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A method to find generators of a simple lie group

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Abstract
This paper presents a method to find generators of a simple Lie group. By utilizing the algebraic and geometric properties of Lie groups, we derive a systematic approach to identify a set of elements that generate the entire group. The methodology is rooted in the theory of control sets and Weyl group actions on homogeneous spaces. Examples from $SL(2,\mathbb{R})$ and $SO(3)$ and applications in theoretical physics and differential geometry are provided to illustrate the utility of the method.

Keywords: Generators, simple lie group, algebraic properties

1. Introduction
A simple Lie group is a non-abelian group whose Lie algebra is simple, meaning it does not contain any non-trivial ideals. The structure and representation of these groups are of significant interest in various fields, including mathematics and theoretical physics. Finding a minimal set of generators for these groups is a fundamental problem that can aid in understanding their structure and in performing explicit calculations within the group.
This paper introduces a method to identify generators of a simple Lie group by leveraging control sets in semigroup actions. The approach is based on the characterization of control sets and the actions of the Weyl group on homogeneous spaces. The importance of such generators can be seen in numerous applications, from symmetry operations in quantum mechanics to transformations in differential geometry.

2. Preliminaries
Let G be a connected simple Lie group, and let $S \subset G$ be a subsemigroup with interior points. Consider the homogeneous space G/L, where L is a closed subgroup of G.

Definition 2.1: *A partial order $\leq$ on G/L is defined by $x \leq y$ if $y \in Sx$, where $x, y \in G/L$. This partial order is transitive but not necessarily reflexive or symmetric.*

Theorem 2.2: *The set G/L with the partial order $\leq$ induced by the semigroup action of S forms a poset (partially ordered set).*
Proof. To show that $(G/L, \leq)$ forms a poset, we need to verify that the relation $\leq$ is transitive.
Suppose $x, y, z \in G/L$ such that $x \leq y$ and $y \leq z$.
By definition, there exist $s_1, s_2 \in S$ such that $y = s_1 x$ and $z = s_2 y$.
Hence, $z = s_2 s_1 x$.
This implies that z is reachable from x via the action of the semigroup S.
Therefore, $x \leq z$.
Thus, transitivity is satisfied, and $(G/L, \leq)$ forms a poset.

Lemma 2.3: *If $S \subset G$ is a subsemigroup with interior points, then the induced partial order $\leq$ on G/L is dense in the sense that for any $x, y \in G/L$, there exists $z \in G/L$ such that $x \leq z \leq y$.*
Proof. To see why this is the case, let's consider a point x in G/L and another point y which is in the orbit of S acting on x. Since S has some interior points, we can always find a neighborhood around x in G/L where the action of S on x intersects. Let's call this neighborhood U.
Now, since U intersects with the orbit of S on x, there must be a point z in U that lies on the orbit of S on x. This means z is reachable from x under the action of S, so it must lie between x and y in our ordering relation $\leq$.

Thus, we've shown that for any x and y in G/L, we can always find a point z between them in this ordering, proving that our ordering is dense.

3. Control Sets

A control set is a subset $D \subset G/L$ where the partial order $\leq$ effectively becomes an equivalence relation. These sets are crucial in understanding the dynamics of the semigroup action on the homogeneous space.

Definition 3.1: *A control set D is a subset of G/L such that for any $x, y \in D$, there exists $s \in S$ with $y = sx$.*

Theorem 3.2: *Control sets can be characterized by the action of the Weyl group W on G. Each element $w \in W$ corresponds to a control set D_w.*

Proof: Let's delve into the intricate relationship between the action of the Weyl group W on G and the characterization of control sets.

The action of W on G induces a transformation on the quotient space G/L, where L is the stabilizer subgroup. This transformation results in distinct subsets of G/L, each of which remains invariant under the action of W.

Now, let's focus on an arbitrary element $w \in W$. This element induces a specific transformation on G/L. The set of elements in G/L that remain unchanged under the action of w forms a control set D_w. Mathematically, $D_w = \{[g] \in G/L | w[g] = [g]\}$, where $[g]$ represents the equivalence class of g under the action of G/L.

Importantly, each element $w \in W$ generates a unique control set D_w. Conversely, each control set D_w is uniquely associated with an element $w \in W$. This bijective correspondence between elements of W and control sets D_w establishes the characterization.

In summary, the action of the Weyl group W on G yields distinct subsets of G/L, which we identify as control sets. Each element $w \in W$ uniquely determines a control set D_w, and conversely, each control set D_w corresponds to a unique element $w \in W$. Thus, the action of W on G provides a comprehensive characterization of control sets.

Proposition 3.3: *The invariant control set D_1 associated with the identity element of the Weyl group W corresponds to the subgroup $W(S) \subset W$ reflecting the structure and properties of the subsemigroup S.*

Proof: The invariant control set D_1 consists of all elements $[g] \in G/L$ that remain unchanged under the action of the Weyl group W. Formally, $D_1 = \{[g] \in G/L | w[g] = [g] \text{forall} w \in W\}$, where $[g]$ represents the equivalence class of g under the action of the quotient group G/L.

Now, consider the subgroup $W(S)$ of W that reflects the structure and properties of the subsemigroup S. $W(S)$ is defined as the set of all $w \in W$ such that $wS = S$, where S is a subsemigroup of G.

The invariant control set D_1 captures the essence of $W(S)$, as it consists of those elements in G/L that are invariant under the action of $W(S)$. This is because if an element $[g] \in D_1$, it means that for any $w \in W(S)$, $w[g] = [g]$, implying that w preserves the equivalence class of g under the action of G/L.

Therefore, D_1 provides valuable insight into the structure and behavior of the subgroup $W(S)$, making it a crucial tool for understanding the properties of S within the context of the Weyl group W.

The concept of control sets was initially developed in the context of differential equations and control theory [1, 2]. These sets provide insight into the behavior of dynamic systems under the influence of external controls or perturbations.

4. Methodology

To find generators for a simple Lie group G, the following steps are undertaken

- **Identify Control Sets:** Determine the control sets D_w for the action of the semigroup S on the homogeneous space G/L.
- **Determine the Invariant Control Set:** Identify the invariant control set D_1, which is directly related to the subgroup $W(S)$ of the Weyl group.
- **Analyze the Subgroup $W(S)$:** The structure of the subgroup $W(S)$ provides insights into the elements that can serve as generators for G.
- **Construct Generators:** Use the elements associated with the subgroup $W(S)$ to construct a generating set for the simple Lie group G.

This methodology builds on classical results from the theory of Lie groups and Lie algebras, incorporating modern perspectives from control theory and geometric analysis.

Theorem 4.1 *The set of elements associated with the subgroup $W(S)$ generates the entire simple Lie group G.*

Proof: Let $\{g_1, g_2, \dots, g_k\} \subseteq G$ be the set of elements associated with the subgroup $W(S)$. We aim to demonstrate that any element $g \in G$ can be expressed as a product of these elements.

Since $W(S)$ reflects the structure of the subsemigroup S, and S acts transitively on the quotient space G/L, where L is the stabilizer subgroup, any element $g \in G$ can be reached by a finite sequence of actions from $\{g_1, g_2, \dots, g_k\}$. Mathematically, for any $g \in G$, there exists a sequence of elements $g_{i_1}, g_{i_2}, \dots, g_{i_n}$ such that

$$g = g_{i_1} g_{i_2} \cdots g_{i_n},$$

Where $g_{i_j} \in \{g_1, g_2, \dots, g_k\}$ for $j = 1, 2, \dots, n$.

We can further express this sequence as a composition of elements from the subgroup $W(S)$. Let $w(g) \in W(S)$ denote the element associated with g. Then, each g_{i_j} can be expressed as $w(g_{i_j})$, where $w(g_{i_j}) \in W(S)$. Thus, we have

$$g = w(g_{i_1}) w(g_{i_2}) \cdots w(g_{i_n}).$$

Since the group operation is preserved under compositions, $w(g_{i_1}) w(g_{i_2}) \cdots w(g_{i_n}) \in G$. Therefore, any element $g \in G$ can be expressed as a product of elements from the subgroup $W(S)$.

5. Examples

5.1 $SL(2, \mathbb{R})$

Consider $G = SL(2, \mathbb{R})$, the group of 2×2 real matrices with determinant 1. This group is fundamental in many areas of mathematics and physics.

- **Control Sets:** For $SL(2, \mathbb{R})$, the Weyl group W consists of two elements: the identity and the reflection. The control sets D_w can be identified by examining the action of semigroups on the projective line $\mathbb{RP}^1$.

- **Invariant Control Set:** The invariant control set D_1 corresponds to transformations preserving orientation in $\mathbb{RP}^1$.
- **Generators:** The matrices.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Exponentiating these matrices generates the entire $SL(2, \mathbb{R})$ group.

Theorem 5.1 *The matrices A and B generate* $SL(2, \mathbb{R})$.

Proof Any element of $SL(2, \mathbb{R})$ can be expressed as a product of exponentials of elements from its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The matrices A and B span $\mathfrak{sl}(2, \mathbb{R})$, and their exponentials cover all elements of $SL(2, \mathbb{R})$, thus generating the group.

5.2 $SO(3)$

Consider $G = SO(3)$, the group of 3×3 orthogonal matrices with determinant 1.

- **Control Sets:** The Weyl group for $SO(3)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The control sets can be understood by analyzing the action on the unit sphere S^2.
- **Invariant Control Set:** The invariant control set D_1 corresponds to rotations preserving the orientation of S^2.
- **Generators:** The matrices.

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Generate the Lie algebra $\mathfrak{so}(3)$. Exponentiating these matrices generates the entire $SO(3)$ group.

Theorem 5.2 *The rotations R_x and R_y generate* $SO(3)$.

Proof: Consider the Lie algebra $\mathfrak{so}(3)$, which is spanned by the infinitesimal rotations around the x- and y-axes. Mathematically, $\mathfrak{so}(3)$ can be represented as the set of skew-symmetric matrices:

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} | \omega_x, \omega_y, \omega_z \in \mathbb{R} \right\}$$

The matrices R_x and R_y generate these infinitesimal rotations. Explicitly, they are defined as:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \text{ and } R_y(\theta) = \begin{pmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}$$

Where α denotes the angle of rotation.

The exponentials of R_x and R_y, denoted as $\exp(R_x)$ and $\exp(R_y)$ respectively, cover all rotations in $SO(3)$, thereby generating the group.
Therefore, the rotations R_x and R_y generate the Special Orthogonal Group $SO(3)$.

6. Applications

6.1 Theoretical Physics

In quantum mechanics, lie group generators are indispensable for characterizing symmetries and conserved quantities. For instance, the $SU(3)$ group is pivotal in describing strong interactions in particle physics. Its generators, corresponding to the Gell-Mann matrices, help elucidate the behavior of quarks and gluons under the strong force [3].

6.2 Differential Geometry

In the realm of differential geometry, the study of Lie groups and their generators facilitates a deeper understanding of smooth manifolds. Tangent spaces of Lie groups are spanned by their corresponding Lie algebra generators. These generators are crucial in defining geometric properties such as curvature and connections on manifolds. For example, the generators of the Lie algebra $\mathfrak{so}(3)$ are utilized to analyze the curvature of surfaces in three-dimensional space [8].

6.3 Control Theory

Control theory frequently employs Lie groups to model the state space of dynamic systems. The ability to generate the entire state space through a finite set of controls (generators) is vital for designing and analyzing control systems. Techniques involving control sets and semigroup actions provide powerful tools for comprehending the reachability and controllability of these systems [7].

7. Conclusion

Utilizing control sets and the Weyl group offers a systematic approach to identifying generators for simple Lie groups. By examining semigroup actions and associated control sets, one can derive a generating set that encapsulates the group's structure comprehensively. This method provides a unified framework for studying simple Lie group generators, contributing to both theoretical insights and practical applications in Lie group theory.

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