma^{*i}_{pl}\right) + U^i_{pk}\left(\dot{\partial}_h\Gamma^{*p}_{jm}\right) + U^i_{jp}\left(\dot{\partial}_h\Gamma^{*p}_{kl}\right) + (\dot{\partial}_p U^i_{jk})\ P^p_{sl}]P^s_{rm}\} = \dot{\partial}_h(a_{lmn})U^i_{jh}$

$+\ a_{lmn}(\dot{\partial}_h U^i_{jk}) + \dot{\partial}_h(b_{lmn})\left(\delta^i_k\ y_j - y^i g_{jk}\right) + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_k g_{jk}\ \right).$

This shows that

(4.19) $(\dot{\partial}_h U^i_{jk})_{|l|m|n} = a_{lmn}(\dot{\partial}_h U^i_{jk}) + b_{lmn}\left(\delta^i_k g_{jh} - \delta^i_k g_{jk}\ \right)$

If and only if

$$(4.20)\ \{[U^r_{jk}(\dot\partial_h\Gamma^{*i}_{rl}) - U^i_{rk}(\dot\partial_h\Gamma^{*r}_{jl}) - U^i_{jr}(\dot\partial_h\Gamma^{*r}_{kl}) - (U^i_{jkr})P^r_{hl}\,]_{|m} + U^r_{jk|l}(\dot\partial_h\Gamma^{*i}_{rm})$$

$$- U^i_{rk|l}(\dot\partial_h\Gamma^{*r}_{jm}) - U^i_{jr|l}(\dot\partial_h\Gamma^{*r}_{km}) - U^i_{jk|r}(\dot\partial_h\Gamma^{*r}_{lm}) - [U^i_{jkr|l} + U^s_{jk}(\dot\partial_s\Gamma^{*i}_{rl}) - U^i_{sk}(\dot\partial_r\Gamma^{*s}_{jl})$$

$$- U^i_{js}(\dot\partial_h\Gamma^{*s}_{kl}) - U^i_{jks}P^s_{rl}\,]P^r_{hm}\,\}_{|n} + U^r_{jk|l|m}(\dot\partial_h\Gamma^{*i}_{rn}) - U^i_{rk|l|m}(\dot\partial_h\Gamma^{*r}_{jn}) - U^r_{jr|l|m}(\dot\partial_h\Gamma^{*r}_{kn})$$

$$- U^i_{jk|r|m}(\dot\partial_h\Gamma^{*r}_{ln}) - U^i_{jk|l|r}(\dot\partial_h\Gamma^{*r}_{mn}) - [U^i_{jkr|l} + U^s_{jk}(\dot\partial_r\Gamma^{*i}_{sl}) - U^i_{sk}(\dot\partial_r\Gamma^{*s}_{jl})$$

$$- U^i_{js}(\dot\partial_r\Gamma^{*s}_{kl}) - U^i_{jks}P^s_{rl}\,]_{|m} - U^s_{jk|l}(\dot\partial_r\Gamma^{*i}_{sm}) + U^i_{sk|l}(\dot\partial_r\Gamma^{*s}_{jm}) + U^i_{js|l}(\dot\partial_r\Gamma^{*s}_{km})$$

$$+ U^i_{jk|s}(\dot\partial_r\Gamma^{*s}_{lm}) + \{[- U^i_{jks|l} - U^p_{jk}(\dot\partial_h\Gamma^{*i}_{pl}) + U^i_{pk}(\dot\partial_h\Gamma^{*p}_{jm}) + U^i_{jp}(\dot\partial_h\Gamma^{*p}_{kl})$$

$$+ U^i_{jkp}P^p_{sl}]P^s_{rm}\} - \dot\partial_h(a_{lmn})\ U^i_{jh} - \dot\partial_h(b_{lmn})(\delta^i_k y_j - y^i g_{jk}) = 0\,.$$

Thus, we get

Theorem 4.3. *In $GU_{|h}$-TRF_n, then the tensor $(\dot\partial_h U^i_{jk})$ is generalized trirecurrent If and only if (4.20) holds.*

5. Conclusion

The introduction of generalized U −trirecurrent Finsler space has significantly enriched the field of Finsler geometry. We study the Douglas tensor D^i_{jkh} and some tensors in $GU_{|h}$-TRF_n by using the Gh −covariant derivative of third order and we studied the relationship between them. Further, we obtained the condition for $(\dot\partial_h U^i_{jk})$ to be trirecurrent. The results presented in this paper open up new avenues for further research.

6. Recommendations for Future Research

- Investigate the relationship between generalized U −trirecurrent Finsler spaces and other geometric structures, such as conformal Finsler geometry.
- Develop numerical methods for studying the properties of generalized U −trirecurrent Finsler space.

7. References

1. Abdallah AA. Study on the relationship between two curvature tensors in Finsler spaces. J Math Anal Model. 2023;4(2):112-120.
2. Ahmed FA, Abdallah AA. The conditions for various tensors to be generalized B-trirecurrent tensor. Int J Adv Res Sci Commun Technol. 2024;4(2):511-517.
3. Al-Qashbari AM. Certain types of generalized recurrent Finsler space [PhD thesis]. Aden (Yemen): University of Aden, 2016.
4. Al-Qashbari AM, Abdallah AA, Ahmed FA. On generalized birecurrent Finsler space of mixed covariant derivatives in Cartan sense. Int J Res Publ Rev. 2024;5(8):2834-2840.
5. Al-Qashbari AM, Abdallah AA, Al-ssallal FA. Recurrent Finsler structures with higher-order generalizations defined by special curvature tensors. Int J Adv Res Sci Commun Technol. 2024;4(1):68-75.
6. Al-Qashbari AM, Abdallah AA, Baleedi SM. A study on the extensions and developments of generalized βK-recurrent Finsler space. Int J Adv Appl Math Mech. 2024;12(1):38-45.
7. Al-Qashbari AM, Abdallah AA, Nasr KS. An extension of generalized U_(|h)-birecurrent Finsler space. Int J Math Stat Invent. 2024;12(5):21-25.
8. Al-Qashbari AM, Saleh S, Ismail I. On some relations of R-projective curvature tensor in recurrent Finsler space. J Nonlinear Model Anal. 2024;6(4):1216-1227.
9. Bacso S, Matsumoto M. On Finsler space of Douglas type: A generalization of the notion of Berwald space. Publ Math Debrecen. 1997;51:385-406.
10. Ghadle KP, Abdallah AA, Navlekar AA. The necessary and sufficient condition for some tensors which satisfy a generalized BP-recurrent Finsler space. Int J Sci Eng Res. 2019;10(11):135-140.
11. Ghadle KP, Abdallah AA, Navlekar AA, Hamoud A. Decomposition for Cartan's second curvature tensor of different order in Finsler spaces. Nonlinear Funct Anal Appl. 2022;27(2):433-448.
12. Ghadle KP, Abdallah AA, Navlekar AA, Hardan B. Fundamentals and recent studies of Finsler geometry. Int J Adv Appl Math Mech. 2022;10(2):27-38.
13. Ghadle KP, Abdallah AA, Navlekar AA, Hardan B. Important results of P-generalized Finsler space with Berwald connection of third order. J Math Probl Equat Stat. 2025;6(1):19-27.
14. Hadi WH. Study of certain types of generalized birecurrent in Finsler space [PhD thesis]. Aden (Yemen): University of Aden, 2016.
15. Kumar A, Shukla HS, Tripathi RP. On the existence of projective affine motion in a W-recurrent Finsler space. Tamkang J Math. 2000;31(1).
16. Kim B, Parw H. On special Finsler space with common geodesies. Comm Korean Math Soc. 2000;15(2):314-338.

17. Nasr KS. Certain types of generalized tensors for third order [MSc dissertation]. Aden (Yemen): University of Aden, 2018.
18. Pandey PN, Saxena S, Goswami A. On generalized H-recurrent Finsler space. J Int Acad Phys Sci. 2011;15:201-211.
19. Qasem FY, Saleem AA. On U-birecurrent Finsler space. Univ Aden J Nat Appl Sci. 2010;14(3):587-596.
20. Rund H. The differential geometry of Finsler space. Springer-Verlag, 1959.
21. Saleem AA. Study on U-trirecurrent Finsler space. Univ Aden J Nat Appl Sci. 2024;28(1):13-20.
22. Saleem AA, Abdallah AA. On generalized U_(|l)-recurrent Finsler space. SHU J Humanit Appl Sci. 2024;2(2):260-270.
23. Saleem AA, Abdallah AA. On U-recurrent Finsler space. Int Res J Innov Eng Technol. 2022;6(1):58-63.
24. Saleem AA, Abdallah AA. Study on U^h-birecurrent Finsler space. Int J Adv Res Sci Commun Technol. 2022;2(3):28-39.