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Numerical solution of non-linear non-local problems for elliptic equations

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Abstract
A non-local problem for an elliptic equation in a rectangular domain was investigated. A rectangular grid for the corresponding difference problem was constructed and the error of the approximate solutions of non-local problems was estimated. Various application problems (heat conductivity [1, 2, 3], fluid mechanics [4], the theory of elasticity and shells [5], etc.) are reduced to non-local boundary value problems. Non-local boundary conditions are especially difficult for justification of classical finite difference schemes due to the complexity of the structure of the matrices obtained from systems of equations. This difficulty manifests itself especially in the justification of numerical methods in the case of non-linear equations. In this paper we consider the non-local boundary value problem for a quasi-linear equation. We found the numerical solutions of stated problem using the finite difference method, and estimated the error of the approximate solutions of non-local problems.

Keywords: Non-local, estimated error, difference problem, difference operator, non-linear.

Introduction

Let $\Omega = \{0 < x < a, 0 < y < b\}$. Denote by $\Gamma^1 = \{0 \le x \le a, y = b\}$, $\Gamma^2 = \{x = 0, 0 < y < b\}$,

$\Gamma^3 = \{0 \le x \le a, y = 0\}$, $\Gamma^4 = \{x = a, 0 < y < b\}$, $\Gamma^l = \{x = l, 0 < y < b,\ 0 < l < b\}$, $\Gamma = \bigcup_{i=1}^{4} \Gamma^i$,

$\sigma = \Gamma^1 \bigcup \Gamma^3$, $\overline{\Omega} = \Omega \cup \Gamma$.

Suppose that $f(x, y, z, p, q)$ is a given continuous function determined $\forall (x, y) \in \overline{\Omega}$ and for all *z*, *p*, *q*. We'll assume that the partial derivatives of $f_z{}', f_p{}', f_q{}'$ exists and satisfies

$$f_z{}' \ge 0, \tag{1}$$

$$| f_p{}'|, | f_q{}'| \le M < \infty. \tag{2}$$

Let $L[u] \equiv \Delta u - f(x, y, u, u_x, u_y)$. Assume that φ, ψ are the given continuous functions of their domain definitions.

We need to find a function $u(x, y)$ continuous in $\overline{\Omega}$, twice continuously differentiable in Ω, satisfying the equation

$$L[u] = 0 \tag{3}$$

And the boundary conditions

$$u|_\sigma = \varphi, \tag{4}$$

$$l[u] = u(l, y) - \alpha(y) u(a, y) = \psi(y),\ 0 < y < b, \tag{5}$$

$$\alpha(y) \geq 1,\ 0 < y < b, \tag{6}$$

$$l^{(1)}[u] = \left. \left(\frac{\partial u}{\partial x} + \beta(y) \frac{\partial u}{\partial y} + \delta(y) u \right) \right|_{\Gamma^2} = \gamma(y),\ \delta(y) \leq 0. \tag{7}$$

Let $h_1 = \dfrac{a}{N_1},\ h_2 = \dfrac{b}{N_2}$. We construct a grid area with lines $x = x_i,\ y = y_j,\ i = \overline{0, N_1},\ j = \overline{0, N_2}$ and let $x_k < l < x_{k+1}$.

We introduce the denotation

$$\Omega_h = \left\{ (x_i, y_j) : i = \overline{1, N_1 - 1},\ j = \overline{1, N_2 - 1} \right\},$$

$$\Gamma_h^1 = \left\{ (x_i, b) : i = \overline{1, N_1} \right\},\ \Gamma_h^2 = \left\{ (0, y_j) :\ j = \overline{1, N_2 - 1} \right\},$$

$$\Gamma_h^3 = \left\{ (x_i, 0) : i = \overline{1, N_1} \right\},\ \Gamma_h^4 = \left\{ (a, y_j) :\ j = \overline{1, N_2 - 1} \right\},$$

$$\sigma_h = \Gamma_h^1 \cup \Gamma_h^3,\ \Gamma_h = \bigcup_{i=1}^{4} \Gamma_h^i,\ \overline{\Omega}_h = \Omega_h \cup \Gamma_h.$$

We approximate the operators *L* and *l* difference operators L_h, l_h defined as follows:

$$L_h[u_{ij}] \equiv \Delta_h[u_{ij}] - f\left(x_i, y_j, u_{ij}, D_{h_1 x^\alpha}[u_{ij}], D_{h_2 y^\alpha}[u_{ij}]\right), \tag{8}$$

$$l_h[u_{N_1 j}] \equiv \frac{l - x_k}{h_1} u_{k+1 j} + \frac{x_{k+1} - l}{h_1} u_k - \alpha_j u_{N_1 j}, \tag{9}$$

Where

$$\begin{cases} \Delta_h[u_{ij}] = u_{\bar{x}x} + u_{\bar{y}y},\ u_{\bar{x}x} = \dfrac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h_1^2} \\ u_{\bar{y}y} = \dfrac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h_2^2},\ D_{h_1 \overset{0}{x}}[u_{ij}] = \dfrac{u_{i+1j} - u_{i-1j}}{2h_1} \\ D_{h_2 \overset{0}{y}}[u_{ij}] = \dfrac{u_{ij+1} - u_{ij-1}}{2h_2}. \end{cases} \tag{10}$$

We formulate a difference problem corresponding to the stated problem to find a function *U* that is defined in $\overline{\Omega}_h$ such that

$$L_h[U_{ij}] = 0 \text{ in } \Omega_h, \tag{11}$$

$$l_h[U_{N_1 j}] = \psi_j \text{ in } \Gamma_h^4, \tag{12}$$

$$U_{ij} = \varphi_{ij} \text{ in } \sigma_h, \tag{13}$$

$$l_h^{(1)}[U_{0j}] = \frac{U_{ij} - U_{0j}}{h_1} + \beta_j^+ \frac{U_{0j+1} - U_{oj}}{h_2} + \beta_j^- \frac{U_{0j} - U_{oj-1}}{h_2} + \delta_j U_{0j} = \gamma_j \text{ in } \Gamma_h^2, \tag{14}$$

Where

$$\beta_j^+ = \frac{\beta_j + |\beta_j|}{2} \geq 0,\ \beta_j^- = \frac{\beta_j - |\beta_j|}{2} \leq 0.$$

We'll assume that the domain $\overline{\Omega}_h$ is connected and the satisfies inequality

$$M_h < 2\theta, \tag{15}$$

Where

h=max $\{h_1, h_2\}$, $0 < \theta < 1$ – a some fixed number.

Results
Consider the linear difference operator

$$\Lambda_h[U_{ij}] = \begin{cases} \Lambda'_h[U_{ij}] \text{ in } \Omega_h \\ l_h[U_{N_1 j}] \text{ in } \Gamma_h^4 \\ l_h^{(1)}[U_{0j}] \text{ in } \Gamma_h^2 \end{cases} \tag{16}$$

Where

$$\Lambda'_h[U_{ij}] = \Delta_h[U_{ij}] + \xi_{ij} D_{\overset{0}{h_1 x}}[U_{ij}] + \eta_{ij} D_{\overset{0}{h_2 y}}[U_{ij}] - \mu_{ij} U_{ij}, \, |\xi_i|, |\eta_{ij}| \le M, \tag{17}$$

$$\mu_{ij} \ge 0. \tag{18}$$

Due to the standard scheme the following lemma is proved.

Lemma 1. Let $V \ne const$ be a function defined in $\overline{\Omega}_h$, and satisfying $\Lambda_h[V] \ge 0$ ($\Lambda_h[V] \le 0$). Then V it may take the greatest positive (least negative) value only at the nodal points of the σ_h.
Let U be an approximate solution of the problem (11)-(14).

Theorem 1. Let the current solution u of (3)-(7) has limited third derivatives in Ω and second derivatives are continuous in $\overline{\Omega}$. Then the error $\varepsilon_{ij} = u_{ij} - U_{ij}$ of the approximate solution satisfies the equation
$\varepsilon_{ij} = O(h)$.

Proof. On the basis of Taylor's formula, we have

$$\begin{cases} \Lambda'_h[\varepsilon_{ij}] = O(h) \text{ in } \Omega_h \\ l_h[\varepsilon_{N_1 j}] = O(h^2) \text{ in } \Gamma_h^4 \\ \varepsilon_{ij} = 0, \quad \text{ in } \sigma_h \\ l_h^{(1)}[\varepsilon_{0j}] = O(h) \text{ in } \Gamma_h^2. \end{cases} \tag{19}$$

We represent the solution of (19) as

$$\varepsilon_{ij} = \varepsilon_{ij}^1 + \varepsilon_{ij}^2, \tag{20}$$

Where

$$\begin{cases} \Lambda'_h[\varepsilon_{ij}^1] = O(h) \text{ in } \Omega_h \\ \varepsilon_{N_1 j}^1 = 0 \quad \text{ in } \Gamma_h^4 \\ \varepsilon_{ij}^1 = 0, \quad \text{ in } \sigma_h \\ l_h^{(1)}[\varepsilon_{0j}^1] = O(h) \text{ in } \Gamma_h^2. \end{cases} \tag{21}$$

$$\begin{cases} \Lambda'_h[\varepsilon^2_{ij}] = 0 & \text{in } \Omega_h \\ l_h[\varepsilon^2_{N_1 j}] = -l_h[\varepsilon^1_{N_1 j}] O(h^2) & \text{in } \Gamma^4_h \\ \varepsilon^2_{ij} = 0, & \text{in } \sigma_h \\ l^{(1)}_h[\varepsilon^2_{0j}] = 0 & \text{in } \Gamma^2_h. \end{cases} \tag{22}$$

First, we estimate the system (21). Consider the function

$$g(x, y) = \frac{1}{K}\left(e^{\nu_0 a} - e^{\nu_0 x}\right),$$

Where

$$\nu_0 = \frac{M}{\theta}\operatorname{arcth}\left(\frac{3\theta - \theta^2}{2}\right), \quad k = \mu_0 \nu_0, \ \mu_0 = \min\left\{1, \frac{M}{2}(1-\theta)\right\}.$$

It is easy to verify, that

$$\begin{cases} \Lambda'_h[g_{ij}] \le -1 \ \text{in}\, \Omega_h \\ l^{(1)}_h[g_{0j}] \le -1 \ \text{in}\, \Gamma^2_h. \end{cases} \tag{23}$$

On the basis of (21), (23) and Lemma 1 we get that the function

$G^{\pm}_{ij} = c \cdot h \cdot g_{ij} \pm \varepsilon^1_{ij}$ is positive on $\overline{\Omega}$ (for the selected finite constant C).

From this inequality it follows that

$$\max_{\overline{\Omega}_h} |\varepsilon^1_{ij}| \le C_1 h, \quad C_1 = const > 0. \tag{24}$$

Denote by $w = \max_{\Gamma^4_h} |\varepsilon^2_{N_1 j}|$ and let the $\overline{\omega}_{ij}$ – be the solution of

$$\begin{aligned} &\Lambda'_h[\overline{\omega}_{ij}] = 0 \ \text{ in } \Omega_h, \\ &\overline{\omega}_{N_1 j} = w \quad \text{in } \Gamma^4_h, \\ &\overline{\omega}_{ij} = 0 \quad \text{in } \sigma_h, \\ &l^{(1)}_h[\overline{\omega}_{0j}] = 0 \ \text{ in } \Gamma^2_h. \end{aligned}$$

Lemma 1 implies that

$$|\varepsilon^2_{ij}| \le \overline{\omega}_{ij} \ \text{ in } \overline{\Omega}_h, \tag{25}$$

$$\overline{\omega}_{ij} \le \tau_i w, \ 0 < \tau_i < 1 \ \text{in } \Omega_h \tag{26}$$

On the other hand

$$l_h[\varepsilon^2_{N_1 j}] = -l_h[\varepsilon^1_{N_1 j}] + O(h^2) \ \text{in } \Gamma^4_h.$$

Hence, respectively to (25), (26) we have

$$\alpha_j |\varepsilon^2_{N_1 j}| \le \frac{l-x_k}{h_1}|\varepsilon^2_{k+1j}| + \frac{x_{k+1}-l}{h_1}|\varepsilon^2_{kj}| + \frac{l-x_k}{h_1}|\varepsilon^1_{k+1j}| + \frac{x_{k+1}-l}{h_1}|\varepsilon^1_{k+1j}| + C_2 h^2$$

Or

$$\alpha_j w \le \tau w + C_1 h + C_2 h,$$

Where

$$\tau = \max\{\tau_{k+1}, \tau_k\}.$$

Hence we have

$$w \le \frac{C_3 h}{\alpha_j - \kappa_i} \le C_4 h, \tag{27}$$

Where

$$C_4 = \frac{C_3}{\min\limits_j(\alpha_j - \tau)}.$$

Then from (25)-(27) we have

$$\max_{\overline{\Omega}_h} |\varepsilon^2_{ij}| \le C_5 h, \quad C_5 = \max_i \tau_i C_4. \tag{28}$$

Based on (20), (24) and (28) we have

$$\max_{\overline{\Omega}_h} |\varepsilon_{ij}| \le C_6 h, \tag{29}$$

Where

$$C_6 = C_1 + C_5.$$

Theorem 1 is proved.
Below we show that by imposing additional conditions on the function $\beta(y)$, $\delta(y)$ the order of accuracy with in h_2 can be improved.

As can be seen from the above, it is sufficient to increase the order of approximation of the operator $l_h^{(1)}$.
Assume, that $h_1 = wh^2$ $(0 < w \le 1)$ and $\beta(y)$, $\delta(y)$ satisfy one of the following conditions

$$|\beta(y)| < w, \tag{30}$$

$$|\beta(y)| \ge w, \ \delta'(y) \le 0, \tag{31}$$

$$|\beta(y)| \le -w, \ \delta'(y) \ge 0. \tag{32}$$

Consider the operators

$$l_{1h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j+1} U_{0j-1}}{2h_2} + \delta_j U_{0j}, \tag{33}$$

$$l_{2h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j+1} - U_{0j}}{h_2} + \delta_j U_{0j}, \tag{34}$$

$$l_{3h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j} - U_{0j-1}}{h_2} + \delta_j U_{0j}. \quad (35)$$

Let

$$\left|\frac{\partial^p u_{0j}}{\partial x^p}\right|_{(0,j)}, \quad \left|\frac{\partial^p u_{0j}}{\partial y^p}\right|_{(0,j)} \le M_j^{(p)}, \ (p \ge 1).$$

Taking into account (3), (7), (33) and applying the Taylor formula is easy to see that

$$\left|\bar{l}_{1h}^{(1)} u_{0j} - \left(l^{(1)} u\right)_{(0,j)}\right| \le c^{(1)} h_2^2, \quad (36)$$

Where

$$\bar{l}_{1h}^{(1)} u_{0j} \cong l_{1h}^{(1)} u_{0j} + \frac{h_1\left(u_{0j+1} - 2u_{0j} + u_{0j-1}\right)}{2h_2^2} - \frac{h_1}{2} f\left\{0, y_j, u_{0j}, D_{h_1 x}[u_{0j}], D_{h_2 y}[u_{0j}]\right\},$$

$$D_{h_1 x}[u_{ij}] = \frac{u_{i+1j} - u_{ij}}{h_1}, \ D_{h_2 y}[u_{ij}] = \frac{u_{i+1j} - u_{ij}}{h_2}, \ C^{(1)} = \max_j \left\{\frac{2\left(w^2 + w + \beta\right) + h_1 M}{12} M_j^{(3)} + \frac{w^2 M}{4} M_j^{(2)}\right\}.$$

Indeed from (33) we have:

$$l_{1h}^{(1)} u_{0j} = \left(l^{(1)} u\right)_{(0,j)} + \frac{h_1}{2}\frac{\partial^2 u}{\partial x^2}\Big|_{(0,j)} + R_j^{(1)}, \ R_j^{(1)} = \frac{h_1^2}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{\left(\xi_0^{(1)}, j\right)} + \frac{h_2^2}{12}\left[\frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(1)}\right)} + \frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(2)}\right)}\right]\beta_j.$$

From (3) we have:

$$\frac{\partial^2 u}{\partial x^2}\Big|_{(0,j)} = -\frac{u_{0j+1} - 2u_{0j} + u_{0j-1}}{h_2^2} + f\left(0, y_j, u_{0j}, D_{h_1 x}[u_{0j}], D_{h_2 \overset{0}{y}}[u_{0j}]\right) -$$

$$-\frac{h_2}{6}\left[\frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(3)}\right)} - \frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(4)}\right)}\right] + f_p'\left(0, y_j, u_{0j}, p_j, q_j\right)\frac{\partial^2 u}{\partial x^2}\Big|_{\left(\xi_0^{(2)}, j\right)}\frac{h_1}{2} +$$

$$+\frac{h_2^2}{12}\left[\frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(3)}\right)} + \frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(4)}\right)}\right] f'_q\left(0, y_j, u_{0j}, p_j q_j\right).$$

Taking into account this $l_{1h}^{(1)} u_{ij}$, we get:

$$l_{1h}^{(1)} u_{0j} = \left(l^{(1)} u\right)_{(0,j)} - \frac{h_1\left(u_{0j+1} - 2u_{0j} + u_{0j-1}\right)}{2h_2^2} + \frac{h_1}{2} f'\left(0, y_j, u_{0j}, D_{h_1 x}[u_{0j}], D_{h_2 \overset{0}{y}}[u_{0j}]\right) + \bar{R}_j^{(1)},$$

Where

$$\bar{R}_j^{(1)} = \frac{h_1^2}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{\left(\xi_i^{(1)}, j\right)} + \frac{h_2^2}{12}\left[\frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(1)}\right)} + \frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(2)}\right)}\right]\beta_j - \frac{h_1 h_2}{12}\left[\frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(3)}\right)} - \frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(4)}\right)}\right] +$$

$$+\frac{h_1^2}{4} f_p'\left(0, y_j, u_{0j}, p_j, q_j\right)\frac{\partial^2 u}{\partial x^2}\Big|_{\left(\xi_0^{(2)}, j\right)} + \frac{h_1 h_2^2}{24} f_q'\left(0, y_j, u_{0j}, p_j, q_j\right)\left[\frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(3)}\right)} + \frac{\partial^3 u}{\partial y^3}\Big|_{\left(0, \eta_j^{(4)}\right)}\right].$$

Hence we find that

$\bar{l}_{1h}^{(1)}u_{0j}=\left(l^{(1)}u\right)_{(0,j)}+\overline{R}_j^{(1)}$, consequently $\left|\bar{l}_{1h}^{(1)}u_{0j}-\left(l^{(1)}u\right)_{(0,j)}\right|\le\left|\overline{R}_j^{(1)}\right|$

And this implies (36).

Now we prove that

$$\left|\bar{l}_{2h}^{(1)}u_{0j}-\left(l^{(1)}u\right)_{(0,j)}\right|\le C^{(2)}h_2^2, \qquad (37)$$

Where

$$\bar{l}_{2h}^{(1)}u_{0j}\equiv l_{2h}^{(1)}u_{0j}+\frac{\beta_j h_2-h_1}{2\beta_j}D_{h_1h_2xy}[u_{0j}]+\frac{\delta_j}{\beta_j}\left(\beta_j h_2-h_1\right)D_{h_2y}[u_{0j}]+\frac{\delta'_j}{2\beta_j}\left(\beta_j h_2-h_1\right)u_{0j}-$$

$$-\frac{\gamma'_j}{2\beta_j}\left(\beta_j h_2-h_1\right)-\frac{h_1}{2}f\left(0,y_j,u_{0j},D_{h_1x}[u_{0j}],D_{h_2y}[u_{0j}]\right),$$

$$C^{(2)}=\max_j\left\{\left[\frac{\beta_j+w^2}{6}+\frac{\left(\beta_j-w\right)\left(1-w\right)}{4\beta_j}+\frac{h_1M}{12}\right]M_i^{(3)}+\left[\frac{|\,\delta_j+\beta'_j\,|\left(\beta_j-w\right)}{4\beta_j}+\frac{w^2M}{4}\right]M_j^{(2)}\right\},$$

$$D_{h_1h_2xy}[u_{0j}]=D_{h_1x}\left\{D_{h_2y}[u_{0j}]\right\}.$$

Suppose that $\beta(y)\neq 0$. Then from (7) we have:

$$\frac{\partial^2u(0,u)}{\partial y^2}=-\frac{1}{\beta(y)}\frac{\partial^2u(0,y)}{\partial x\partial y}-\frac{\delta'(y)}{\beta(y)}u(0,y)-\frac{\delta(y)+\beta'_j}{\beta(y)}\frac{\partial u(0,y)}{\partial y}+\frac{\gamma'(y)}{\beta(y)}. \qquad (38)$$

Obviously

$$l_{2h}^{(1)}u_{0j}=\left(l_3u\right)_{(0,j)}+\frac{h_1}{2}\frac{\partial^2u}{\partial x^2}\Big|_{(0,j)}+\frac{h_2}{2}\beta_j\frac{\partial^2u}{\partial y^2}\Big|_{(0,j)}+R_j^{(2)},$$

Where

$$R_j^{(2)}=\frac{h_1^2}{6}\frac{\partial^3u}{\partial x^3}\Big|_{\left(\xi_0^{(1)},j\right)}+\beta_j\frac{h_2^2}{6}\frac{\partial^3u}{\partial x^3}\Big|_{\left(0,\eta_0^{(1)}\right)}.$$

From (3) we get:

$$\frac{\partial^2u}{\partial x^2}\Big|_{(0,j)}=-\frac{\partial^2u}{\partial y^2}\Big|_{(0,j)}+f\left(0,y_j,u_{0j},\frac{\partial u}{\partial x}\Big|_{(0,j)},\frac{\partial u}{\partial y}\Big|_{(0,j)}\right).$$

Then

$$l_{2h}^{(1)}u_{0j}=\left(l^{(1)}u\right)_{(0,j)}+\frac{1}{2}\left(\beta_j h_2-h_1\right)\frac{\partial^2u}{\partial y^2}\Big|_{(0,j)}+\frac{h_1}{2}f\left(0,y_j,u_{0j},\frac{\partial u}{\partial x}\Big|_{(0,j)},\frac{\partial u}{\partial y}\Big|_{(0,j)}\right)+R_j^{(2)}.$$

Taking into account (38)

$$l_{2h}^{(1)}u_{0j}=\left(l^{(1)}u\right)_{(0,j)}+\frac{1}{2}\left(\beta_j h_2-h_1\right)\left[-\frac{1}{\beta_j}\frac{\partial^2u}{\partial x\partial y}\Big|_{(0,j)}-\frac{\delta'_j}{\beta_j}u_{0j}-\frac{\delta_j+\beta'_j}{\beta_j}\frac{\partial u}{\partial y}\Big|_{(0,j)}+\frac{\gamma_j}{\beta_j}\right]+$$

$$+\frac{h_1}{2}f\left(0,y_j,u_{0j},\frac{\partial u}{\partial x}\Big|_{(0,j)},\frac{\partial^2 u}{\partial y}\Big|_{(0,j)}\right)+R_j^{(2)}=\left(l^{(1)}u\right)_{(0,j)}-\frac{1}{2\beta_j}\left(\beta_j h_2-h_1\right)\frac{\partial^2 u}{\partial x\partial y}\Big|_{(0,j)}-$$

$$-\frac{\delta_j+\beta_j}{2\beta_j}\left(\beta_j h_2-h_1\right)\frac{\partial u}{\partial y}\Big|_{(0,j)}-\frac{\delta_j'}{2\beta_j}\left(\beta_j h_2-h_1\right)u_{0j}+\frac{\gamma_j'}{2\beta_j}\left(\beta_j h_2-h_1\right)+$$

$$+\frac{h_1}{2}f\left(0,y_j,u_j,\frac{\partial u}{\partial x}\Big|_{(0,j)},\frac{\partial u}{\partial y}\Big|_{(0,j)}\right)+R_j^{(2)}=\left(l^{(1)}u\right)_{(0,j)}-\frac{\beta_j h_2-h_1}{2\beta_j}D_{h_1h_2xy}[u_{0j}]-$$

$$-\frac{\delta_j+\beta_j'}{\beta_j}\left(\beta_j h_2-h_1\right)D_{h_2y}[u_{0j}]-\frac{\delta_j'}{2\beta_j}\left(\beta_j h_2-h_1\right)u_{0j}+\frac{\gamma_j'}{2\beta_j}\left(\beta_j h_2-h_1\right)+$$

$$+\frac{h_1}{2}f\left(,y_j,u_{0j},D_{h_1x}[u_{0j}],D_{h_2y}[u_{0j}]\right)+\overline{R}_j^{(2)},$$

Where

$$\overline{R}_j^{(2)}=R_j^{(2)}-\frac{\beta_j h_2-h_1}{4\beta_j}\left[\frac{\partial^3 u}{\partial x^2\partial y}h_1-\frac{\partial^3 u}{\partial x\partial y^2}h_2\right]-\frac{\delta_j+\beta_j'}{4\beta_j}\left(\beta_j h_2-h_1\right)h_2\frac{\partial^2 u}{\partial y^2}\Big|_{(0,\eta_j^{(2)})}+$$

$$+\frac{h_1^2}{4}f_p'\left(0,y_j,u_{0j},p_j,q_j\right)\frac{\partial^2 u}{\partial x^2}\Big|_{(\xi_0^{(2)},j)}+\frac{h_1h_2^2}{24}f_q'\left(0,y_j,u_{0j},p_j,q_j\right)\left[\frac{\partial^3 u}{\partial y^3}\Big|_{(0,\eta_j^{(2)})}+\frac{\partial^3 u}{\partial y^3}\Big|_{(0,\eta_j^{(3)})}\right].$$

Then

$$\left|\bar{l}_{2h}^{(1)}u_{0j}-\left(l_3u\right)_{(0,j)}\right|\le\left|\overline{R}_j^{(2)}\right|.$$

This implies (37).

Finally we prove that

$$\left|\bar{l}_{3h}^{(1)}u_{0j}-\left(l_3u\right)_{(0,j)}\right|\le C^{(2)}h_2^2, \tag{39}$$

Where

$$C^{(3)}=\left(\frac{|w^2+\beta|}{6}+\frac{|w+\beta|\left(w+1\right)}{2|\beta|}+\frac{h_1M}{12}\right)M_3+\left(\frac{|w+\beta\|\delta+\beta'|}{4|\beta|}+\frac{w^2M}{4}\right)M_2,$$

$$\bar{l}_{3h}^{(1)}u_{0j}\equiv l_{3h}^{(1)}u_{0j}-\frac{\beta_j h_2-h_1}{2\beta_j}D_{h_1h_2x\bar{y}}[u_{0j}]-\frac{\delta_j}{\beta_j}\left(\beta_j h_2+h_1\right)D_{h_2\bar{y}}[u_{0j}]-\frac{\delta_j'}{2\beta_j}\left(\beta_j h_2+h_1\right)u_{0j}+$$

$$+\frac{\gamma_j'}{2\beta_j}\left(\beta_j h_2-h_1\right)+\frac{h_1}{2}f\left(0,y_j,u_{0j},D_{h_1x}[u_{0j}],D_{h_2y}[u_{0j}]\right),\quad D_{h_1h_2xy}[u_{0j}]=D_{h_1x}\left\{D_{h_2\bar{y}}[u_{0j}]\right\}.$$

Indeed

$$l_{3h}^{(1)}u_{0j}=\left(l^{(1)}u\right)_{(0,j)}-\frac{h_1+h_2\beta_j}{2}\frac{\partial^2 u}{\partial y^2}\Big|_{(0,j)}-\frac{h_1}{2}f\left(0,y_j,u_{0j},\frac{\partial u}{\partial x}\Big|_{(0,j)},\frac{\partial u}{\partial y}\Big|_{(0,j)}\right)+R_j^{(3)},$$

$$R_j^{(3)} = \frac{h_1^2}{6}\frac{\partial^3 u}{\partial x^3}\Big|_{(\xi_0^{(1)},j)} + \beta_j \frac{h_2^2}{6}\frac{\partial^3 u}{\partial y^3}\Big|_{(0,\eta_j^{(1)})}.$$

Taking into account (38)

$$l_{3h}^{(1)}u_{0j} = \left(l^{(1)}u\right)_{(0,j)} + \frac{h_1 + h_2\beta_j}{2\beta_j}\frac{\partial^2 u}{\partial x\partial y} + \frac{h_1 + h_2\beta_j}{2}\frac{\delta'_j}{\beta_j}u_{0j} + \frac{h_1 + h_2\beta_j}{2}\cdot\frac{\delta_j + \beta'_j}{\beta_j}\frac{\partial u}{\partial y}\Big|_{(0,j)} -$$

$$-\frac{h_1 + h_2\beta_j}{2}\cdot\frac{\gamma'_j}{\beta_j} - \frac{h_1}{2} f\left(0, y_j, u_{0j}, \frac{\partial u}{\partial x}\Big|_{(0,j)}, \frac{\partial u}{\partial y}\Big|_{(0,j)}\right) + R_j^{(3)},$$

$$\bar{l}_{3h}^{(1)}u_{0j} \equiv l_{3h}^{(1)}u_{0j} + \frac{\beta_j h_2 + h_1}{2\beta_j}D_{h_1h_2x\bar{y}}[u_{0j}] + \frac{\left(\delta_j + \beta'\right)}{2\beta_j}\left(\beta_j h_2 + h_1\right)D_{h_2\bar{y}}[u_{0j}] + \frac{\delta'_j}{2\beta_j}\left(\beta_j h_2 + h_1\right)u_{0j} -$$

$$-\frac{\gamma'_j}{2\beta_j}\left(\beta_j h_2 + h_1\right) - \frac{h_1}{2} f\left(0, y_j, u_{0j}, D_{h_1x}[u_{0j}], D_{h_2y}[u_{0j}]\right) + \bar{R}_j^{(3)},$$

Where

$$\bar{R}_j^{(3)} = R_j^{(3)} + \frac{h_1 + h_2\beta_j}{2\beta_j}\left[\frac{\partial^3 u}{\partial x^2\partial y}h_1 + \frac{\partial^3 u}{\partial x\partial y^2}h_2\right] + \frac{\left(h_1 + h_2\beta_j\right)\left(\delta_j + \beta'_j\right)}{2\beta_j}h^2\frac{\partial^2 u}{\partial y^2}\Big|_{(0,\eta_j^{(2)})} -$$

$$-\frac{h_1^2}{4} f'_p\left(0, y_j, u_{0j}, p_j, q_j\right)\frac{\partial^2 u}{\partial x^2}\Big|_{(\xi_0^{(2)},j)} - \frac{h_1h_2^2}{24} f'_q\left(0, y_j, u_{0j}, p_j, q_j\right)\left[\frac{\partial^3 u}{\partial y^3}\Big|_{(0,\eta_j^{(3)})} + \frac{\partial^3 u}{\partial y^3}\Big|_{(0,\eta_j^{(4)})}\right].$$

Consequently,

$$\left|\bar{l}_{3h}^{(1)u_{0j}} - \left(l^{(1)}u\right)_{(0,j)}\right| \le \left|\bar{R}_j^{(3)}\right|,$$

Which was required to prove.
We now state the difference problem corresponding to the problem (3)-(7).
It is required to find a discrete function $U_{ij}^{(k)}$ $(k = 1,2,3)$ determined in $\overline{\Omega}_h$ satisfying the properties (11) - (13), and one of the following conditions

$$\bar{l}_{kh}^{(1)}U_{0j} = \gamma_j \quad \left(j = \overline{1, N_2 - 1},\ k = 1,2,3\right) \tag{40}$$

Respectively, when one of the conditions (30), (31) and (32) is satisfied. The solution of the difference scheme (11) - (13), with one of the conditions (40) will be taken as an approximate solution of the problem (3) - (7) at the points $\overline{\Omega}_h$. Consider the following linear difference operators:

$$\Lambda_h^{(k)}[U_{ij}] = \begin{cases} \bar{L}_h[U_{ij}], \\ l_h[U_{N_1 j}], \\ \bar{l}_{kh}^{(1)}[U_{0j}],\ (k = 1,2,3) \end{cases}$$

Where

$$\bar{L}_h[U_{ij}] \equiv \Delta_h[U_{ij}] + \xi_{ij}D_{h_1\overset{0}{x}}[U_{ij}] + \eta_{ij}D_{h_2\overset{0}{y}}[U_{ij}] - \mu_{ij}U_{ij},$$

$$\bar{l}_{1h}^{(1)}[U_{0j}] \equiv l_{1h}^{(1)}[U_{0j}] + \frac{h_1\left(U_{0j+1} + 2U_{0j} + U_{0j-1}\right)}{2h_2^2} - \frac{h_1}{2}\left[\xi_{0j} D_{h_1 x}[U_{0j}] + \eta_{0j} D_{\overset{0}{h_2 y}}[U_{0j}] - \mu_{0j} U_{0j}\right],$$

$$\bar{l}_{2h}^{(1)}[U_{0j}] \equiv l_{2h}^{(1)}[U_{0j}] + \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[U_{0j}] + \frac{\delta_j}{\beta_j}\left(\beta_j h_2 - h_1\right) D_{h_2 y}[U_{0j}] + \frac{\delta'_j}{2\beta_j}\left(\beta_j h_2 - h_1\right) U_{0j} -$$

$$-\frac{h_1}{2}\left[\xi_{0j} D_{h_1 x}[U_{0j}] + \eta_{0j} D_{h_2 y}[U_{0j}] - \mu_{0j} U_{0j}\right],$$

$$\bar{l}_{3h}^{(1)}[U_{0j}] \equiv l_{3h}^{(1)}[U_{0j}] - \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[U_{0j}] - \frac{\delta_j}{\beta_j}\left(\beta_j h_2 + h_1\right) D_{h_2 y}[U_{0j}] + \frac{\delta'_j}{2\beta_j}\left(\beta_j h_2 + h_1\right) U_{0j} +$$

$$+\frac{h_1}{2}\left[\xi_{0j} D_{h_1 x}[U_{0j}] + \eta_{0j} D_{h_2 y}[U_{0j}] - \mu_{0j} U_{0j}\right].$$

We assume that if (30) is satisfied, then

$$Mh_2 < 2\left(1 - \sup|\beta(x)|\right), \tag{41}$$

and if the (31), (32) are satisfied, then

$$\overline{M} h_2 < 1, \tag{42}$$

Where

$$\overline{M} = \max\left\{\frac{M}{1 + \left(\sup|\beta|\right)^{-1}}, \frac{M + \sup\left(\dfrac{|\beta|+1}{|\beta|}|\beta' + \delta|\right)}{\inf|\beta| + \left(\sup|\beta|\right)^{-1}}\right\}.$$

Lemma 2. Let *V*≠ *const* be a function defined in $\overline{\Omega}_h$, that satisfies the inequality $\Lambda_h^{(k)}[V_{ij}] \ge 0$ $\left(\Lambda_h^{(k)}[V_{ij}] \le 0\right)$ $k = 1,2,3$. Then *V* may take the greatest positive (least negative) value only at the points σ_h.

Proof. It's obvious that

$$\bar{l}_{1h}^{(1)}[U_{ij}] \equiv A_{1j}^{(1)} U_{ij} + A_{2j}^{(1)} U_{0j-1} + A_{3j}^{(1)} U_{0j+1} - A_{0j}^{(1)} U_{0j},$$

$$\bar{l}_{2h}^{(1)}[U_{ij}] \equiv A_{1j}^{(2)} U_{ij} + A_{2j}^{(2)} U_{0j-1} + A_{3j}^{(2)} U_{0j+1} - A_{0j}^{(2)} U_{0j},$$

$$\bar{l}_{3h}^{(1)}[U_{ij}] \equiv A_{1j}^{(3)} U_{ij} + A_{2j}^{(3)} U_{0j-1} + A_{3j}^{(3)} U_{0j+1} - A_{0j}^{(3)} U_{0j},$$

Where

$$A_{0j}^{(1)} = \frac{h_1}{h_2^2} + \frac{1}{h_1} - \delta_j + \frac{\xi_j}{2} + \frac{h_1}{2}\mu_j, \quad A_{1j}^{(1)} = \frac{1}{h_1}\left(1 - \frac{h_1}{2}\xi_j\right),$$

$$A_{2j}^{(1)} = \frac{1}{2h_2}\left(\frac{h_1}{h_2} - \beta_j - \frac{h_1}{2}\eta_j\right), \quad A_{3j}^{(1)} = \frac{1}{2h_2}\left(\frac{h_1}{h_2} + \beta_j + \frac{h_1}{2}\eta_j\right),$$

$$A_{0j}^{(2)} = \beta_j\left(\frac{1}{h_1} + \frac{1}{h_2}\right) - \delta_j - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2\beta_j}\left(\beta_j h_2 - h_1\right) - \frac{\delta'_j}{2\beta_j}\left(\beta_j h_2 - h_1\right) - \frac{\xi_j}{2} - \frac{h_1\eta_j}{2h_2} - \frac{h_1}{2}\mu_j,$$

$$A_{1j}^{(2)} = \frac{\beta_j}{h_1} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} - \frac{\xi_j}{2},\quad A_{2j}^{(2)} = \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1},\quad A_{3j}^{(2)} = \frac{\beta_j}{h_2} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2\beta_j}\left(\beta_j h_2 - h_1\right) - \frac{h_1\eta_j}{2h_2},$$

$$A_{0j}^{(3)} = \frac{1}{h_1} - \frac{\beta_j}{h_2} - \delta_j - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2\beta_j}\left(\beta_j h_2 + h_1\right) + \frac{\delta'_j}{2\beta_j}\left(\beta_j h_2 + h_1\right) + \frac{\xi_j}{2} - \frac{h_1\eta_j}{2h_2} + \frac{h_1}{2}\mu_j$$

$$A_{1j}^{(3)} = \frac{1}{h_1} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\xi_j}{2},\quad A_{2j}^{(3)} = -\frac{\beta_j}{h_2} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2\beta_j}\left(\beta_j h_2 + h_1\right) - \frac{h_1\eta_j}{2h_2},\quad A_{3j}^{(3)} = \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1}.$$

All these coefficients are positive and satisfy the following conditions:

$$A_{0j}^{(1)} - A_{1j}^{(1)} - A_{2j}^{(1)} - A_{3j}^{(1)} = -\delta_j + \frac{h_1}{2}\mu_j \ge 0,$$

$$A_{0j}^{(2)} - A_{1j}^{(2)} - A_{2j}^{(2)} - A_{3j}^{(2)} = -\delta_j - \frac{h_1}{2}\mu_j - \frac{\delta'_j}{2\beta_j}\left(\beta_j h_2 - h_1\right) \ge 0,$$

$$A_{0j}^{(3)} - A_{1j}^{(3)} - A_{2j}^{(3)} - A_{3j}^{(3)} = -\delta_j + \frac{\delta'_j}{2\beta_j}\left(\beta_j h_2 + h_1\right) + \frac{h_1}{2}\mu_j \ge 0.$$

Taking into account these properties of the coefficients, applying Lemma 1 we obtain Lemma 2.

Corollary. Lemma 2 implies that the solution of (11)-(13) (40) is unique.

Theorem 2. Let u the exact solution of the problem (3)-(7) limited the fourth derivatives and continued in the third derivative $\overline{\Omega}$. Then the error $\varepsilon_{ij} = u_{ij} - U_{ij}$, where U_{ij} - the approximate solution of (11)-(13), (40), the estimate $\varepsilon = O(h_2)$.

Proof. With the help of Taylor's formula for the error $\varepsilon_{ij} = u_{ij} - U_{ij}$ we have:

$$\begin{cases} \overline{L}_h[\varepsilon_{ij}] = O\left(h^2\right) & \text{in } \Omega_h \\ l_h[\varepsilon_{N_1 j}] = O\left(h^2\right) & \text{in } \Gamma_h^4 \\ \varepsilon_{ij} = 0, & \text{in } \sigma_h \\ \bar{l}_{kh}^{(1)}[\varepsilon_{0j}] = O\left(h^2\right), & \text{in } \Gamma_h^2,\ k = 1,2,3. \end{cases} \tag{43}$$

As in the proof of Theorem 1, we represent the solution of the system (43) of the form $\varepsilon_{ij} = \varepsilon_{ij}^1 + \varepsilon_{ij}^2$,

Where

$$\begin{cases} \overline{L}_h[\varepsilon_{ij}^1] = O\left(h^2\right) & \text{in } \Omega_h \\ l_h[\varepsilon_{N_1 j}^1] = 0 & \text{in } \Gamma_h^4 \\ \varepsilon_{ij}^1 = 0, & \text{in } \sigma_h \\ \bar{l}_{kh}^{(1)}[\varepsilon_{0j}^1] = O\left(h^2\right), & \text{in } \Gamma_h^2,\ k = 1,2,3. \end{cases} \tag{44}$$

$$\begin{cases} \overline{L}_h[\varepsilon_{ij}^2] = 0 & \text{in } \Omega_h \\ l_h[\varepsilon_{N_1 j}^2] = -l_h[\varepsilon_{N_1 j}^1] + O\left(h^2\right) & \text{in } \Gamma_h^4 \\ \varepsilon_{ij}^2 = 0, & \text{in } \sigma_h \\ \bar{l}_{kh}^{(1)}[\varepsilon_{0j}^2] = 0, & \text{in } \Gamma_h^2,\ k = 1,2,3. \end{cases} \tag{45}$$

An estimate of $\max_{\overline{\Omega}_h} |\varepsilon_h^1| \le c_7 h^2$ for the solutions system of (44) is obtained on the basis of Lemma 2, due to scheme of proof of Theorem 1 by the majorant function $g(x,y) = \dfrac{e^{\nu_0 a} - e^{\nu_0 x}}{k}$ and the parameters k and ν_0 are selected as follows:

$$k = \mu_0 \nu_0, \;\; \mu_0 = \min\{\alpha^0, M\beta^0\}, \;\; \alpha^0 = \begin{cases} \sup|\beta| & \text{if } |\beta| < 1 \\ \dfrac{1-\theta}{2} & \text{if } |\beta| \ge 1 \end{cases} \quad \beta^0 = \begin{cases} \sup|\beta| & \text{if } |\beta| < 1 \\ 1-\theta & \text{if } |\beta| \ge 1 \end{cases}$$

$$\nu_0 = \frac{2M}{\overline{\delta}} \operatorname{arcth}\left(\frac{2\overline{\delta} - \overline{\delta}^2}{2} \right), \qquad \overline{\delta} = \begin{cases} 1 - \sup|\beta| & \text{if } |\beta| < 1 \\ \theta & \text{if } |\beta| \ge 1. \end{cases}$$

An estimate of $\max_{\overline{\Omega}_h} |\varepsilon_h^2| \le c_8 h^2$ for the solutions of the system (45) is obtained by the same way as the estimate of the solution of system (22) in the proof of Theorem 1.

Theorem 2 is proved.

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