E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2024; 5(2): 97-100

www.mathematicaljournal.com
Received: 05-07-2024
Accepted: 11-08-2024

M Geetha
Head and Assistant Professor, Department of Mathematics, Saradha Gangadharan College, Pondicherry, India

Corresponding Author:
M Geetha
Head and Assistant Professor, Department of Mathematics, Saradha Gangadharan College, Pondicherry, India

Characterization of symmetric stable processes with finite mean

M Geetha

Abstract
This paper demonstrates the characterization of symmetric stable processes with finite mean, focusing on how symmetric stable processes can be identified when the mean is finite. Various properties related to stochastic processes and integrals are discussed.

Keywords: Symmetric stable processes and finite mean

Introduction
Laha studied the characterization of symmetric stable laws through regression properties. This study characterize symmetric stable process with finite mean. Some results concurring stochastic processes and stochastic integrals are discussed.

Theorem 1
Let X and Y be two independent random variables whose expectations exist and are zero.
Suppose $U = aX + bY, V = cX + dY$, where $a, b, c, d \neq 0$ and $ad \neq$. Then both X and Y have symmetric stable distribution with the same exponent $\lambda > 1$ if and only if
There exists a constant $\delta > 0$ such that the relation $(V/U) = \beta U$ a. e holds for all a such that $0 < |a| < \delta$ and $\beta = (ca^{-1}\alpha_1 |a|^\lambda + db^{-1}\alpha_2 |b|^\lambda)(\alpha_1 |a|^\lambda + \alpha_2|b|^\lambda)^{-1}$
where α_1 and α_2 are the scale parameters of the distribution of X and Y respectively.

Definition 1
Homogeneous Process
A stochastic process $((t), t \in \mathrm{T}), T = [0,1]$ is said to be a homogeneous
Process with independent increments if the distribution of the increments $(t + \mathrm{h}) - (t)$ depends only on h but is independent of t and if the increments over non overlapping intervals are independent

Definition 2
Symmetric stable process
Let $\{(t), t \in T\}$ be a continuous homogeneous process with independent increments. Let $(u\colon \mathrm{h})$ denote the characteristic function of $(t + \mathrm{h}) - (t)$, $X(0) = 0$, $\phi(u\colon \mathrm{h})$ is infinitely divisible, $\phi(u\colon \mathrm{h}) = [\phi(u\colon 1)]^{\mathrm{h}}$
$\int^1 \mathrm{a}(t)X(t)$ is defined in the sense of convergence in probability. The process is said to be a symmetric stable process with exponent a if the increments of the process have symmetric stable laws with exponent α.

Theorem 2
Let $\{(t), t \in T\}$ be a homogeneous process with independent increments.
Suppose that
$X(0) = 0$
$E\ [X(t)] = 0$ for all t and
The increments of the process have non degenerate symmetric distributions. Let

$$Y_\lambda = \int_0^1 t^\lambda \, \mathrm{d}X(t), for\ any\ \lambda > 0 \ (1)$$

Then the process is a symmetric stable process with exponent α > 1 if and only if for some positive numbers λ and, $\lambda, u \neq \mu$, $[Y_\lambda / Y_\mu] = \beta Y_\mu$ (2) for some constant β depending on λ, μ. Furthermore α, λ, μ and β are connected by the relation

$$(\mu\alpha + 1) = (\lambda - \mu + \mu\alpha + 1)$$

Lemma 2

Let U and V be two random variables with $(U) = (V) = 0$. Let (u, v) denote the characteristic function of (U, V). Then $(U/V) = \beta u\ a.e$ if and only if.

$$\frac{\partial h(u,v)}{\partial v}\Big]_{v=0} = \beta \frac{dh(u,0)}{du}$$

for all real u.

Proof

Let $h(u, v)$ is the characteristic function of (U, V)

$$h(u, v) = E(e^{iux+ivy})$$

$$= \int e^{iux+ivy}\ dF(x,y)$$

$$= \int e^{iux}\ [\int e^{ivy}\ dF_x(y)]\, dF(x)$$

where $F_x(y)$ represents the conditional distribution function of y for fixed x.

The regression of y on x is linear,

$$\frac{\partial h(u,v)}{\partial v}\Big|_{v=0} = i \int e^{iux}\ [\int y\ dF_x(y)]\, dF(x)$$

$$= i\beta \int e^{iux}\ x\, dF(x)$$

$$= \beta \frac{\partial h(u,0)}{\partial u}$$

Since the regression of y on x is assumed to exist

$$E_x(y) = \int y\ dF_x(y)$$

$$\frac{\partial h(u,v)}{\partial v}\Big|_{v=0} = i \int e^{iux}\ E_x(y)]dF(x)$$

$$\frac{\partial h(u,v)}{\partial v}\Big]_{v=0} = \beta \frac{dh(u,0)}{du}$$

Lemma 2

Let $\{(t),\ t \in T\}$ be a homogeneous process with independent increments.

Let $Y = \int_0^1 a(t)dX(t)$, $Z = \int_0^1 b(t)dX(t)$, for any two infinitely differentiable functions $a(t)$ and $b(t)$ on [0,1]. Let $(u{:}\ h)$, (u, v) denote the characteristic functions of $X(t + h) - X(t)$ and (Y, Z) respectively. Then (u, v) is different from zero for all real U and V and

$$\log\ \theta(u, v) = \int_0^1 \psi[u\, a(t) + v\, b(t)]dt$$

where $\psi(u) = \log\ \theta(u, 1)$.

Proof of theorem 2: Let $\{(t),\ t \in T\}$ be a symmetric stable process with $X(0) = 0$ and $[X(t)] = 0$ for all t. Let (u, v) denote the log of the characteristic function of $(Y_\lambda, Y) \cdot \theta(u, v)$ is well defined since the process $\{X(t),\ t \in T\}$ is infinitely divisible. Let (u) denote the logarithm of the characteristic function of

Let h(u, v) is the characteristic function of (U, V)

$$h(u, v) = E(e^{iux+ivy})$$

$$= \int e^{iux+ivy}\ dF(x, y)$$

$$= \int e^{iux}\ [\int e^{ivy}\ dF_x(y)]dF(x)$$

Where $F_x(y)$ represents the conditional distribution function of y for fixed x. The regression of y on x is linear,

$$\frac{\partial h(u,v)}{\partial v}\Big|_{v=0} = i \int e^{iux}\ [\int y\ dF_x(y)]\, dF(x)$$

$$= i\beta \int e^{iux}\ x\, dF(x)$$

$$= \beta \frac{\partial h(u,0)}{\partial u}$$

Since the regression of y on x is assumed to exist

$$E_x(y) = \int y\ dF_x(y)$$

$$\frac{\partial h(u,v)}{\partial v}\Big|_{v=0} = i \int e^{iux}\ E_x(y)]dF(x)$$

$$\frac{\partial h(u,v)}{\partial v}\Big]_{v=0} = \beta \frac{dh(u,0)}{du}$$

where β satisfies (3).

$\Longrightarrow E[Y_\lambda/Y_\mu] = \beta Y_\mu\ a.e$ using Lemma (1).

Let us define (u, v) and (u) as before. Since $[Y_\lambda/Y] = \beta Y_\mu\, a.e$ using Lemma (1).

$$\frac{\partial \theta(u,v)}{\partial u}\Big]_{u=0} = \beta \frac{d\theta(0,v)}{dv}\ \ for\ all\ v.$$

$$\frac{\partial}{\partial u}[\int_0^1 \psi(ut^\lambda + vt^\mu)\, dt]\Big]_{u=0} = \beta \frac{d}{dv}(\int_0^1 \psi\ [vt^\mu]dt)$$

Using Lemma (2).

Since the process is infinitely divisible with finite mean, it follows that Y is differentiable and using (7).

$$\int_0^1 t^{\lambda}\,\psi'\,[vt^{\mu}]dt = \beta \int_0^1 t^{\lambda}\,\psi'\,[vt^{\mu}]dt$$

for all v.

Integrating both sides with respect to v,

$$\int_0^1 t^{\lambda-\mu}\,\psi\,[vt^{\mu}]dt = \beta \int_0^1 \psi\,[vt^{\mu}]dt$$

for all v.

Since $\psi(0) = 0$, using

$$\int_0^v s^{\frac{1}{2}(\lambda+1-2\mu)}\,\psi(s)\,ds = \beta v^{\frac{(\lambda-\mu)}{\mu}-1}\int_0^v s^{\frac{1}{\mu}}\,\psi(s)\,ds$$

for any $v > 0$.

Differentiating with respective to v

$$v^{\mu^{-1}}\,\psi(v)(1-\beta) = \beta(\lambda-\mu)\,\mu^{-1}\int_0 s^{(1-\mu)\mu^{-1}}\psi(s)\,ds \qquad (13)$$

Differentiating again with respect to v

$$\mu(1-\beta)\,v\,\psi'(v) = \psi(v)[\beta\,(\lambda-\mu)-(1-\beta)] \qquad (14)$$

Since $X(1)$ has a non-degenerate distribution it follows that $\beta \neq 1$.

$$\psi'(v)[\psi(v)]^{-1} = [\beta\,(\lambda-\mu)-(1-\beta)][\mu(1-\beta)]^{-1} \qquad (15)$$

for all $v > 0$.

$$\text{Let } \alpha = \beta\,(\lambda-\mu)-(1-\beta)\,[\mu(1-\beta)]^{-1} \qquad (16)$$

$\psi\,(\cdot)$ is continuous at the origin. Using equation (15)

$$\psi(t) = -cv^{\alpha} \qquad (17)$$

where c is a constant different from zero.

Since ψ is the logarithm of the characteristic function of a symmetric distribution with finite mean.

$$c > 0, \alpha > 1, \psi(v) = -c|v|^{\mu} \text{ for all } v \qquad (18)$$

$\psi\,(\cdot)$ is the characteristic function of a symmetric stable law with exponent α.

Hence $\{X(t), t \in T\}$ is a symmetric stable process with finite mean. Using (18) $\Rightarrow$ (16).

Conclusion

This study provides a characterization of symmetric stable processes with finite mean, building on the foundational work by Laha on symmetric stable laws. The theorems and lemmas presented reinforce that for symmetric stable processes, specific conditions related to independent increments and homogeneous properties must hold. The results confirm that both independent random variables and their characteristic functions follow predictable structures, as defined by symmetric stable distributions. Through these findings, the study contributes to a deeper understanding of stochastic processes and expands the theoretical framework for processes exhibiting symmetry and stability with finite expectations.

References

1. Laha RG. On a characterization of the stable law with finite expectation. Ann Math Stat. 1955;26:187-195.
2. Doob JL. Stochastic Processes. New York: Wiley; 1972.
3. Rao BP. On a characterization of symmetric stable processes with finite mean. The Annals of Mathematical Statistics. 1968 Oct 1:1498-501.
4. Lukacs E. A characterization of stable processes. Journal of Applied Probability. 1969 Aug;6(2):409-418.
5. Kuelbs J. A representation theorem for symmetric stable processes and stable measures on H. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete. 1973 Dec;26(4):259-271.