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On the use of implicit method for solving parabolic PDE temperature in a slender road

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Abstract
In this paper, we discuss the "finite difference method" (FDM) to solve temperature distribution with Implicit Method. The temperature distribution in a slender rod of unit length has been described by the one- dimensional heat equation. The finite difference method (FDM) seems to be the simplest approach for the numerical solution of PDEs. The partial-derivatives are replaced by 'finite difference approximations' that lead to a system of linear algebraic equations. In the process we introduce important concepts as stability used in analyzing finite difference methods and to show how to solve parabolic equations.

Keywords: Temperature, time, finite difference method, implicit method

1. Introduction
In most of the research work in the fields of applied elasticity, theory of plates and shells hydrodynamics, quantum mechanics, the problem reduces to partial differential equations. The heat equation is fundamental in diverse scientific fields, which describe the distribution of heat (Or variation of temperature) in a given region over time. In the heat equation, there are derivatives with respect to time and derivatives with respect to space. Different combinations of mesh points in different formule result in different schemes as used. As the mesh spacing (x and t) goes to zero, the numerical solution obtained with any useful scheme will approach to the true solution of the original differential equation. However, the rate at which the numerical solution approaches the true solution varies with the scheme. In addition, some practically useful schemes can fail to yield a solution for bad combinations of x and t. Four different schemes for the solution to heat equation are developed. The implicit Euler schemes are constructed and investigated for heat conduction

2. Temperature Distribution in a Slender Road with Implicit Method
Given a slender rod of unit length, its temperature can be explained by the 1-D heat equation

$$u_t = \alpha u_{xx}, \text{for } 0 < x < 1,\ 0 < t < \text{T} \tag{1}$$

Replacing the space derivative by a centered difference at the forward time step $j+1$ and the time derivative by a forward difference, we get

$$\frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] = \alpha \frac{1}{h^2}\left[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right] \tag{2}$$

Or

$$\begin{aligned} u_{1,j} &= -rg_{0,j+1} + (1+2r)u_{1,j+1} - ru_{2,j+1} \\ u_{i,j} &= -ru_{i-1,j+1} + (1+2r)u_{i,j+1} - ru_{i+1,j+1} \qquad (\text{for} \quad i = 2,\dots,n-2) \\ u_{n-1,j} &= -ru_{n-2,j+1} + (1+2r)u_{n-1,j+1} - rg_{1,j+1} \end{aligned} \tag{3}$$

Where $g_{0,j}$ and $g_{1,j}$ denote values given by the boundary conditions and $r = \frac{\alpha k}{h^2}$. This method is unconditionally stable.

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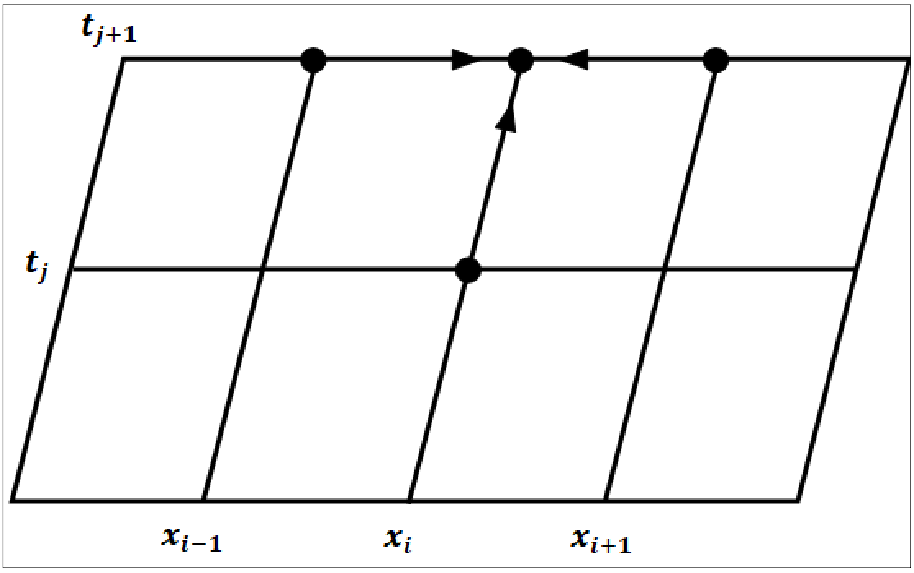

Fig 1: Values needed for computation at time j+1

The points involved in the calculations are illustrated in Figure 1. Unlike calculations in the explicit method, calculations for a point at the $(j+1)^{th}$ time level depend both on the results from one point at the j^{th} time level and on other points at the $(j+1)^{th}$ level.
The tridiagonal system must be solved in each time step, with a different right hand side, so LU factorization of the tridiagonal system is an efficient approach.

3. Discussion
The finite difference representations of the partial derivatives in the heat equation are same as given for the explicit method except that the second derivative is approximated at step $j+1$, rather than at step j. Thus we have

$$u_t(x_i, t_j) = \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] + o(k) \tag{4}$$

And

$$u_{xx}(x_{i,}t_{j+1}) = \frac{1}{h^2}\left[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right] + o(h^2) \tag{5}$$

Substituting into the PDE and simplifying shows that the truncation error for the implicit method is the same as for the explicit method, $i.e. O(h^2 + k)$.
To show that the implicit method is unconditionally stable, let us consider the matrix vector form of the process:

$$\begin{bmatrix} 1+2r & r & \cdot & \cdot & \cdot \\ r & 1+2r & r & \cdot & \cdot \\ \ddots & \ddots & \ddots & \cdot & \vdots \\ \vdots & r & 1+2r & r & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & r & 1+2r & r \\ \cdot & \cdot & \cdot & r & 1+2r \end{bmatrix} \begin{bmatrix} u(1,j+1) \\ u(2,j+1) \\ \vdots \\ u(i,j+1) \\ \vdots \\ u(n-1,j+1) \\ u(n,j+1) \end{bmatrix} = \begin{bmatrix} u(1,j) \\ u(2,j) \\ \vdots \\ u(i,j) \\ \vdots \\ u(n-1,j) \\ u(n,j) \end{bmatrix} \tag{6}$$

or $Au(:,j+1) = u(:j)$

For analysis, we write the latter equation as

$A^{-1}u(:,j) = u(:j+1)$

So that by the same reasoning as for the explicit method, stability is assured if λ,the dominant eigenvalue of A^{-1}, satisfies $|\lambda| \leq 1$. In terms of the eigenvalues of A, the condition becomes

$|\mu| \geq 1$

Where μ is the eigenvalue of A with the smallest magnitude.
By the Gerschgorian theorem, this condition is true regardless of the value of r. Each of the circles determined by the Gerschgorin theorem is centered at (1+2r, 0). The radii of the first and last circles are r; all other circles have radius 2r. Hence, all eigenvalues are greater than or equal to 1.

4. Numerical Example

Let us consider the temperature in a rod of unit length be given by the PDE

$$u_t = \alpha u_{xx} \text{for} 0 < x < 1, 0 < t < T$$

The initial conditions are

$$u(x,0) = x^4 \; for \; 0 \leq x \leq 1 \quad (7)$$

And the boundary conditions are

$$u(0,t) = 0, u(1,t) = 1, for \; 0 < t < T \quad (8)$$

Suppose a fairly coarse mesh, with $n = 5$, so that $h = \Delta x = 0.2$ and take $m = 5$ and $T = 0.2$, so that $k = \Delta t = 0.04$ with these parameter values we get

$$r = \frac{ck}{h^2} = \frac{\Delta t}{(\Delta x)^2} = 1.0$$

and the general equation

$$u_{i,j} = -ru_{i-1,j+1} + (1+2r)u_{i,j+1} - ru_{i+1,j+1} \quad (9)$$

simplifies to

$$-u_{i-1,j+1} + 3u_{i,j+1} - u_{i+1,j+1} = u_{i,j} \quad (10)$$

We must find values of u at the node points $i = 1,2,3, and\; 4$ for each time step; u is given by the boundary conditions for $i = 0 \; and \; i = 5$. To go from the initial conditions $(t = 0)$ to the solution at the first time step $(t = 0.04)$, we requires the following tridiagonal system to be solved:

$$3u_{1,1} - u_{2,1} = u_{1,0} + u_{0,1} = 0.0016 + 0.0$$

$$-u_{1,1} + 3u_{2,1} - u_{3,1} = u_{2,0} = 0.0256$$

$$-u_{2,1} + 3u_{3,1} - u_{4,1} = u_{3,0} = 0.1296 \quad (11)$$

$$-u_{3,1} + 3u_{4,1} = u_{4,0} + u_{5,1} = 0.4096 + 1.0$$

The computed values are

$$u_{1,1} = 0.037033, u_{2,1} = 0.1095, u_{3,1} = 0.26586, u_{4,1} = 0.55849 \quad (12)$$

Repeating the calculations using the right hand side found from the solution at the previous time step gives the solution with values listed in the Table 1.

Table 1: Temperature in rod; solution for t = 0.0 to t = 0.20

0	0.1475	0.3140	0.5125	0.7452	1
0	0.1286	0.2819	0.4785	0.7229	1
0	0.1039	0.2387	0.4305	0.6903	1
0	0.0729	0.1817	0.3629	0.6404	1
0	0.0370	0.1095	0.2659	0.5585	1
0	0.0016	0.0256	0.1296	0.4096	1

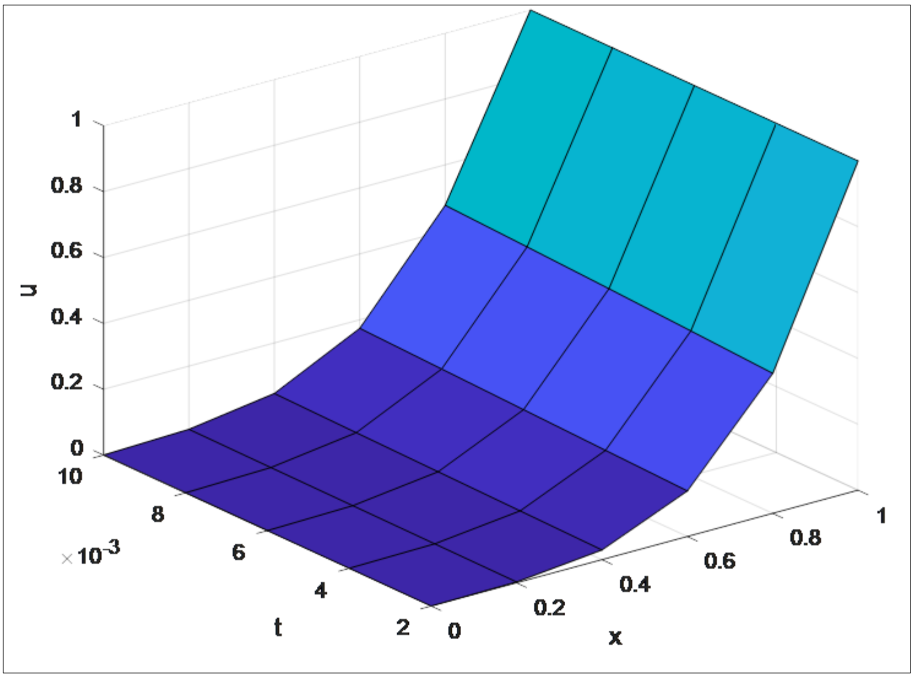

Fig 2: Temperature in rod

5. Conclusion

We present a simple case study of temperature in a slender rod of unit length with implicit method which is unconditionally stable and we observed from Table1 that with increase in time temperature increases. Therefore temperature is directly proportional to time.

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