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Important results of $P-$ Generalized Finsler space with Berwald connection of third order

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Abstract
The generalized recurrent and birecurrent spaces for P^i_{jkh} discussed by Finslerian geometrics. Now, in this paper, we generalize these spaces by using the covariant derivative of third order in sense of Berwald. The necessary and sufficient condition for P^i_{jkh} satisfying the generalized trirecurrence property, is obtained. In addition, we obtain the covariant derivative in sense of Berwald coincides with $h-$covariant derivative in sense of Cartan. Various identities concerning of such space have established.

Keywords: Generalized $\mathcal{BP}-$trirecurrent space, Berwald covariant derivative $\mathcal{B}_k$, $h-$covariant derivative, Cartan's second curvature tensor P^i_{jkh}.

Introduction

The concept of trirecurrent and generalized trirecurrent spaces have been discussed by [11, 17, 19, 32]. The generalized recurrent space for H^i_{jkh}, K^i_{jkh}, R^i_{jkh} and P^i_{jkh} in sense of Berwald studied by [1, 6, 10, 13, 22, 30]. Also, the generalized property for normal projective curvature tensor N^i_{jkh} in sense of Berwald has been introduced by [15]. Furthermore, the generalized birecurrent space for different curvature tensors in sense of Berwald studied [3, 12, 18, 20]. Srivastava [29] defined special R-generalized recurrent Finsler spaces of the order two.

In the same regard, Qasem and Ahmed [16] studied the generalized $\mathcal{B}H-$trirecurrent space. The generalized trirecurrent space for K^i_{jkh} and R^i_{jkh} in sense of Cartan have been discussed by Nasr [26] and Husien [28], respectively.

Let F_n be an $n-$dimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [4, 31]. The vector y_i is defined by

(1.1) $y_i = g_{ij}(x,y)y^j$.

Two sets of quantities g_{ij} and its associative g^{ij} are connected by

(1.2) $g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 \ if \ j = k \,, \\ 0 \ if \ j \neq k \,. \end{cases}$

In view of Eqs. (2.1) and (2.2), we have

(1.3) a) $\delta^i_k \, y^k = y^i$, b) $\delta^i_j g_{ir} = g_{jr}$ and c) $\delta^i_k y_i = y_k$.

The $(h)hv-$torsion tensor C^i_{jk} is the associate tensor of the tensor C_{ijk} which are defined by [5, 9, 25, 27]

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$, b) $C^i_{jk}y^j = C^i_{kj}\, y^j = 0$ and c) $\delta^i_j C_{kil} = C_{kjl}$,

where

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2 \,.$$

Cartan $h-$ covariant derivative (Cartan's second kind covariant differentiation) with respect to x^k given by [7, 21]

(1.5) $X^i_{|k} = \partial_k X^i - (\dot{\partial}_r X^i) G^r_k + X^r \Gamma^{*i}_{rk}$,

where Γ^{*i}_{rk} is a function called it *Cartan*'s *connection parameter*. The $h-$ covariant derivative of the vector y^i vanish identically $i.e.$

(1.6) $y^i_{|k} = 0$.

The connection parameter G^i_{jk} of Berwald connected with Cartan's connection parameter Γ^{*i}_{jk} by

(1.7) $G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|h}\, y^h$.

Berwald's covariant derivative $\mathcal{B}_k$ of an arbitrary tensor field T^i_j with respect to x^k is given by [21]

(1.8) $\mathcal{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}$.

The processes of Berwald's covariant differentiation and the partial differentiation commute according to

(1.9) $(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$

for an arbitrary tensor field T^i_j .

Berwald's covariant derivative of the vector y^i and metric tensor g_{ij} satisfy

(1.10) a) $\mathcal{B}_k y^i = 0$ and b) $\mathcal{B}_k g_{ij} = -2C_{ijk|h} y^h = -2y^h \mathcal{B}_h C_{ijk}$.

The $h-$ curvature tensor (Cartan's third curvature tensor) is defined by [21]

$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + (\partial_l \Gamma^{*i}_{jk}) G^l_h + C^i_{jm}(\partial_k G^m_h - G^m_{kl} G^l_h) + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - k/h^*$.

The curvature tensor R^i_{jkh}, $R-$Ricci tensor R_{jk} and curvature vector R_k satisfy

(1.11) a) $R^i_{jkh} y^j = H^i_{kh} = K^i_{jkh} y^j$, b) $R^i_{jkh} = K^i_{jkh} + C^i_{js} H^s_{kh}$

c) $R_{jk} = K_{jk} + C^r_{js} H^s_{kr}$ and d) $R_{jk} y^j = R_k$.

The tensor P^i_{jkh} called $hv-$ *curvature tensor* (*Cartan's second curvature tensor*) is positively homogeneous of degree -1 in y^i and defined by [21, 23]

$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^r_{kh} - C^i_{jh|k}$.

The associate tensor P_{ijkh}, torsion tensor P^i_{kh} and $P-$Ricci tensor P_{jk} of $hv-$curvature tensor P^i_{jkh} satisfies the relations

(1.12) a) $P^i_{jkh}\, y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r$, c) $P^i_{jki} = P_{jk}$, d) $P^i_{ki} = P_k$,

e) $P_k\, y^k = P$ and f) $P^i_{jk} y^j = 0$.

Using Eq. (2.12) in Eq. (2.7), we get

(1.13) $P^i_{kh} = G^i_{kh} - \Gamma^{*i}_{kh}$.

The $hv-$ curvature tensor P^i_{jkh} satisfies the following:

(1.14) $P^i_{jkh} - P^i_{jhk} = -S^i_{jkh|r}\, y^r$

And

(1.15) $P^i_{jkh} - P^i_{kjh} = C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k$.

Alaa *et al.* [2, 3, 24] introduced the generalized $\mathcal{BP}$ $-$recurrent space and generalized $\mathcal{BP}$ $-$birecurrent space which are characterized by the conditions

(1.16) $\mathcal{B}_n P^i_{jkh} = \lambda_n P^i_{jkh} + \mu_n(\delta^i_j g_{kh} - \delta^i_k g_{jh}), P^i_{jkh} \neq 0$

and

(1.17) $\mathcal{B}_m \mathcal{B}_n P^i_{jkh} = u_{mn} P^i_{jkh} + v_{mn}(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2y^t \mu_n \mathcal{B}_t(\delta^i_j C_{khm} - \delta^i_k C_{jhm})$,

where $u_{mn} = \mathcal{B}_m \lambda_n + \lambda_m \lambda_n$ and $v_{mn} = \lambda_n \mu_m + \mathcal{B}_m \mu_n$ are non – zero covariant tensors field of second order.
In this paper, we focus on a Finsler space that P^i_{jkh} satisfies the generalized trirecurrent. Additionally, we obtain the relationship between two types of curvature tensors and covariant derivatives.

Main Results

In this section, we discuss a Finsler space that P^i_{jkh} satisfies the generalized trirecurrent. Important theorems concerned with this space have been established and proved. Taking $\mathcal{B}$ $-$covariant derivative for the condition (1.17) with respect to x^l and using the condition (1.16) and Eq. (1.10), we get

(2.1) $\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n P^i_{jkh} = a_{lmn} P^i_{jkh} + b_{lmn}(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2y^t c_{mn} \mathcal{B}_t(\delta^i_j C_{khl} - \delta^i_k C_{jhl})$

$-2\, y^t d_{ln} \mathcal{B}_t(\delta^i_j C_{khm} - \delta^i_k C_{jhm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t(\delta^i_j C_{khm} - \delta^i_k C_{jhm})$,

where $a_{lmn} = \mathcal{B}_l u_{mn} + u_{mn}\lambda_l$ and $b_{lmn} = \mathcal{B}_l v_{mn} + u_{mn}\mu_l$ are non – zero covariant tensors field of third order, $c_{mn} = v_{mn}$ and $d_{ln} = \mathcal{B}_l \mu_n$ are non – zero covariant tensor field of second order. Also, $\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n$ is the differential operator in sense of Berwald with respect to x^l, x^m and x^n, successively.

Definition 2.1. A Finsler space F_n which Cartan's second curvature tensor P^i_{jkh} satisfies the condition (2.1) will be called *a generalized* $\mathcal{BP}$ *$-$ trirecurrent space* and the tensor will be called *a generalized* $\mathcal{B}$ *$-$ trirecurrent* This space and tensor denote them briefly by $G(\mathcal{BP}) - TRF_n$ and $G\mathcal{B} - TRF_n$, respectively.
In next theorems we obtain that some tensors are non-vanishing.

Theorem 2.1. *In* $G(\mathcal{BP}) - TRF_n$, Berwald's covariant derivative of the third order for the torsion tensor P^i_{kh} and $(-S^{\ i}_{jkh|r} y^r)$ *are* given by

(2.2) $\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n P^i_{kh} = a_{lmn} P^i_{kh} + b_{lmn}(y^i g_{kh} - \delta^i_k y_h) - 2y^t c_{mn} \mathcal{B}_t(y^i C_{khl})$

$-2\, y^t d_{ln} \mathcal{B}_t(y^i C_{khm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t(y^i C_{khm})$.

And

(2.3) $\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n(-S^{\ i}_{jkh|r} y^r) = a_{lmn}(-S^{\ i}_{jkh|r} y^r) + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2y^t c_{mn} \mathcal{B}_t(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})$

$-2\, y^t d_{ln} \mathcal{B}_t(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})$,

respectively.

Proof. Let us consider a $G(\mathcal{BP}) - TRF_n$ which characterized by the condition (2.1). Transvecting the condition (2.1) by y^j, using Eqs. (1.10), (1.2), (1.3), (1.1) and (1.4), we get Eq. (2.2).
Taking $\mathcal{B}$ $-$covariant derivative for Eq. (1.14) thrice with respect to x^n, x^m and x^l, successively, using the condition (2.1), we get

$\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n(-S^{\ i}_{jkh|r} y^r) = a_{lmn}(P^i_{jkh} - P^i_{jhk}) + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2y^t c_{mn} \mathcal{B}_t(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})$

$-2\, y^t d_{ln} \mathcal{B}_t(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})$.

Using Eq. (1.14) in above equation, we get Eq. (2.3). Hence, we have proved this theorem.

Theorem 2.2. *The tensor* $(C^i_{jk|h} + C^i_{hs} P^s_{jk} - h/j)$ is a generalized $\mathcal{B}$ $-$ trirecurrent in $G(\mathcal{BP}) - TRF_n$.

Proof. Assume that $G(\mathcal{BP}) - TRF_n$. Taking $\mathcal{B}$ $-$covariant derivative for Eq. (1.15) thrice with respect to x^n, x^m and x^l, successively, using the condition (2.1), we get

$\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n(C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k) = a_{lmn}(P^i_{jkh} - P^i_{hjk}) + \propto_{lmn} (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

$-2y^t\gamma_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right)-2\,y^t\Omega_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\sigma_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right),$

where $2b_{lmn}=\propto_{lmn}$, $2c_{mn}=\gamma_{mn}$, $2d_{ln}=\Omega_{ln}$ and $2\mu_n=\sigma_n$.

Using Eq. (1.15) in above equation, we get

(2.4) $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(C^i_{kh|j}+C^i_{sj}P^s_{kh}-j/k)=a_{lmn}(C^i_{kh|j}+C^i_{sj}P^s_{kh}-j/k)+\propto_{lmn}\left(\delta^i_j g_{kh}-\delta^i_k g_{jh}\right)$

$-2y^t\gamma_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right)-2\,y^t\Omega_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\sigma_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right).$

The above equation refers to this theorem have been proved.
Now, we have a corollary related to previous theorems. Contracting the indices i and h in the condition (3.1) and Eq. (3.2), respectively. And using Eqs. (2.12), (2.3), (2.4) and (2.1), we get

(2.5) $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P_{jk}=a_{lmn}P_{jk}$

And

(2.6) $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P_k=a_{lmn}P_k$.

Transvecting Eq. (2.6) by y^k , using Eqs. (1.10) and (1.12), we get

(2.7) $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P=a_{lmn}P$.

The equations (2.5), (2.6) and (2.7) show that P_{jk}, P_k and P behave as trirecurrent. Thus, we conclude the following corollary:

Corollary 2.1. In $G(\mathcal{B}P)-TRF_n$, the behavior of Ricci tensor P_{jk} , curvature vector P_k and curvature scalar P are tirecurrent.
In next three theorems we focus on about the necessary and sufficient condition for R^i_{jkh} , H^i_{kh} and K^i_{jkh} that satisfy the generalized trirecurrence property.

Theorem 2.3. In $G(\mathcal{B}P)-TRF_n$, *for* $n=4$, Cartan's third curvature tensor R^i_{jkh} is a generalized trirecurrent if and only if the tensor $\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right)$ is trirecurrent.

Proof. Since in Riemannian space V_4 , the projective curvature tensor P^i_{jkh} is defined as follows [8]

(2.8) $P^i_{jkh}=R^i_{jkh}-\frac{1}{3}\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right)$.

Taking $\mathcal{B}$ –covariant derivative for Eq. (2.8) thrice with respect to x^n, x^m and x^l, successively, using the condition (2.1), we get

$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n R^i_{jkh}=a_{lmn}P^i_{jkh}+b_{lmn}\left(\delta^i_j g_{kh}-\delta^i_k g_{jh}\right)-2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right)$

$-2\,y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)+\frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right).$

Using Eq. (2.8) in above equation, we get

$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n R^i_{jkh}=a_{lmn}R^i_{jkh}+b_{lmn}\left(\delta^i_j g_{kh}-\delta^i_k g_{jh}\right)-2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl}-\delta^i_k C_{jhl}\right)$

$-2\,y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm}-\delta^i_k C_{jhm}\right)$

if and only if

$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right)=a_{lmn}\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right).$

The above equation means that the tensor $\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right)$ behaves as trirecurrent. Hence, we have proved this theorem.

Theorem 2.4. *In* $G(\mathcal{B}P)-TRF_n$, *for* $n=4$, Berwald's covariant derivative of third order for the torsion tensor H^i_{kh} of Berwald curvature tensor are given by

(2.9) $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n H^i_{kh}=a_{lmn}H^i_{kh}+b_{lmn}\left(y^i g_{kh}-\delta^i_k y_h\right)-2y^t c_{mn}\mathcal{B}_t(y^i C_{khl})-2\,y^t d_{ln}\mathcal{B}_t(y^i C_{khm})-2y^t\mu_n\mathcal{B}_l\mathcal{B}_t(y^i C_{khm})$

if and only if the tensor $\left(\delta^i_h R_{jk}-\delta^i_k R_{jh}\right)$ is trirecurrent.

Proof. Transvecting Eq. (3.8) by y^j, using Eqs. (1.10), (1.12) and (1.11), we get

(2.10) $P^i_{kh} = H^i_{kh} - \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right).$

Taking $\mathcal{B}$ −covariant derivative for Eq. (2.10) thrice with respect to x^n, x^m and x^l, successively, using Eq. (2.2), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n H^i_{kh} = a_{lmn}P^i_{kh} + b_{lmn}\left(y^i g_{kh} - \delta^i_k y_h\right) - 2y^t c_{mn}\mathcal{B}_t(y^i C_{khl}) - 2\, y^t d_{ln}\mathcal{B}_t(y^i C_{khm})$$

$$-2y^t \mu_n \mathcal{B}_l\mathcal{B}_t(y^i C_{khm}) + \frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_k - \delta^i_k R_h\right).$$

Using Eq. (2.10) in above equation, we get Eq. (2.9) if and only if

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(\delta^i_h R_k - \delta^i_k R_h\right) = a_{lmn}\left(\delta^i_h R_k - \delta^i_k R_h\right).$$

The above equation refers to the tensors $\left(\delta^i_h R_k - \delta^i_k R_h\right)$ behave as trirecurrent. Hence, we have proved this theorem.

Theorem 2.5. *In* $G(\mathcal{B}P) - TRF_n$ *, for* $n = 4$, Cartan's fourth curvature tensor K^i_{jkh} is a generalized $\mathcal{B}$ − trirecurrent if and only if

(2.11) $\frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left\{\left(K_{jk} + C^r_{jm}\, H^m_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{jm}\, H^m_{hr}\right)\delta^i_k\right\} - \mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(C^i_{jm}\, H^m_{kh}\right)$

$= \frac{1}{3}a_{lmn}\left\{\left(K_{jk} + C^r_{jm}\, H^m_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{jm}\, H^m_{hr}\right)\delta^i_k\right\} - a_{lmn}\left(C^i_{jm}\, H^m_{kh}\right).$

Proof. Using Eq. (1.11) in Eq. (2.8), we get

(2.12) $P^i_{jkh} = K^i_{jkh} + C^i_{jm}\, H^m_{kh} - \frac{1}{3}\{\left(K_{jk} + C^r_{jm}\, H^m_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{jm}\, H^m_{hr}\right)\delta^i_k\}.]$

Taking $\mathcal{B}$ −covariant derivative for Eq. (2.12) thrice with respect to x^n, x^m and x^l, successively, using Eq. (2.8), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n K^i_{jkh} = a_{lmn}P^i_{jkh} + b_{lmn}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - 2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right)$$

$$-2\, y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right) - 2y^t \mu_n \mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right)$$

$$-\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(C^i_{jm}\, H^m_{kh}\right) + \frac{1}{3}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left\{\left(K_{jk} + C^r_{jm}\, H^m_{kr}\right)\delta^i_h - \left(K_{jh} + C^r_{jm}\, H^m_{hr}\right)\delta^i_k\right\}.$$

Using Eq. (2.12) in above equation, we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n K^i_{jkh} = a_{lmn}K^i_{jkh} + b_{lmn}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - 2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right)$$
$$-2\, y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right) - 2y^t \mu_n \mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k C_{jhm}\right)$$

if and only if Eq. (2.11) holds. Hence, we have proved this theorem.
In next theorem we find the necessary and sufficient condition for the tensors $\left(\dot\partial_h P_{jk}\right)$ and $\left(\dot\partial_h P_k\right)$ that behave as trirecurrent.

Theorem 2.6. In $G(\mathcal{B}P) - TRF_n$, the behavior of the tensors $\left(\dot\partial_h P_{jk}\right)$*and* $\left(\dot\partial_h P_k\right)$ are trirecurrent *if* and only *if if* and only *if*

(2.13) $(\mathcal{B}_l\mathcal{B}_m P_{rk})G^r_{hnj} + (\mathcal{B}_l P_{rk})(\mathcal{B}_m G^r_{hnj}) + (\mathcal{B}_m P_{rk})(\mathcal{B}_l G^r_{hnj}) + P_{rk}(\mathcal{B}_l\mathcal{B}_m G^r_{hnj}) + (\mathcal{B}_l\mathcal{B}_m P_{jr})G^r_{hnk}$

$+(\mathcal{B}_l P_{jr})(\mathcal{B}_m G^r_{hnk}) + (\mathcal{B}_m P_{jr})(\mathcal{B}_l G^r_{hnk}) + P_{jr}(\mathcal{B}_l\mathcal{B}_m G^r_{hnk}) + (\mathcal{B}_l\mathcal{B}_r P_{jk})G^r_{hmn} + (\mathcal{B}_r P_{jk})(\mathcal{B}_l G^r_{hmn})$

$+(\mathcal{B}_l\mathcal{B}_n P_{rk})G^r_{hmj} + (\mathcal{B}_n P_{rk})(\mathcal{B}_l G^r_{hmj}) + (\mathcal{B}_l\mathcal{B}_n P_{jr})G^r_{hmk} + (\mathcal{B}_n P_{jr})(\mathcal{B}_l G^r_{hmk})$

$+\left(\mathcal{B}_r\mathcal{B}_n P_{jk}\right)G^r_{hlm} + \left(\mathcal{B}_m\mathcal{B}_r P_{jk}\right)G^r_{hln} + (\mathcal{B}_m\mathcal{B}_n P_{rk})G^r_{hlj} + \left(\mathcal{B}_m\mathcal{B}_n P_{jr}\right)G^r_{hlk} - (\dot\partial_h a_{lmn})\, P_{jk} = 0$

And

(2.14) $(\mathcal{B}_l\mathcal{B}_m P_r)G^r_{hnk} + (\mathcal{B}_l P_r)(\mathcal{B}_m G^r_{hnk}) + (\mathcal{B}_m P_r)(\mathcal{B}_l G^r_{hnk}) + P_r(\mathcal{B}_l\mathcal{B}_m G^r_{hnk}) + (\mathcal{B}_r\mathcal{B}_l P_k)G^r_{hmn} + (\mathcal{B}_r P_k)(\mathcal{B}_l G^r_{hmn}) + (\mathcal{B}_l\mathcal{B}_n P_r)G^r_{hmk} + (\mathcal{B}_n P_r)(\mathcal{B}_l G^r_{hmk}) + (\mathcal{B}_r\mathcal{B}_n P_k)G^r_{hlm} + (\mathcal{B}_m\mathcal{B}_r P_k)G^r_{hln} + (\mathcal{B}_m\mathcal{B}_n P_r)G^r_{hlk} - (\dot\partial_h a_{lmn})P_k = 0,$

respectively.

Proof. Differentiating Eq. (2.5) partially with respect to y^h, we get

$\dot{\partial}_h(\mathcal{B}_l\mathcal{B}_m\mathcal{B}_nP_{jk}) = (\dot{\partial}_h a_{lmn})\, P_{jk} + a_{lmn}\, \dot{\partial}_h P_{jk}$

Using the commutation formula exhibited by Eq. (1.9) for the tensor $(\mathcal{B}_m\mathcal{B}_nP_{jk})$ in above equation, we get

$\mathcal{B}_l\dot{\partial}_h\big(\mathcal{B}_m\mathcal{B}_nP_{jk}\big) - \big(\mathcal{B}_r\mathcal{B}_nP_{jk}\big)G^r_{hlm} - \big(\mathcal{B}_m\mathcal{B}_rP_{jk}\big)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_{rk})G^r_{hlj}$

$-\big(\mathcal{B}_m\mathcal{B}_nP_{jr}\big)G^r_{hlk} = (\dot{\partial}_h a_{lmn})\, P_{jk} + a_{lmn}\, \dot{\partial}_h P_{jk}.$

Again applying the commutation formula exhibited by Eq. (1.9) for the tensor $(\mathcal{B}_nP_{jk})$ in above equation, we get

$\mathcal{B}_l[\mathcal{B}_m\dot{\partial}_h\big(\mathcal{B}_nP_{jk}\big) - \big(\mathcal{B}_rP_{jk}\big)G^r_{hmn} - (\mathcal{B}_nP_{rk})G^r_{hmj} - \big(\mathcal{B}_nP_{jr}\big)G^r_{hmk}] - \big(\mathcal{B}_r\mathcal{B}_nP_{jk}\big)G^r_{hlm}$

$-\big(\mathcal{B}_m\mathcal{B}_rP_{jk}\big)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_{rk})G^r_{hlj} - \big(\mathcal{B}_m\mathcal{B}_nP_{jr}\big)G^r_{hlk} = (\dot{\partial}_h a_{lmn})\, P_{jk} + a_{lmn}\, \dot{\partial}_h P_{jk}.$

Also, applying the commutation formula exhibited by Eq. (1.9) for the tensor P_{jk} in above equation, we get

$\mathcal{B}_l[\mathcal{B}_m\, \{\mathcal{B}_n\dot{\partial}_hP_{jk} - P_{rk}G^r_{hnj} - P_{jr}G^r_{hnk}\} - \big(\mathcal{B}_rP_{jk}\big)G^r_{hmn} - (\mathcal{B}_nP_{rk})G^r_{hmj} - \big(\mathcal{B}_nP_{jr}\big)G^r_{hmk}]$

$-\big(\mathcal{B}_r\mathcal{B}_nP_{jk}\big)G^r_{hlm} - \big(\mathcal{B}_m\mathcal{B}_rP_{jk}\big)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_{rk})G^r_{hlj} - \big(\mathcal{B}_m\mathcal{B}_nP_{jr}\big)G^r_{hlk} = (\dot{\partial}_h a_{lmn})\, P_{jk} + a_{lmn}\, \dot{\partial}_h P_{jk},$

which can be written as

$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(\dot{\partial}_hP_{jk}) - (\mathcal{B}_l\mathcal{B}_mP_{rk})G^r_{hnj} - (\mathcal{B}_lP_{rk})(\mathcal{B}_mG^r_{hnj}) - (\mathcal{B}_mP_{rk})(\mathcal{B}_lG^r_{hnj}) - P_{rk}(\mathcal{B}_l\mathcal{B}_mG^r_{hnj})$

$-(\mathcal{B}_l\mathcal{B}_mP_{jr})G^r_{hnk} - (\mathcal{B}_lP_{jr})(\mathcal{B}_mG^r_{hnk}) - (\mathcal{B}_mP_{jr})(\mathcal{B}_lG^r_{hnk}) - P_{jr}(\mathcal{B}_l\mathcal{B}_mG^r_{hnk}) - (\mathcal{B}_l\mathcal{B}_rP_{jk})G^r_{hmn}$

$-(\mathcal{B}_rP_{jk})(\mathcal{B}_lG^r_{hmn}) - (\mathcal{B}_l\mathcal{B}_nP_{rk})G^r_{hmj} - (\mathcal{B}_nP_{rk})(\mathcal{B}_lG^r_{hmj}) - (\mathcal{B}_l\mathcal{B}_nP_{jr})G^r_{hmk} - (\mathcal{B}_nP_{jr})(\mathcal{B}_lG^r_{hmk})$

$-\big(\mathcal{B}_r\mathcal{B}_nP_{jk}\big)G^r_{hlm} - \big(\mathcal{B}_m\mathcal{B}_rP_{jk}\big)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_{rk})G^r_{hlj} - \big(\mathcal{B}_m\mathcal{B}_nP_{jr}\big)G^r_{hlk} = (\dot{\partial}_h a_{lmn})\, P_{jk} + a_{lmn}\, \dot{\partial}_h P_{jk}.$

This shows that

(2.15) $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\big(\dot{\partial}_hP_{jk}\big) = a_{lmn}\big(\dot{\partial}_hP_{jk}\big)$

if and only if Eq. (2.13) holds.

Differentiating Eq. (2.6) partially with respect to y^h, we get

$\dot{\partial}_h(\mathcal{B}_l\mathcal{B}_m\mathcal{B}_nP_k) = (\dot{\partial}_h a_{lmn})P_k + a_{lmn}\, \dot{\partial}_h P_k$

Using the commutation formula exhibited by Eq. (1.9) for the tensor $(\mathcal{B}_m\mathcal{B}_nP_k)$ in above equation, we get

$\mathcal{B}_l\dot{\partial}_h(\mathcal{B}_m\mathcal{B}_nP_k) - (\mathcal{B}_r\mathcal{B}_nP_k)G^r_{hlm} - (\mathcal{B}_m\mathcal{B}_rP_k)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_r)G^r_{hlk} = (\dot{\partial}_h a_{lmn})P_k + a_{lmn}\, \dot{\partial}_h P_k.$

Again applying the commutation formula exhibited by Eq. (1.9) for the tensor $(\mathcal{B}_nP_k)$ in above equation, we get

$\mathcal{B}_l[\mathcal{B}_m\dot{\partial}_h(\mathcal{B}_nP_k) - (\mathcal{B}_rP_k)G^r_{hmn} - (\mathcal{B}_nP_r)G^r_{hmk}] - (\mathcal{B}_r\mathcal{B}_nP_k)G^r_{hlm} - (\mathcal{B}_m\mathcal{B}_rP_k)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_r)G^r_{hlk}$
$= (\dot{\partial}_h a_{lmn})\, P_k + a_{lmn}\, \dot{\partial}_h P_k.$

Also, applying the commutation formula exhibited by Eq. (1.9) for the tensor P_k in above equation, we get

$\mathcal{B}_l[\mathcal{B}_m\, \{\mathcal{B}_n\dot{\partial}_hP_k - P_rG^r_{hnk}\} - (\mathcal{B}_rP_k)G^r_{hmn} - (\mathcal{B}_nP_r)G^r_{hmk}] - (\mathcal{B}_r\mathcal{B}_nP_k)G^r_{hlm} - (\mathcal{B}_m\mathcal{B}_rP_k)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_r)G^r_{hlk}$
$= (\dot{\partial}_h a_{lmn})\, P_k + a_{lmn}\, \dot{\partial}_h P_k.$

Which can be written as

$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(\dot{\partial}_hP_k) - (\mathcal{B}_l\mathcal{B}_mP_r)G^r_{hnk} - (\mathcal{B}_lP_r)(\mathcal{B}_mG^r_{hnk}) - (\mathcal{B}_mP_r)(\mathcal{B}_lG^r_{hnk}) - P_r(\mathcal{B}_l\mathcal{B}_mG^r_{hnk})$

$-(\mathcal{B}_r\mathcal{B}_lP_k)G^r_{hmn} - (\mathcal{B}_rP_k)(\mathcal{B}_lG^r_{hmn}) - (\mathcal{B}_l\mathcal{B}_nP_r)G^r_{hmk} - (\mathcal{B}_nP_r)(\mathcal{B}_lG^r_{hmk}) - (\mathcal{B}_r\mathcal{B}_nP_k)G^r_{hlm}$

$-(\mathcal{B}_m\mathcal{B}_rP_k)G^r_{hln} - (\mathcal{B}_m\mathcal{B}_nP_r)G^r_{hlk} = (\dot{\partial}_h a_{lmn})\, P_k + a_{lmn}\, \dot{\partial}_h P_k.$

This shows that

(2.16) $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(\dot{\partial}_h P_k) = a_{lmn}(\dot{\partial}_h P_k)$

if and only if Eq. (2.14) holds.
The equations (2.15) and (2.16) refer to the tensors $(\dot{\partial}_h P_{jk})$ and $(\dot{\partial}_h P_k)$ behave as trirecurrent if and only if Eqs. (2.13) and (2.14) hold.

In next theorems we obtain the relation between the covariant derivative in sense of Berwald and Cartan for different tensors.

Theorem 2.7. In $G(\mathcal{B}P) - TRF_n$, when the covariant derivative in sense of Berwald with respect to x^l, x^m coincide with the $h-$ covariant derivative in sense of Cartan with respect to x^n , then P^r_{jkh} is given by

(2.17) $\mathcal{B}_l\mathcal{B}_m P^i_{jkh|n} = a_{lmn}P^i_{jkh} + b_{lmn}(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2y^t c_{mn}\mathcal{B}_t(\delta^i_j C_{khl} - \delta^i_k C_{jhl})$

$-2\, y^t d_{ln}\mathcal{B}_t(\delta^i_j C_{khm} - \delta^i_k C_{jhm}) - 2y^t \mu_n \mathcal{B}_l\mathcal{B}_t(\delta^i_j C_{khm} - \delta^i_k C_{jhm})$

is a generalized birecurrent if and only if

(2.18) $\mathcal{B}_l\mathcal{B}_m(P^r_{jkh}P^i_{rn} - P^i_{rkh}P^r_{jn} - P^i_{jrh}P^r_{kn} - P^i_{jkr}P^r_{hn}) = 0.$

Proof. Using the definition of the covariant derivative in sense of Berwald for P^i_{jkh} exhibited by Eq. (1.8) with respect to x^l, we get

$$\mathcal{B}_n P^i_{jkh} = \partial_n P^i_{jkh} + P^r_{jkh}G^i_{rn} - P^i_{rkh}G^r_{jn} - P^i_{jrh}G^r_{kn} - P^i_{jkr}G^r_{hn} - (\dot{\partial}_r P^i_{jkh})G^r_n .$$

Using Eq. (1.13) in above equation, we get

$$\mathcal{B}_n P^i_{jkh} = \partial_n P^i_{jkh} + P^r_{jkh}\Gamma^{*i}_{rn} - P^i_{rkh}\Gamma^{*r}_{jn} - P^i_{jrh}\Gamma^{*r}_{kn} - P^i_{jrh}\Gamma^{*r}_{hn} - (\dot{\partial}_r P^i_{jkh})G^r_n + P^r_{jkh}P^i_{rn} - P^i_{rkh}P^r_{jn} - P^i_{jrh}P^r_{kn} - P^i_{jkr}P^r_{hn}.$$

Using the definition of $h-$ covariant derivative in sense of Cartan exhibited by Eq. (1.5) with respect to x^n in above equation, we get

(2.19) $\mathcal{B}_n P^i_{jkh} = P^i_{jkh|n} + P^r_{jkh}P^i_{rn} - P^i_{rkh}P^r_{jn} - P^i_{jrh}P^r_{kn} - P^i_{jkr}P^r_{hn}$

Taking $\mathcal{B}-$ covariant derivative for Eq. (2.19) twice with respect to x^m and x^l, using the condition (2.1), we get

$$\mathcal{B}_l\mathcal{B}_m P^i_{jkh|n} = a_{lmn}P^i_{jkh} + b_{lmn}(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2y^t c_{mn}\mathcal{B}_t(\delta^i_j C_{khl} - \delta^i_k C_{jhl})$$

$$-2\, y^t d_{ln}\mathcal{B}_t(\delta^i_j C_{khm} - \delta^i_k C_{jhm}) - 2y^t \mu_n \mathcal{B}_l\mathcal{B}_t(\delta^i_j C_{khm} - \delta^i_k C_{jhm}) + \mathcal{B}_l\mathcal{B}_m(P^r_{jkh}P^i_{rn} - P^i_{rkh}P^r_{jn} - P^i_{jrh}P^r_{kn} - P^i_{jkr}P^r_{hn})$$

Thus, we obtain Eq. (2.17) if and only if Eq. (2.18) holds. Hence, we have proved this theorem.

Theorem 2.8. In $G(\mathcal{B}P) - TRF_n$, when the covariant derivative in sense of Berwald with respect to x^l, x^m coincide with the $h-$ covariant derivative in sense of Cartan with respect to x^n , then P^i_{kh} is given by

(2.20) $\mathcal{B}_l\mathcal{B}_m P^i_{kh|n} = a_{lmn}P^i_{kh} + b_{lmn}(y^i g_{kh} - \delta^i_k y_h) - 2y^t c_{mn}\mathcal{B}_t(y^i C_{khl}) - 2\, y^t d_{ln}\mathcal{B}_t(y^i C_{khm}) - 2y^t \mu_n \mathcal{B}_l\mathcal{B}_t(y^i C_{khm})$

if and only if

(3.21) $\mathcal{B}_l\mathcal{B}_m(P^r_{kh}P^i_{rn} - P^i_{rh}P^r_{kn} - P^i_{kr}P^r_{hn}) = 0.$

Proof. Transvecting Eq. (3.19) by y^j, using Eqs. (1.6), (1.10) and (1.12), we get

(2.22) $\mathcal{B}_n P^i_{kh} = P^i_{kh|n} + P^r_{kh}P^i_{rn} - P^i_{rh}P^r_{kn} - P^i_{kr}P^r_{hn}$.

Taking $\mathcal{B}-$ covariant derivative for Eq. (2.22) twice with respect to x^m and x^l, using Eq. (2.2), we get

$$\mathcal{B}_l\mathcal{B}_m P^i_{kh|n} = a_{lmn}P^i_{kh} + b_{lmn}(y^i g_{kh} - \delta^i_k y_h) - 2y^t c_{mn}\mathcal{B}_t(y^i C_{khl}) - 2\, y^t d_{ln}\mathcal{B}_t(y^i C_{khm})$$

$$-2y^t \mu_n \mathcal{B}_l\mathcal{B}_t(y^i C_{khm}) + \mathcal{B}_l\mathcal{B}_m(P^r_{kh}P^i_{rn} - P^i_{rh}P^r_{kn} - P^i_{kr}P^r_{hn}).$$

Thus, we obtain Eq. (2.20) if and only if Eq. (2.21) holds. Hence, we have proved this theorem.

Now, we have a corollary related to previous theorems. Contracting the indices i and h in Eq. (2.19) and using Eq. (1.12), we get

$$\mathcal{B}_n P_{jk} = P_{jk|n} + P^r_{jki} P^i_{rn} - P_{rk} P^r_{jn} - P_{jr} P^r_{kn} - P^i_{jkr} P^r_{in} .$$

Taking $\mathcal{B}$ −covariant derivative for above equation twice with respect to x^m and x^l , using Eq. (2.5), we get

$$\mathcal{B}_l \mathcal{B}_m P_{jk|n} = a_{lmn} P_{jk}$$

if and only if

(2.23) $\mathcal{B}_l \mathcal{B}_m (P_{rk} P^r_{jn} - P_{jr} P^r_{kn}) = 0.$

Contracting the indices i and h in Eq. (2.22) and using Eq. (1.12), we get

$$\mathcal{B}_n P_k = P_{k|n} + P^r_{ki} P^i_{rn} - P_r P^r_{kn} - P^i_{kr} P^r_{in} .$$

Taking $\mathcal{B}$ −covariant derivative for above equation twice with respect to x^m and x^l, using Eq. (2.6), we get

$$\mathcal{B}_l \mathcal{B}_m P_{k|n} = a_{lmn} P_k$$

if and only if

(2.24) $\mathcal{B}_l \mathcal{B}_m (P_r P^r_{kn}) = 0.$

Thus, we conclude the following corollary:

Corollary 2.2. In $G(\mathcal{B}P) - TRF_n$, when the covariant derivative in sense of Berwald with respect to x^l, x^m coincide with the $h-$ covariant derivative in sense of Cartan with respect to x^n , then P_{jk} and P_k behave as trirecurrent if and only if Eqs. (2.23) and (2.24) hold, respectively.

Conclusion

We introduced a Finsler space that P^i_{jkh} satisfies the generalized trirecurrence property in $G(\mathcal{B}R) - TRF_n$. The relationship between P^i_{jkh} and R^i_{jkh} have been studied in the above mentioned space.

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