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Functional substitution method for differential equations with physical applications

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Abstract
Functional substitution method is proposed to solve ordinary differential equations. The method depends on making a general functional substitution and then simplifying the equation by selecting a special form of the function and its argument. The method is applied to linear and nonlinear ordinary differential equations of first and second order. Mathematical models representing physical phenomena of dynamics and heat transfer are treated with this method.

Keywords: Ordinary differential equations, exact solutions, transformation of the equations, physical problems, heat transfer

1. Introduction

Differential equations are widely used in modeling physical problems. The best solutions are the exact ones, despite their rare availability. If an exact solution is not possible, then one resorts to semi-analytical approximate solutions as the next choice. Searching for numerical solutions is the third choice. Tremendous amount of work has already been recorded in all categories whether exact analytical, semi-analytical or numerical.

The theory is well established in search of exact solutions for the constant coefficient linear ordinary differential equations. For variable coefficient linear equations as well as the nonlinear equations, the cases with exact solutions decline much. A unified approach to determine the exact solutions is the Lie Group theory approach (Bluman and Kumei, 1989) [9] which requires extensive calculations. In the case of approximate analytical solutions, the series solution methods (O'Neil, 1991) and perturbation methods (Nayfeh, 1981) [12] are the oldest and widely used methods with tremendous amounts of variants developed in the past. Since the perturbation solutions heavily depend on a small parameter, the range of validity is limited. Recently, perturbation-iteration method and its variants has been developed to extend the validity of the results (Aksoy & Pakdemirli, 2010; Pakdemirli, 2013; Singh and Reddy, 2020; Srivastava *et al*., 2021) [4, 15, 18, 19]. Variational Iteration Method (He, 1999; Anjum & He, 2019) [9, 6] Homotopy Perturbation Method (He, 2003; Abbasbandy, 2006) [10, 1] and Homotopy Analysis Method (Liao, 2004; Abbasbandy, 2007) [11, 2] are some of the examples for the semi-analytical methods. Numerical techniques are employed when analytical and semi-analytical techniques are not the choices. However, they are used as reference tools for the check of the semi-analytical solutions also. In this work, functional substitution method is proposed in search of exact analytical solutions for ordinary differential equations. The method mainly depends on the substitution of an arbitrary functional form with an arbitrary function argument into the equations, determining the simplifications by selecting a suitable form of this function and the argument, and solving the simplified equations to reach the solution of the original equation. Results presented here can be retrieved by other analytical techniques such as integrating factors, reduction of orders, variation of parameters, symmetries of the equations etc. The method is not straightforward, needs some experience in determining the substitution function, may led to unsolvable equations and may fail to work if an exact solution is unavailable.

MSC2020: 34A05, 34A25, 34A30, 34A34, 70B05, 80A19

2. Preliminaries: For the k'th order differential equation

$$F\left(x, y, y', y'', \dots y^{(k)}\right) = 0 \tag{2.1}$$

A substitution is sought of the form

$$g\left(y, y', y'', \ldots y^{(m)}\right) = f(u) \; (m<k) \tag{2.2}$$

Which may simplify the equation and make it solvable. Usually, the function g is a linear function of one of its arguments albeit more complex forms may also appear depending on the characteristics of the equation. The arbitrary function f can be selected via inspection of the transformed equation in terms of u or via a differential equation, so that another solvable equation in terms of the argument u can be retrieved. There may be occasions where u=u(x) is first determined and then the equation for f can be solved. Back substitution then yields the solution for the original equation. The method will be exploited for linear and nonlinear equations of first and second orders.

3. First Order Equations

Sample linear and nonlinear equations are solved by employment of the method.

3.1 *Linear Equations*

Consider the general linear non-homogenous equation

$$y' + a(x)y = h(x) \quad . \tag{3.1}$$

If the transformation $y(x) = f(u(x))$ is substituted

$$f_u u' + a(x)f = h(x) \tag{3.2}$$

and selecting

$$u' = a(x) \quad \rightarrow \quad u = \int a(x)dx \tag{3.3}$$

simplifies the equation into

$$f_u + f = \frac{h}{a} \tag{3.4}$$

with a solution

$$y = f = e^{-u}\left(\int \frac{h}{a} e^u du + c\right) \tag{3.5}$$

Substituting (3.3) into (3.5) yields the solution

$$y = e^{-\int a(x)dx}\left(\int h e^{\int a(x)dx} dx + c\right) \tag{3.6}$$

3.2 Non-Linear Equations

Two sample equations are treated in this section.

Example 3.1

Consider the nonlinear equation

$$y' + a(x)y^2 = 0 \quad . \tag{3.7}$$

A substitution of $y = f(u)$ leads to

$$f_u u' + a(x)f^2 = 0 \tag{3.8}$$

Selecting

$$f_u = f^2 \tag{3.9}$$

or

$$f = -\frac{1}{u} \tag{3.10}$$

the equation simplifies into

$$u' + a(x) = 0 \tag{3.11}$$

Solving for u and substituting back into (3.11) leads to

$$y = f = \frac{1}{c+\int a(x)dx} \tag{3.12}$$

Example 3.2

Consider the nonlinear equation

$$xy' + cos^2 y = 0 \; y(1) = 0 \quad . \tag{3.13}$$

A substitution of $y = f(u)$ leads to

$$xf_u u' + cos^2 f = 0 \tag{3.14}$$

Selecting

$$f_u = cos^2 f \tag{3.15}$$

or

$$f = Arctan(u) \tag{3.16}$$

the equation simplifies into

$$xu' + 1 = 0 \tag{3.17}$$

Solving the equation for u

$$u = -lnx + c \tag{3.18}$$

and substituting back into (3.16) together with the application of the condition leads to the exact solution of the problem

$$y = Arctan(-lnx) \tag{3.19}$$

4. Second Order Equations

First the linear equations and then the non-linear second order equations will be treated with the method.

4.1 *Linear Equations*

Consider the general linear variable coefficient homogenous equation

$$y'' + p(x)y' + q(x)y = 0 \quad . \tag{4.1}$$

A substitution of $y = f(u)$ transforms the equation into

$$f_{uu}u'^2 + f_u u'' + p(x)f_u u' + q(x)f = 0 \tag{4.2}$$

The choice of $f_{uu} = f_u = f$ is an apparent simplification which determines the function

$$y = f = e^u \tag{4.3}$$

and leaves the remaining equation for u

$$u'^2 + u'' + p(x)u' + q(x) = 0 \quad . \tag{4.4}$$

Defining $v = u'$, the equation transforms into a Riccati equation

$$v' + p(x)v + v^2 + q(x) = 0 \quad . \tag{4.5}$$

If the Riccati equation possesses an analytical solution, then back substitution yields

$$u = \int vdx \tag{4.6}$$

and from (4.3) one solution is

$$y = e^{\int vdx} \tag{4.7}$$

Hence, solution of a variable coefficient second order linear equation reduces to the solution of the Riccati equation. For a recent detailed analysis on Riccati equations, see Ndiaye (2022) [13].
The method also leads to the well-known exponential solutions for constant coefficient second order equations for the special case of the variable coefficients.

Example 4.1
Consider the second order equation

$$x^2 y'' + xy' - 9y = 0 \tag{4.8}$$

which upon dividing by the coefficient x^2 yields

$$p(x) = \frac{1}{x}, \quad q(x) = -\frac{9}{x^2} \tag{4.9}$$

The Riccati equation to be solved is

$$v' + \frac{1}{x} v + v^2 - \frac{9}{x^2} = 0 \tag{4.10}$$

The trial function $v = c/x$, leads to $c = \mp 3$. Hence

$$u = \int v dx = \mp 3 ln x + c \tag{4.11}$$

and from (4.3) the solution space is spanned by the functions

$$y = c_1 x^3 + c_2 x^{-3} \tag{4.12}$$

Example 4.2
Consider the second order equation

$$y'' - 2xy' - 2y = 0 \tag{4.13}$$

for which

$$p(x) = -2x, \quad q(x) = -2 \tag{4.14}$$

The Riccati equation is

$$v' - 2xv + v^2 - 2 = 0 \tag{4.15}$$

One obvious solution is

$$v = 2x, \rightarrow \ u = x^2 \tag{4.16}$$

The other solution can be retrieved from the transformation $v = 2x + \frac{1}{p}$ which eventually leads to

$$v = 2x + \frac{1}{e^{x^2} \int e^{-x^2} dx}, \rightarrow \ u = x^2 + \int \frac{dx}{e^{x^2} \int e^{-x^2} dx} \tag{4.17}$$

The final solution is

$$y = c_1 exp(x^2) + c_2 exp\left(x^2 + \int \frac{dx}{e^{x^2} \int e^{-x^2} dx}\right) \tag{4.18}$$

4.2 *Non-Linear Equations*
A general algorithm does not exist for the nonlinear equations. Instead, inspection and experience play an important role in search of exact solutions by this method. A number of sample problems will be treated.

Example 4.3
Consider the second order non-linear equation

$$yy'' + y'^2 = 0 \quad y(0) = 1, y'(0) = 2 \tag{4.19}$$

for which the substitution $y = f(u)$ leads to

$f(f_{uu}u'^2 + f_u u'') + f_u^2 u'^2 = 0$ (4.20)

The equation can be simplified if $ff_{uu} = ff_u = f_u^2$ or.

$y = f = e^u$ (4.21)

The remaining equation is

$u'' + 2u'^2 = 0$ (4.22)

which possesses a solution

$u = \frac{1}{2}\ln(2x + c_1) + c_2$ (4.23)

The final solution satisfying the conditions is

$y = \sqrt{4x + 1}$ (4.24)

Example 4.4
Consider the second order non-linear equation

$yy'' + y'^2 + a(x)yy' = 0$ (4.25)

for which the substitution

$yy' = f(u)$ (4.26)

leads to

$f_u u' + af = 0$ (4.27)

which suggests a selection

$f = e^u$. (4.28)

The remaining equation is

$u' + a = 0$ (4.29)

which possesses a solution

$u = -\int a(x)dx + lnc_1$ (4.30)

From (4.26) and (4.28)

$yy' = f(u) = c_1 e^{-\int a(x)dx}$ (4.31)

with a direct integral of

$y = \left(c_1 e^{-\int a(x)dx} + c_2\right)^{1/2}$ (4.32)

Example 4.5
Consider the second order non-linear equation

$xy'' + (2y + x)y' + y'^2 + y^2 = 0$ (4.33)

For which the substitution

$y + y' = f(u)$ (4.34)

Transforms the equation

$xf_u u' + f^2 = 0$ (4.35)

Which suggests a selection

$$f = \frac{1}{u} \tag{4.36}$$

For simplification. The remaining equation is

$$xu' = 1 \tag{4.37}$$

Which possesses a solution

$$u = lnx + c_1 \tag{4.38}$$

From (4.34) and (4.36)

$$y' + y = \frac{1}{c_1 + lnx} \tag{4.39}$$

Which is a first order linear equation with a solution

$$y = e^{-x} \int \frac{e^x}{c_1 + lnx} dx + c_2 e^{-x} \tag{4.40}$$

5. Applications to Physical Problems

Three applied problems derived from physical phenomena will be solved by the method.

5.1 *Optimum Path of a Flying Object*

In an exponentially decaying density medium, the path for minimum work of a flying object against the drag force of air has been dictated by the ordinary differential equation (Pakdemirli, 2009; Abbasbandy *et al*., 2009) [15, 3].

$$y'' + \varepsilon(1 + \beta^2 y'^2) = 0 \quad y(0) = 0, y(1) = 1 \tag{5.1}$$

Where $\varepsilon = \frac{\alpha R^2}{h}$ is the dimensionless parameter related to the density decay, $\beta = \frac{h}{R}$ is the ratio of the final height over the final distance.

Assume a functional substitution of the form

$$y' = f(u) \tag{5.2}$$

which transforms the equation into

$$f_u u' + \varepsilon(1 + \beta^2 f^2) = 0 \tag{5.3}$$

To simplify the equation, one may select

$$f_u = 1 + \beta^2 f^2 \tag{5.4}$$

with a solution for f

$$f = \frac{1}{\beta} \tan(\beta u + c_1) \tag{5.5}$$

The remaining equation for u is trivial

$$u' + \varepsilon = 0 \quad \rightarrow u = -\varepsilon x + c_2 \tag{5.6}$$

Substituting (5.6) and (5.5) into (5.2) and integrating once, the solution is

$$y = \frac{1}{\varepsilon\beta^2} ln[cos(c_1 - \varepsilon\beta x)] + c_2 \quad . \tag{5.7}$$

Imposing the boundary conditions given in (5.1) yields the final physical solution

$$y = \frac{1}{\varepsilon\beta^2} ln\left[cos\varepsilon\beta x + sin\varepsilon\beta x \frac{e^{\varepsilon\beta^2} - cos\varepsilon\beta}{sin\varepsilon\beta}\right] \tag{5.8}$$

which is given by Pakdemirli (2009) [3].

5.2 *Bratu's Initial Value Problem:* Bratu type problems arise in a wide range of applications ranging from combustion theory, heat transfer, and expansion of the universe to electrospinning processes (Boyd, 1986; Wan *et al*., 2004; Wazwaz, 2005; Aksoy & Pakdemirli, 2010) [7, 20, 21, 4]. The Bratu's initial value problem is.

$$u'' - 2e^{u} = 0 \quad 0 \le x \le 1, u(0) = u'(0) = 0 \tag{5.9}$$

We assume a transformation

$$u = c\, lnf(v) \tag{5.10}$$

Which when substituted into the equation yields the transformed equation

$$c\frac{f_{vv}v'^2}{f^2} + c\frac{f_v v''}{f} - c\frac{f_v^2 v'^2}{f^2} - 2f^c = 0 \tag{5.11}$$

Selecting $c = -2$ and multiplying the equation by f^2 introduces a simplification

$$-2f_{vv}v'^2 - 2f_v f v'' + 2f_v^2 v'^2 - 2 = 0 \tag{5.12}$$

One may assume $f = v$ for simplicity, reducing the equation to

$$-2v''v + 2v'^2 - 2 = 0 \tag{5.13}$$

With a solution of

$$v = cosx \tag{5.14}$$

Back-substituting yields the exact solution

$$u = -2ln(cosx) \tag{5.15}$$

Which satisfies the conditions also.

5.3 *Heat Transfer Conduction in a Slab*

The heat transfer conduction in a slab with temperature dependent thermal conductivity is expressed by the boundary value problem (Rajabi *et al*., 2007; Aksoy *et al*., 2012) [17, 5].

$$\frac{d}{dy}\left[(1+\varepsilon u)\frac{du}{dy}\right] = 0 \quad u(0) = 1, u(1) = 0 \tag{5.16}$$

Assuming a transformation

$$1 + \varepsilon u = f(v) \tag{5.17}$$

the equation reduces to

$$\frac{d}{dy}\left[f\frac{f_v v'}{\varepsilon}\right] = 0 \tag{5.18}$$

Selecting $f = v$ simplifies the equation

$$vv' = \varepsilon c_1 \tag{5.19}$$

Solving and back-substituting, the exact solution for the problem satisfying the boundary conditions is

$$u = \frac{1}{\varepsilon}\left\{\sqrt{(1-(1+\varepsilon)^2)y + (1+\varepsilon)^2} - 1\right\} \tag{5.20}$$

6. Concluding Remarks

Functional substitution method is proposed in search of analytical solutions to the ordinary differential equations. The method depends on assuming a functional form of an arbitrary transfer function with an arbitrary function as its argument, separating the equations for the functional form and the argument so that more simplified equations readily solvable can be obtained and finally back-substituting all to find the exact solution. The method is exploited for first and second order linear and non-linear differential equations. Three differential equations stemming from physical applications are solved by the method also. The method is an attempt to systemize transformation techniques although some intuition and experience is needed to determine a suitable form for the transformations.

6.1 Declaration
Author declares no conflict of interest.

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