Journal of Mathematical Problems, Equations and Statistics

E-ISSN: 2709-9407
P-ISSN: 2709-9393
JMPES 2024; 5(2): 01-07

www.mathematicaljournal.com
Received: 02-06-2024
Accepted: 05-07-2024

Rachna Rathore
Research Scholar, Shrimant Madhav Rao Scindia Govt. Model Science College, Jiwaji University, Gwalior, Madhya Pradesh, India

Dr. RK Shrivastava
Principal, Dr. Bhagwat Sahay Govt. College, Gwalior, Madhya Pradesh, India

Corresponding Author:
Rachna Rathore
Research Scholar, Shrimant Madhav Rao Scindia Govt. Model Science College, Jiwaji University, Gwalior, Madhya Pradesh, India

Analysis of M/M/1 queueing model with removable and unreliable server, partial breakdown during working vacation, setup with repair

Rachna Rathore and Dr. RK Shrivastava

DOI: https://doi.org/10.22271/math.2024.v5.i2a.141

Abstract
This paper investigates a M/M/1 queueing Model with unreliable and removable server in which Partial breakdown happens during working vacation, set up and repair. Customers arrival follow Poisson's distribution with rate λ. When system becomes empty server moves for a single working vacation with rate θ. During working vacation, server faces an unpredictable partial breakdown due to which server provides service at some reduced rate η. On completion of vacation, if server find customers waiting for service, then server resumes normal busy period otherwise moves to closed down state with rate γ. If customers come during closed down state, the server needs to restart. During restarting of server, the server may fail with probability q otherwise join regular busy period with probability p. The closed form expression of various steady state probabilities along with various performance measures like waiting time in system, expected queue length, system probabilities at different server states have been evaluated. The impact of some parameters on various system performances have been presented numerically in tabular form and illustrated graphically.

Keywords: Partial breakdown, removable server, repair, setup time, working vacation, unreliable server

1. Introduction
In real life, no industry, Business or organization wants to lose their customers at any cost because it affects them not only financially but also put negative effects on their reputation and on customers goodwill. We have seen many times in our daily life when the customers abandon the system due to service interruption. One of the main reasons of service interruption is Breakdown during service mechanism due to which customer have to wait long for their service. The reason of server breakdown may be software crashes, hardware malfunction, some technical issue, maintenance or unexpected outages. Some real-life examples where service interruption occurs due to breakdowns in the service mechanism are Airline travel, Internet service provider, Public Transportation, Hospital emergency room, retail store checkout, Banks, machine failure in industries. A partial Breakdown refers to a situation where servers capacity of serving is reduced but it still remains partially operational and provide some level of service. When a server breakdown, it ceases to serve customers entirely until it is repaired or replaced. Repair time is the time required to fix a server that has experienced a breakdown or failure and is no longer able to server customers. In queueing theory, Setup time refers to the duration which is needed to prepare a server before it can start serving customers. This preparation may involve initializing the server, loading necessary data or software, configuring settings and ensuring it is ready to process incoming requests. In queueing theory, the term unreliable server is used for the server that may fail or become unavailable unexpectedly. This could be due to any disruptions like breakdown, hardware failure, software crashes etc. An unreliable server may affect queue dynamics and performance measures of system like queue length, waiting time, throughput etc. A removable server in queueing theory refers to a server that can be taken out of service temporarily due to maintenance, breakdown, Fatigue or some other reasons. We can switch on or off the removable server for Power savings. For restarting of server, setup time is required. In queueing theory, a setup time refers to a situation needed to prepare a server before it can start serving customers. This preparation may involve initializing the server, loading of necessary data or software, configuring settings and ensuring it is ready to process incoming requests.

2. Brief Review of Literature

A review of the literature related to our research Topic in brief is discussed here:

Yadin *et al.* [16] determined system characteristics and optimized cost structure of a single server markovian queue when server is in partially closed down and setup state. Neuts *et al.* [10] analyzed M/M/N queue with N servers which are subject to breakdown having c repair facilities. Federgruen *et al.* [1] mentioned transient probabilities and various performance measures in a finite Markovian system with breakdown during service with two type of repair facilities. Wang *et al.* [8] determined the steady state condition and total expected cost function during optimum operation of an unreliable and removable server in a single server markovian queueing system. Hsieh *et al.* [15] developed a cost model for determining the behavior of an unreliable and removable server economically. Wang *et al.* [9] analyzed single, unreliable and removable server in a finite and infinite queueing system with hyper exponential distribution of service of k type under N policy. They developed cost structure for both finite and infinite queueing system. Charanjeet *et al.* [6] analyzed the transient behavior of several performance metrics by using Laplace transform in a controllable and non-reliable single server queueing system. Begum *et al.* [3] analyzed a retrial queueing system in which arrival comes in batches and optional service is provided under N policy with unpredictable breakdown with immediate repair. Liou *et al.* [5] investigated M/M/1/∞ queue with a single non reliable server which may breakdown during working. They also discussed customers impatience behavior during breakdown. Richa [14] focused on Impatient behavior of customers for a non-reliable server under N Policy with vacation interruption. Ma *et al.* [13] derived the social welfare and throughput of the queueing system with non-reliable server under threshold policy. Bharthidas *et al.* [4] analyzed Bulk queueing model with Erlangian service with k service phases with system breakdown which leads to repair. Qing Melikov *et al.* [2] analyzed a Bernoulli retrial queue with multiple servers where a non-reliable server is subject to breakdown during both service and idle time of server. Poonam [11] studied retrial policy with the concept of feedback, interruption during working vacation with repair of failed server with setup time. Further, Gupta *et al.* [12] introduced the concept of waiting server along with breakdown and repair of server during busy period. Kim *et al.* [7] extended M^a/M/c queue with static and dynamic groups of servers which may have same or different service rates. Further, they compared their model with classical queue.

2.1 Model Description

In this paper a M/M/1 Queueing Model with unreliable and removable server in which Partial breakdown occurs during working vacation along with concept of setup with repair is analyzed. Arrivals of customers follows poissons distribution with rate λ while rate of providing service follows an exponential distribution with rate μ during busy period and with rate η, where(η<μ), during working vacation period respectively. The server goes for a single working vacation with parameter θ during idle period. During working vacation server faces a partial breakdown or soft failure due to which customers get service with lower service rate than regular busy period, as no physical repair is required due to partial breakdown. When server becomes free during working vacation period, he moves to closed down state with rate γ which follows exponential distribution, where the server is turned off as server is removable server. On completion of vacation, server resumes normal busy period. Those customers who comes when server is off, waits for their turn in front of server until it is turned on. It takes some time known as Setup Time with exponentially distributed rate s to restart the server. Since the server is considered to be unreliable, therefore, during setup activation server may fail with probability $q = 1 - p$, otherwise server switches back to busy period with probability p. The failed or damaged server is sent for repair with exponentially distributed rate r. The inter arrival time, service time, vacation time, repair time and setup time all are identically and independently distributed. The transition rate diagram of the model is shown in the figure 1 given below:

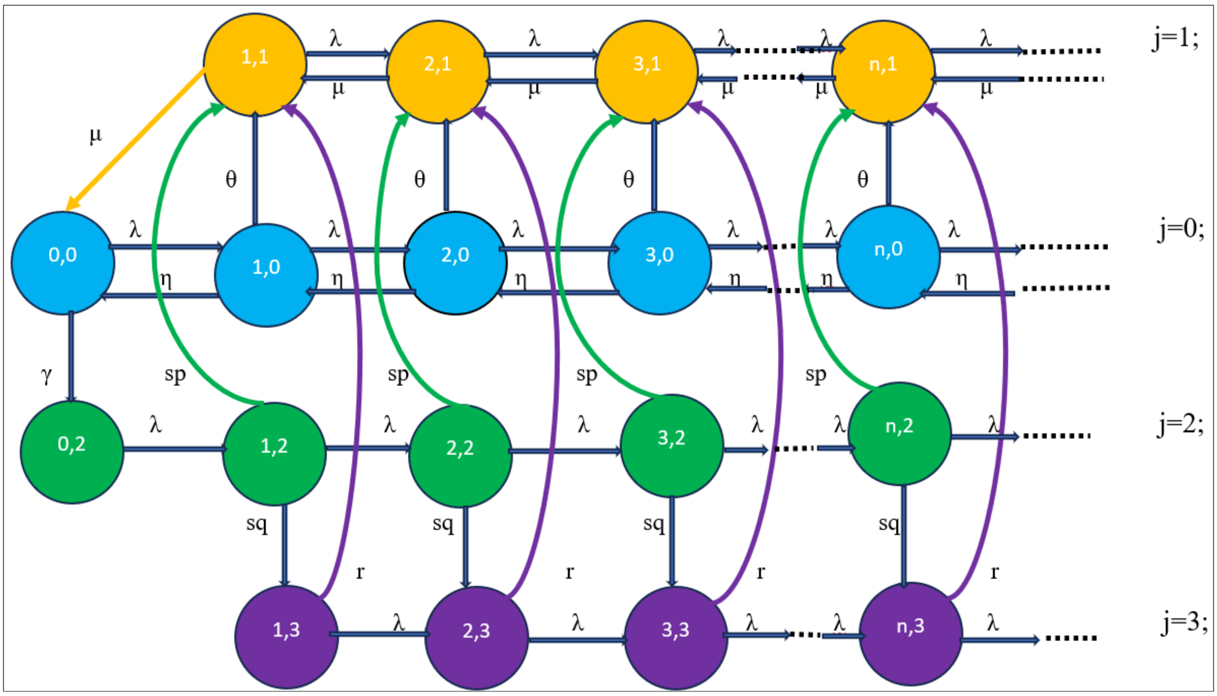

Fig 1: Transition rate diagram of different states of server.

2.2 Mathematical Formulation

Let P_{nj} denotes Probability of n customers during j state of server, where j= 0,1,2,3 and n= 0,1,2,3……

Here j=0 shows working vacation state
j=1 shows busy period state
j=2 shows closed down state
j=3 shows repair state.
Balance Equations for each server state are given as follows:
For j=1:

$P_{11}(\lambda+\mu)=\mu P_{21}+\theta P_{10}+spP_{12}+rP_{13}$, n=1 (1)

$P_{n1}(\lambda+\mu)=\mu P_{n+1\,1}+\theta P_{n0}+spP_{n2}+rP_{n3}+\lambda P_{n-1\,1}$, n≥2 (2)

For j=0:

$P_{00}(\lambda+\gamma)=\eta P_{10}+\mu P_{11}$, n=0 (3)

$P_{n0}(\lambda+\eta+\theta)=\eta P_{n+1\,0}+\lambda P_{n-10}$, n≥1 (4)

For j=2:

$\gamma P_{00}=\lambda P_{02}$, n=0 (5)

$P_{n2}(\lambda+sq+sp)=\lambda P_{n-12}$, n≥1 (6)

For j=3:

$P_{13}(\lambda+r)=sqP_{12}$, n=1 (7)

$P_{n\,3}(\lambda+r)=sqP_{n\,2}+\lambda P_{n-1\,3}$, n≥2 (8)

2.3 Steady State Probabilities and some System Performances

Consider PGF $F_0(z)=\sum_{n=0}^{\infty}P_{n0}\,z^n$, $F_0'(z)=\sum_{n=0}^{\infty}nP_{n0}\,z^{n-1}$ (9)

$F_1(z)=\sum_{n=1}^{\infty}P_{n1}z^n$, $F_1'(z)=\sum_{n=1}^{\infty}nP_{n1}z^{n-1}$ (10)

$F_2(z)=\sum_{n=0}^{\infty}P_{n2}z^n$, $F_2'(z)=\sum_{n=0}^{\infty}nP_{n2}z^{n-1}$ (11)

$F_3(z)=\sum_{n=1}^{\infty}P_{n3}z^n$, $F_3'(z)=\sum_{n=1}^{\infty}nP_{n3}z^{n-1}$ (12)

Such that $F_0(1)+F_1(1)+F_2(1)+F_3(1)=1$. (13)

Multiply equation (2) by z^n and taking summation over n, we get,

$(\lambda+\mu)\sum_{n=2}^{\infty}P_{n1}\,z^n=\mu\sum_{n=2}^{\infty}P_{n+1\,1}\,z^n+\theta\sum_{n=2}^{\infty}P_{n0}\,z^n+sp\sum_{n=2}^{\infty}P_{n2}z^n+r\sum_{n=2}^{\infty}P_{n\,3}z^n+\lambda\sum_{n=2}^{\infty}P_{n-1\,1}z^n$

On simplifying above equation, we get

$F_1(z)(\lambda z-\mu)=z[\theta F_0(z)-\theta P_{00}+spF_2(z)-spP_{02}+rF_3(z)-2\mu P_{11}]$ (14)

Taking limit $z\to 1$ in (14) we get,

$F_1(1)=\frac{1}{(\lambda-\mu)}[\theta F_0(1)-\theta P_{00}+spF_2(1)-spP_{02}+rF_3(1)-2\mu P_{11}]=P(B)$ (15)

Differentiate equation (14) both side w.r.t. z and taking limit $z\to 1$, we get,

$F_1'(1)=\frac{1}{(\lambda-\mu)}[spF_2(1)+spF_2'(1)-spP_{02}+\theta F_0(1)+\theta F_0'(1)-\theta P_{00}+rF_3(1)+rF_3'(1)-2\mu P_{11}-\lambda G_1(1)]=E(L_1)$ (16)

Multiply equation (4) by z^n and taking summation over n,we get,

$$(\lambda+\eta+\theta)\sum_{n=1}^{\infty}P_{n0}z^n=\eta\sum_{n=1}^{\infty}P_{n+1\,0}\,z^n+\lambda\sum_{n=1}^{\infty}P_{n-10}z^n$$

On simplifying above equation, we get

$F_0(z)=\frac{[(\lambda z+\eta z+\theta z-\eta)P_{00}-\eta z P_{10}]}{(\lambda z+\eta z+\theta z-\eta-\lambda z^2)}$ (17)

Taking limit $z \to 1$ in (17) we get,

$F_0(1)=\frac{(\lambda+\theta)P_{00}-\eta P_{10}}{\theta}=P(WV)$ (18)

Differentiate equation (17) both side w.r.t. z and taking limit $z \to 1$, we get,

$F_0'(1)=\frac{1}{\theta}[(\lambda+\eta+\theta)P_{00}-\eta P_{10}-F_0(1)(\eta+\theta-\lambda)]=E(L_0)$ (19)

Multiply equation (6) by $z^n$and taking summation over n, we get,

$$(\lambda+sp+sq)\sum_{n=1}^{\infty}P_{n2}z^n=\lambda\sum_{n=1}^{\infty}P_{n-12}z^n$$

On simplifying above equation, we get

$F_2(z)=\frac{(\lambda+sp+sq)P_{02}}{(\lambda+sp+sq-\lambda z)}$ (20)

Taking limit $z \to 1$ in (20) we get,

$F_2(1)=\frac{(\lambda+sp+sq)}{(sp+sq)}P_{02}=P(S)$ (21)

Differentiate equation (20) both side w.r.t. z and taking limit $z \to 1$, we get,

$F_2'(1)=\frac{\lambda G_2(1)}{(sp+sq)}=E(L_2)$ (22)

Multiply equation (8) by $z^n$and taking summation over n,we get,

$$(\lambda+r)\sum_{n=2}^{\infty}P_{n3}z^n=sq\sum_{n=2}^{\infty}P_{n2}z^n+\lambda\sum_{n=2}^{\infty}P_{n-1\,3}z^n$$

On simplifying above equation, we get

$F_3(z)=\frac{sqG_2(z)-sqP_{02}-sqzP_{12}+(\lambda+r)zP_{13}}{(\lambda-\lambda z+r)}$ (23)

Taking limit $z \to 1$ in (23) we get,
$F_3(1)=\frac{sqG_2(1)-sqP_{02}-sqP_{12}+(\lambda+r)P_{13}}{r}=P(R)$ (24)

Differentiate equation (23) both side with respect to z, taking limit $z \to 1$ we get,

$F_3'(1)=\frac{\lambda G_3(1)+sqG_2'(1)-sqP_{12}+(\lambda+r)P_{13}}{r}=E(L_3)$ (25)

Now, by using recurrence relation (1), (2), (3), (4), (5),(6),(7),(8) we get,

$P_{00}=\frac{\lambda}{\gamma}P_{02}=K_1P_{02}$,

$P_{12}=\left(\frac{\lambda}{\lambda+sp+sq}\right)P_{02}=K_2P_{02}$

$P_{13}=\frac{\lambda sq}{(\lambda+r)(\lambda+sp+sq)}P_{02}=K_3P_{02}$

$P_{10}=\frac{\lambda^2}{(\eta+\theta)\gamma}P_{02}=K_4P_{02}$

$$P_{11}=\frac{\lambda}{\mu\gamma}\left[(\lambda+\gamma)-\frac{\eta\lambda}{(\eta+\theta)}\right]P_{02}=K_5P_{02}$$

Where $K_1 = \frac{\lambda}{\gamma}$, $K_2 = \frac{\lambda}{(\lambda+sp+sq)}$, $K_3 = \frac{\lambda sq}{(\lambda+r)(\lambda+sp+sq)}$, $K_4 = \frac{\lambda^2}{(\eta+\theta)\gamma}$,

$K_5 = \frac{\lambda}{\mu\gamma}\left[(\lambda+\gamma) - \frac{\eta\lambda}{(\eta+\theta)}\right]$

By using above P_{nj}'s, we can rewrite $F_0(1)$, $F_1(1)$, $F_2(1)$, $F_3(1)$ in terms of P_{02} as follows,

$F_0(1) = M_0 P_{02}$, $M_0 = \left\{\frac{(\lambda+\theta)k_1 - \eta k_4}{\theta}\right\} P_{02}$

$F_1(1) = M_1 P_{02}$, $M_1 = \frac{1}{(\lambda-\mu)} \left[\theta M_0 - K_1 + \mathrm{sp} M_2 - \mathrm{sp} + \mathrm{r} M_3 - 2\mu K_5\right]$

$F_2(1) = M_2 P_{02}$, $M_2 = \left\{\frac{\lambda+sp+sq}{sq+sp}\right\}$

$F_3(1) = M_3 P_{02}$, $M_3 = \left\{\frac{sqM_1 - sq - sqK_2 + (\lambda+r)K_3}{r}\right\}$

Since $F_0(1)$, $F_1(1)$, $F_2(1)$, $F_3(1)$ and all P_{nj}'s are expressed in terms of P_{02}, therefore we need to calculate P_{02} which can be determined by using Normalization condition,

$$\mathrm{F}_0(1) + \mathrm{F}_1(1) + \mathrm{F}_2(1) + \mathrm{F}_3(1) = 1.$$

$$M_0 P_{02} + M_1 P_{02} + M_2 P_{02} + M_3 P_{02} = 1$$

$$\therefore P_{02} = [M_0 + M_1 + M_2 + M_3]^{-1}$$

3. Numerical Analysis

Table 1: Impact of repair rate r on various system Performances.

r	E (L_0)	E (L_1)	E (L_2)	E (L_3)	E(L)	W
0.1	0.133335	0.386145	0.72	9.79958	11.03906	2.7597
0.2	0.133335	0.18666	0.72	2.9	3.939995	0.98499
0.4	0.133335	0.086668	0.72	0.95	1.890003	0.47250
0.6	0.133335	0.045368	0.72	0.499	1.397703	0.34942
0.9	0.133335	0.025626	0.72	0.288	1.166961	0.29174

Table 1 represents Impact of repair rate r on different system Performances E (L_0), E (L_1), E (L_2), E (L_3), E(L), W for λ=4, μ=6, η=5, p=0.5, q=0.5, s=0.5, θ=0.3, γ=0.2. As the value of r increases E (L_0), E (L_2) remains constant while E (L_3), E (L_1), E (L),W decreases continuously.

Table 2: Impact of set up activation probability p on different system Probabilities.

p	P(WV)	P(C)	P(R)	P(B)
0.1	0.35095	0.1433	0.06	0.008527
0.3	0.35095	0.11	0.05	0.01019
0.5	0.35095	0.09	0.04	0.010195
0.7	0.35095	0.076	0.03	0.009245
0.9	0.35095	0.067	0.028	0.0100

Table 2 shows Impact of set up activation probability p on P(WV), P(B), P(C), P(R) for λ=4,μ=6,η=5,q=0.5,s=0.5,θ=1,r=0.5, γ=0.2.As p increases, P(WV) remains constant, P(C) and P(R) decreases gradually and a very small increment is noticed in P(B).

4. Graphical Illustration

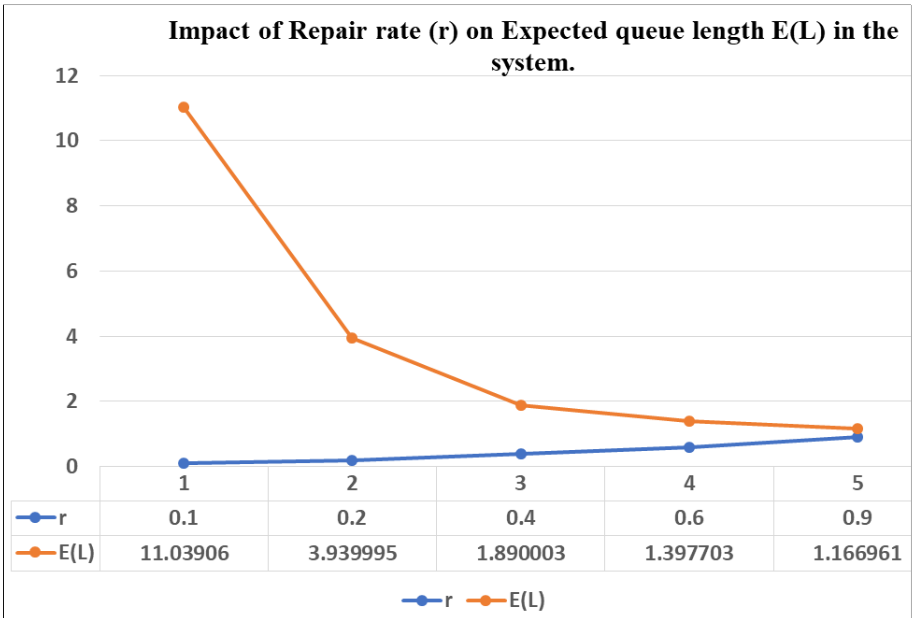

Fig 1: Shows graphical illustration of Repair rate (r) on Expected queue length E(L) in the system.

It is clear that Fig 1 shows impact of Repair rate r on Expected queue length E(L) in the system for $\lambda=4$, $\mu=6$, $\eta=5$, $p=0.5$, $q=0.5$, $s=0.5$, $\theta=0.3$, $\gamma=0.2$. It have been observed that as value of r increases, value of E(L) decreases continuously.

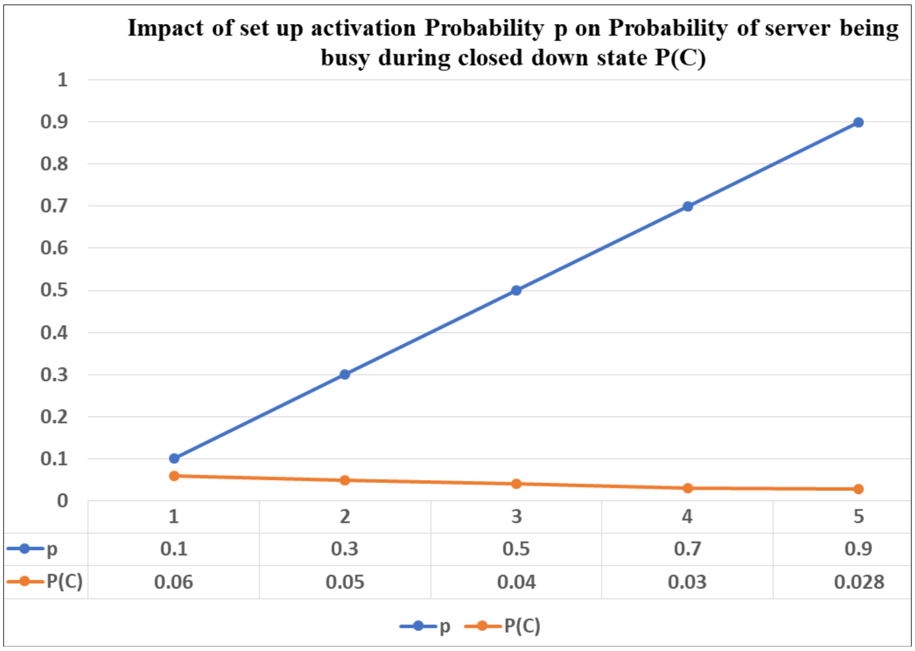

Fig 2: Shows graphical illustration of Set up activation probability p on Probability of server being in closed down state.

It is clear that Fig 2 shows graphical illustration of the effect of Set up activation probability p on Probability of server being in closed down state P(C) for $\lambda=4$, $\mu=6$, $\eta=5$, $q=0.5$, $s=0.5$, $\theta=1$, $r=0.5$, $\gamma=0.2$. It has been observed as the value of p increases, the value of P(C) decreases gradually.

5. Application of the Model

This model can be applied in various real-life scenarios. Here an example of Retail shop is considered;

5.1 Scenario

Consider a retail shop in which customer arrives in a random order to purchase different items and there is only one counter for checkout. It is assumed that server at checkout counter is removable i.e. it may be possible that server may leave counter temporarily for performing other duties, breaks, meetings etc. Also, server may be unexpectedly absent due to personal emergencies, fatigue or illness etc. as we consider server is unreliable. The server may encounter issues related to cash change, barcode scanner, Payment transaction which result in slow service, temporary unavailability of certain service. The repair process may involve replacement of equipment, short out problem of cash change, calling a specialized technician or simply troubleshooting is initiated to restoring functionality by fixing the issue.

6. Model Application

This model can be used to analyze the performance measures of above-mentioned scenario by evaluating mean wait time, average queue length, repair times, cashier's (Server) availability during different situation by evaluating probabilities of server busy at different states. It will help in analyzing the impact of cashier unavailability and breakdown on customers and allows the shop owner to identify Possible obstruction and implement contingency plans which may involve other trained staff members or backup systems as backup server to handle transactions, scanning of items etc. during downtime to mitigate risks. It will also help in optimizing staffing levels and schedules for break to minimize customer wait times and maintain service levels during peak hours.

7. Conclusion

In this paper we have analyzed a single server Markovian queueing model with removable and unreliable server, partial breakdown during working vacation, setup and repair. We have evaluated analytically closed form expression of different steady state probabilities of the system and various performance measures like expected queue length during busy period, working vacation, repair and closed down state of server. The impact of some parameters on some performance measures have been investigated numerically and illustrated graphically. Finally, it has been concluded that breakdown, long setup and repair time introduce uncertainty and variability into the queueing system, as they directly influence system performance, customer satisfaction and operational costs. Therefore, it is essential for organizations to proactively manage and mitigate the interruption due to breakdown, long repair and setup time through proper planning, redundancy, maintenance schedules and disaster recovery. Effective management of these parameters can lead to smoother operation and improve overall efficiency.

8. Future Scope: This work can be further extended by introducing the concept of Bulk arrival, Threshold Policy for service.

9. References

1. Federgruen A, So KC. Optimality of threshold policies in single-server queueing systems with server vacations. Adv Appl Prob. 1991;23(2):388-405.
2. Melikov A, Aliyeva S, Sztrik J. Retrial queues with unreliable servers and delayed feedback. Math. 2021;9(19):2415. Available from: https://doi.org/10.3390/math9192415
3. Begum A, *et al*. N-policy for repairable bulk arrival queueing model with setup, second multi optional service facility under restricted number of server vacations. Int J Innov Res Sci Eng Technol. 2014;3:14614-14626.
4. Bharathidass S, Arivukkarasu V, Ganesan V. Bulk service queue with server breakdown and repairs. Int J Stat Appl. Math. 2018;3(1):136-142.
5. Liou C-D. Markovian queue optimization analysis with an unreliable server subject to working breakdowns and impatient customers. Int J Syst Sci. 2015;46(12):2165-2182.
6. Singh CJ, Jain M. Single server unreliable queueing model with removable service station. Int J Oper. Res. 2013;18(4):452-466.
7. Kim JJ, Down DG, Chaudhry M, Banik AD. Difference equations approach for multi-server queueing models with removable servers. Methodol Comput Appl Prob. 2022;24(3):1297-1321.
8. Wang K-H. Optimal operation of a Markovian queueing system with a removable and non-reliable server. Microelectron Reliab. 1995;35(8):1131-1136.
9. Wang K-H, Kao H-T, Chen G. Optimal management of a removable and non-reliable server in an infinite and a finite M/Hk/1 queueing system. Qual Technol Quant Manag. 2004;1(2):325-339.
10. Neuts MF, Lucantoni DM. A Markovian queue with N servers subject to breakdowns and repairs. Manag Sci. 1979;25(9):849-861.
11. Gupta P. Study of feedback retrial queueing system with working vacation, setup time and perfect repair. Ratio Math. 2021;41:291-307.
12. Gupta P, Kumar N. Performance analysis of retrial queueing model with working vacation, interruption, waiting server, breakdown and repair. J Sci. Res. 2021;13(3):833-844.
13. Ma Q, Zhang X. Analysis and comparison of queue with N-policy and unreliable server. Hindawi Discrete Dyn Nat Soc. 2020;2020:6195080. Available from: https://doi.org/10.1155/2020/6195080
14. Sharma R. Optimal N-policy for unreliable server queue with impatient customer and vacation interruption. J Reliab Stat Stud. 2017;10(1):83-96.
15. Hsieh W-F, Wang K-H. Optimal control of a removable and non-reliable server in a Markovian queueing system with finite capacity. Microelectron Reliab. 1995;35(2):189-196.
16. Yadin M, Naor P. Queueing systems with a removable service station. J Oper. Res Soc. 1963;14:393-405.