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Common fixed point theorems with applications

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Abstract

Fixed point theorems play a fundamental role in various branches of mathematics and have significant applications in real-world problem-solving. This paper explores common fixed point theorems and their diverse applications across different mathematical domains. Beginning with an overview of foundational concepts, including fixed point theorems and their significance, the paper delves into specific common fixed point theorems such as the Banach Fixed Point Theorem, the Kakutani Fixed Point Theorem, and the Browder Fixed Point Theorem. Each theorem is examined in terms of its assumptions, conditions, and implications, providing insights into their applicability in different contexts.

Furthermore, this paper discusses applications of common fixed point theorems in areas such as optimization, economics, game theory, and computer science. These applications demonstrate the versatility and utility of fixed point theorems in modeling and solving real-world problems. Moreover, the paper highlights recent developments and extensions of common fixed point theorems, as well as open problems and avenues for future research in this field.

By elucidating the theoretical underpinnings and practical applications of common fixed point theorems, this paper aims to contribute to a deeper understanding of their significance in mathematics and their relevance to diverse interdisciplinary fields.

Keywords: Fixed point theorems, real-world problem-solving, mathematics

Introduction

It is well known that many problems in topology, non-linear analysis, mathematical economics and game theory give rise to fixed point problems for some uni-valued as set-valued mappings. The fixed point theory of set valued mapping began in 1937 when, for proving his famous minimax theorem, von Neumann [1] reduced the problem of existence of a saddle point for a given bifurcation to that of existence of a fixed point for an associated set – valued mapping. Then in 1941 [2], in order to give a simpler proof of von Neumann's Minimax theorem, kalcutanl established a fixed point theorem for an upper semi continuous set valued mapping with nov empty, compact and convex values defined or a compact and convex subset of Euclidean space.

Kakutani's theorem was extended to Banach spaces by Bohenenblust and Karlin in 1950 and two years later, independently, by Fan and Glicksberg to locally convex topological vector spaces. The result of Fan and Glicksberg, Known today as the Kakutani-Fan-Glicksberg fixed point theorem, is a powerful tool in showing the existence of solutions of several problem in pure and applied Mathematics. This fact determined the appearance of large number of versions of generalisations of the Kakutani – Fan – Glicksberg fixed point theorem.

The Fan – Glicksberg Fixed Point Theorem

In constructive mathematics a non empty set is called an inhabited set. A sets is inhabited if there exists an element of s. In constructive mathematics compactness of a set means total boundedness with completeness. A set s is finitely enumerable if there exist a natural number N and a mapping of the set {1,2,.....,N} onto s. An ∈ - approximation to s, set in a metric space, is a subset of S such that for each X E S there exists y in that ∈ - approximation with (x-y | < ∈ | x-y | is the distance between x and y). s is totally bounded if for each ∈ >0 there exists a finitely enumerable ∈ - approximation to s. Completeness of a set of course means that every Cauchy sequence in the set converges.

We consider an n-dimensional simplex Δ as a compact metric space.

Lemma 1: For each - ∈ >0 there exist totally bounded sets H_1, H_2,.......... H_n each of diameter less than or equal to ∈ such that $\Delta = U_{i=J}^{n}$ H_i.

The notion that a function of from Δ to it self has at most one fixe point by [3-7] define as –

Definition 1 (At most one fixe point): For all x,y ∈ Δ, if x ≠ y then f (x) ≠ x $\theta 2$ f (y) ≠ y.

Definition 2 (Sequential local non-constancy of functions)
There exists $\bar{\epsilon}$ > o with the following property. For each ∈ = o then or equal to $\bar{\epsilon}$ there exist totally bounded H_1, H_2, H_m, each of diameter less than or equal to ε, such that Δ= $U_{i=J}^{n}$ H_i, and if for all sequences (x_n) n ≥ 1, (Y_n) n ≥ 1 in each Hi, If (Xn) – Xn |→o and if (Yn)-Yn | → O, then | Xn – Yn | → O

Definition 3 (Sequential local non constancy of Multi functions)
There exists $\bar{\varepsilon} \leq$ O with the following property. For each $\varepsilon > o$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets H_1, H_2,......H_n
Each of diameter less than or equal to, such that Δ= $U_{i=J}^{n}$ H_i, and if for all sequences (X_n)n ≥ 1, (Y_n) n > 1 in each H_i | ∅ (Yn) | → O then | Xn – Yn | → O

Definition 4: (Approximate fixe point of a multifunction):
For each $\varepsilon > 0x$ is an appropriate fixed point of a multi – function ∅ from X to the collection of its inhabited subsets if

$$\sum_{iEf} Pi\ \big(x* - \emptyset(x*)\big) < \epsilon$$

For each finitely enumerable FCI, where Pi (x* - $\emptyset\ (x*)$) = inf Y E $\emptyset(x*) pi\ (x* - y)$. This imfimum exists because a totally bounded set in a locally convex space is located.
Theorem 1: (The Fan – Glicksberge fixe point theorem for sequentially locally non constant multi functions).
Let x be a compact (totally bounded and complete) and convex subset of a locally convex space E, and ∅ be a convex and compact value, sequentially locally non-constant multi function with uniformally closed graph from x to the collection of inhabited subsets of x. Then ∅ has a fixed point.

Proof
1. According to $\emptyset x_v$ has an appropriate fixe d point, that is, for any n > o there exists x* such that

$$\sum_{i \in F} Pi\ \big(x* - \emptyset xv\ (x*)\big) < n$$

For each finitely enumerable FCI. Then,

$$\sum_{i \in F} Pi\ \big(x* - \emptyset xv\ (x*)\big) < n + \tau$$

For $\tau > o$. Let $\in = n + \tau$ we have

$$\sum_{i \in F} Pi\ \big(x* - \emptyset\ (x*)\big) < \in$$

Since n and τ are arbitrary and so ∈ is arbitrary, we obtain,

$$\underset{x \in Hj}{inf} \sum_{i \in F} Pi\ \big(x* - \emptyset xv\ (x*)\big) = 0$$

For some Hj such that X = $U_{j=1}^{m} Hj$

2. Choose a sequence (Zn)n ≥1 in some Hi such that

$$\sum i \in Fpi\ (\emptyset(Zn) - Zn) \to O$$

For each $\in > O$ there exists n > O such that if x, y ∈ Hi, $\sum i \in Fpi\ (\emptyset(y) - y) < n$, than $\sum i \in Fpi\ (x - y) \leq \in$
Assume that the set

$$K = \{(x, y) \in Hi\ X\ Hi: \sum_{i \in F} Pi\ \big((x - y) \geq \in\big)\}$$

Is non empty and compact.

Since the mapping (x,y) → max ($\sum i \in Fpi\ (\emptyset(x) - x)$,

$\sum i \in Fpi\ (\emptyset(y) - y)$ is uniformly continuous we can construct an increasing binary sequence $(\gamma n)n \geq 1$ such that γn = 0

=inf max $\sum_{i \in F} Pi\ (\emptyset\ (x) - x), \sum_{i \in F} Pi\ (\emptyset\ (y) - y)$ <2^{-n}
(x,y)∈ K

$\gamma n = 1$

= inf max ($\sum_{I \in F} pi\ (\emptyset(x) - x)$, $\sum_{I \in F} pi\ (\emptyset(y) - y$ <2^{-n-1}

(x,y) 6k

It suffices to find n such that $\gamma n = 1$. In that case, if $\sum i \in F\ pi\ (\emptyset(x) - x)$ <2 $^{-n-1}$, $\sum i \in F\ pi\ (\emptyset(y) - y)$ <2 $^{-n}$, we have (x,y) k and $\sum i \in F\ pi\ (x - y) \leq \in$.

Assume $\gamma 1 = 0. If\ \gamma n = O, choose\ (xn, yn) \epsilon K$ such that max. ($\sum i \in F\ pi\ (\emptyset(xn) - xn), \sum i \in F\ pi\ (\emptyset(Yn) - Yn)$ <2 $^{-n}$ and if γn =1, set Xn = Yn = 2n. Then, $\sum i \in F\ pi\ (\emptyset(2n) - xn) \to O$

Computing N, such that

$$\sum i \in F\ pi\ (xn - Yn) < \in$$

We must have $\gamma n = 1$.

Applications
In general equilibrium theory in economics, Kakutani's theorem has been used to prove the ex is tense of set of prices which simultaneously equate supply with demand in all markets of an economy. The first proof of this result was constructed by Lionet Mckenzie.

Fair division
This fixe point theorem is used in proving the existence of cake allocations that are both envy free Pareto efficient. This result is known as Weller's theorem.

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